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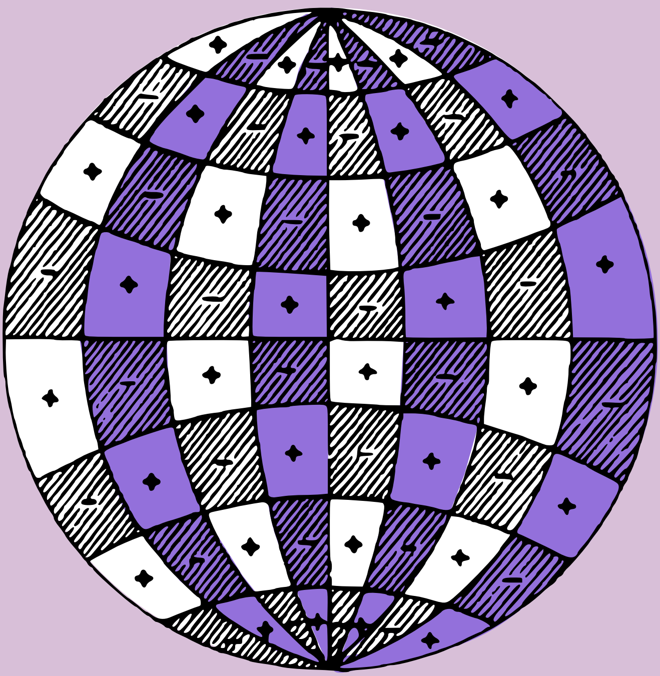

DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS

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Translated by SCRIPTA TECHNICA, INC.

Translation Editor
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Department of Mathematics,
Brown University, Providence, R.I.

1964

NORTH-HOLLAND PUBLISHING COMPANY—AMSTERDAM

PUBLISHERS:
NORTH-HOLLAND PUBLISHING COMPANY, AMSTERDAM

SOLE DISTRIBUTORS FOR U.S.A.:
INTERSCIENCE PUBLISHERS, A DIVISION OF
JOHN WILEY & SONS, INC., NEW YORK

Printed in The Netherlands

PREFACE

The rapid development of contemporary technology requires ever more extensive mathematical preparation for engineers. This has resulted in a demand for a more complete exposition of the applications of the fundamental mathematical disciplines for engineers, technicians, and students in technological institutions.

The present book examines a number of physical and technical problems which involve second-order partial differential equations. Considerable attention is also given to the theory of such equations. In addition, the text includes several chapters and sections of a general nature (indicated by an asterisk). The material in these sections does not as yet have direct application; nonetheless, it is important for an understanding of contemporary scientific literature on mathematical physics.

Among the applications studied are the vibrations of strings, membranes, and shafts; electric oscillations in lines; the electrostatic problem; the basic gravimetric problem; the emission of electromagnetic waves and their distribution along wave guides and in horns; the emission and dispersion of sound; gravity waves on the surface of a liquid; heat flow in a solid body, and so forth. Solutions are given to both very simple and more complicated problems, making it possible for the reader to master the methods considered in the book and also the physics of the phenomena in question. In almost every chapter, there are problems whose basic purpose is to develop the reader's technical skill.

Approximate methods for solving problems in mathematical physics are not discussed, since their exposition would require a considerable increase in the size of the book. Also excluded are certain specialized problems (for example, those associated with the physics of atomic reactors) that have arisen only in the last few years.

The preparation of the book was carried out under the guidance and with the cooperation of Member-Correspondent of the Academy of Sciences of the USSR, Professor Nikolai Sergeevich Koshlyakov, whose untimely death occurred before publication of the book. A noted specialist in the field of analytic number theory and higher transcendental functions, Prof. Koshlyakov published a number of works in the field of mathematical physics. In the course of his career, which included more than 30 years of scientific and pedagogical activity, as well as 15 years of research in applied problems, Prof. Koshlyakov always devoted a great deal of attention to the mathematical education of engineers. An excellent lecturer and teacher, he enjoyed the constant respect and devotion of his listeners and students. His textbook Osnovnye Differentsial'nye Uravneniy̆a Matematicheskoi Fiziki (Basic Differential Equations of Mathematical Physics),

several chapters of which are used in the present book, has seen four editions (the latest in 1936).

The authors of the Introduction and Parts I and III are N. S. Koshlyakov and M. M. Smirnov; the authors of parts II and IV are N. S. Koshlyakov and E. B. Gliner.

We take this occasion to express our deep gratitude to I. M. Gel'fand, G. I. Zel'tser and G. P. Samosyuk for graciously reading the individual sections of the book, to G. Yu. Dzhanelidze and S. I. Amosov for making a thorough review of the manuscript, and especially to Scientific Instructor G.P. Akilov. All of these made a number of valuable comments leading to an improvement in the text and to the correction of a number of errors.

Leningrad, September 5, 1951

E. B. Gliner
M. M. Smirnov

CONTENTS

INTRODUCTION

1. *The fundamental differential equations of mathematical physics*

Many problems in mechanics and physics involve the *study* of second-order partial differential equations. The following are some examples: (1) The study of various types of waves – elastic, acoustic, and electromagnetic – and of other oscillational phenomena leads to the *wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{1}$$

where c is the velocity of propagation of the wave in the given medium. (2) The processes of heat flow in a homogeneous isotropic body and other diffusion phenomena are described by the *heat-flow equation:*

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right). \tag{2}$$

(3) Study of a steady thermal state in a homogeneous isotropic body leads to *Poisson's equation:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = - f(x, y, z) . \tag{3}$$

In the absence of internal heat sources, eq. (3) becomes *Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 . \tag{4}$$

The potentials of a gravitational or of a constant electric field in which there are no masses or electric charges also satisfy Laplace's equation.

Equations (1) - (4) are often called the fundamental equations of mathematical physics. A detailed study of these equations makes possible the theoretical treatment of a large number of physical phenomena and the solution of many physical and technical problems.

Each of eqs. (1) - (4) has an infinite number of particular solutions. In solving a specific physical problem, it is necessary to choose from among these solutions the one that satisfies certain additional conditions imposed by the physical situation. Thus, *problems in mathematical physics reduce to finding solutions to partial differential equations that satisfy certain additional conditions*. The most common of these additional conditions are the so-called *boundary conditions* (conditions that must be satisfied at the boundary of the medium in question) and *initial conditions* (which must be satisfied at the particular instant of time at which consideration of a physical process begins).

Let us note one very important point. *A problem in mathematical phys-*

ics is considered to be stated correctly if the problem has exactly one stable solution satisfying all the conditions. By "stable", we mean that small changes in any of the given conditions of the problem must cause a correspondingly small change in the solution. The existence and uniqueness requirement means that among the given conditions there are none that are incompatible and that these conditions are sufficient to determine a unique solution. The stability requirement is necessary for the following reason. In the given conditions for a specific problem, especially if they are obtained from experiment, there is always some error, and it is necessary that a small error in the given conditions causes only a small inaccuracy in the solution. This requirement expresses the physical determinacy of the stated problem.

Determining whether a problem in mathematical physics is stated correctly is a very important and at the same time extremely difficult question in the theory of partial differential equations. We shall not make a complete study of this question in the present book.

The following three sections are devoted to the classification of second-order equations of the form

$$\sum_{i,j=1}^{n} A_{ij}(x_1,\dots,x_n)\frac{\partial^2 u}{\partial x_i\,\partial x_j} + F\left(x_1,\dots,x_n,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right) = 0 ,$$

and, in the case of two independent variables, to their reduction to canonical form.

2. *The reduction of second-order equations to canonical form*

Let us examine a second-order equation with two independent variables that is linear with respect to the second-order derivatives:

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) = 0 , \tag{5}$$

where A, B, and C are functions of x and y that have continuous derivatives up to the second order.

Let us replace x and y by the new independent variables ξ and η. Suppose that

$$\xi = \varphi_1(x,y) , \qquad \eta = \varphi_2(x,y) \tag{6}$$

are twice continuously differentiable functions and that the Jacobian

$$\frac{\partial(\varphi_1,\varphi_2)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial\varphi_1}{\partial x} & \dfrac{\partial\varphi_1}{\partial y} \\ \dfrac{\partial\varphi_2}{\partial x} & \dfrac{\partial\varphi_2}{\partial y} \end{vmatrix} \neq 0 \tag{7}$$

throughout the region in question.

The derivatives with respect to the old variables are expressed in terms of the derivatives with respect to the new variables according to the formulae

$$
\begin{gathered}
\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}, \qquad \frac{\partial u}{\partial y}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y},\\
\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2+2\frac{\partial^2 u}{\partial \xi\,\partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}+\frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2+\frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2},\\
\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2+2\frac{\partial^2 u}{\partial \xi\,\partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}+\frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2+\frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2}+\frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2},\\
\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}+\frac{\partial^2 u}{\partial \xi\,\partial \eta}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}+\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)+\frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}+\frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x\,\partial y}+\frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x\,\partial y}.
\end{gathered}
\tag{8}
$$

When we substitute the values of the derivatives in (8) into eq. (5) we obtain

$$
\bar{A}(\xi,\eta)\frac{\partial^2 u}{\partial \xi^2}+2\bar{B}(\xi,\eta)\frac{\partial^2 u}{\partial \xi\,\partial \eta}+\bar{C}(\xi,\eta)\frac{\partial^2 u}{\partial \eta^2}+F\left(\xi,\eta,u,\frac{\partial u}{\partial \xi},\frac{\partial u}{\partial \eta}\right)=0, \tag{9}
$$

where

$$
\begin{aligned}
\bar{A}(\xi,\eta)&=A\left(\frac{\partial \xi}{\partial x}\right)^2+2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}+C\left(\frac{\partial \xi}{\partial y}\right)^2,\\
\bar{B}(\xi,\eta)&=A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}+B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}+\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)+C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y},\\
\bar{C}(\xi,\eta)&=A\left(\frac{\partial \eta}{\partial x}\right)^2+2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}+C\left(\frac{\partial \eta}{\partial y}\right)^2
\end{aligned}
\tag{10}
$$

By a direct substitution, we can easily verify that

$$
\bar{B}^2-\bar{A}\bar{C}=(B^2-AC)\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}-\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)^2 \tag{11}
$$

In the transformation (6), the two functions $\varphi_1(x,y)$ and $\varphi_2(x,y)$ are at our disposal. Let us show that it is possible to choose them so that only one of the following conditions will be satisfied:

$$
1)\ \bar{A}=0,\ \bar{C}=0, \qquad 2)\ \bar{A}=0,\ \bar{B}=0, \qquad 3)\ \bar{A}=\bar{C},\ \bar{B}=0.
$$

Then, obviously, the transformed equation (9) will take the simplest form.

Let us examine the first-order differential equation

$$
A\left(\frac{\partial \varphi}{\partial x}\right)^2+2B\frac{\partial \varphi}{\partial x}\frac{\partial \varphi}{\partial y}+C\left(\frac{\partial \varphi}{\partial y}\right)^2=0. \tag{12}
$$

We must examine separately the cases $B^2-AC>0$, $B^2-AC<0$, and $B^2-AC=0$ throughout the entire region. The case in which the expression B^2-AC changes sign in the region will be examined later.

CASE I: B^2-AC greater than zero. In this case, eq. (5) is said to be of the *hyperbolic* type. We may assume that either $A\neq 0$ or $C\neq 0$. We shall examine separately the case when $A=C=0$. Without loss of generality, we may assume that $A\neq 0$ everywhere. Then, eq. (12) may be written in the form

$$\left\{A\frac{\partial\varphi}{\partial x}+(B+\sqrt{B^2-AC})\frac{\partial\varphi}{\partial y}\right\}\left\{A\frac{\partial\varphi}{\partial x}+(B-\sqrt{B^2-AC})\frac{\partial\varphi}{\partial y}\right\}=0\ .$$

This equation can be separated into two equations:

$$A\frac{\partial\varphi}{\partial x}+(B+\sqrt{B^2-AC})\frac{\partial\varphi}{\partial y}=0\ , \tag{12a}$$

$$A\frac{\partial\varphi}{\partial x}+(B-\sqrt{B^2-AC})\frac{\partial\varphi}{\partial y}=0\ . \tag{12b}$$

Consequently, the solutions of each of eqs. (12a) and (12b) will be solutions of eq. (12).

To integrate eqs. (12a) and (12b), we set up corresponding systems of differential equations [1])

$$\frac{dx}{A}=\frac{dy}{B+\sqrt{B^2-AC}}\ ,\qquad \frac{dx}{A}=\frac{dy}{B-\sqrt{B^2-AC}}\ ,$$

or

$$\begin{aligned}A\,dy-(B+\sqrt{B^2-AC})\,dx&=0\ ,\\ A\,dy-(B-\sqrt{B^2-AC})\,dx&=0\ .\end{aligned} \tag{13}$$

We note that eqs. (13) may be written in the form of a single equation

$$A\,dy^2-2B\,dx\,dy+C\,dx^2=0\ . \tag{13a}$$

Suppose that

$$\varphi_1(x,y)=\text{constant},\qquad \varphi_2(x,y)=\text{constant} \tag{14}$$

are solutions of eq. (13). Then, as we know, their left members will be solutions of eqs. (12a) and (12b) and hence of eq. (12).

The curves representing (14) are called the *characteristic curves* or simply the *characteristics* of eq. (5), and eq. (12) is called the *equation of the characteristics*.

For an equation of the hyperbolic type ($B^2-AC>0$), the solutions (14) will be real and distinct. Here, we have two distinct families of real characteristics.

In eq. (6), let us set

$$\xi=\varphi_1(x,y)\ ,\qquad \eta=\varphi_2(x,y)\ ,$$

where $\varphi_1(x,y)$ and $\varphi_2(x,y)$ are solutions of eq. (12). Then, on the basis of (10), $\bar{A}=\bar{C}=0$ in eq. (9). The coefficient $\bar{B}$ is everywhere different from zero in the region in question – a consequence of (7) and (11). Dividing eq. (9) by the coefficient $2\bar{B}\neq 0$, we reduce it to the form

$$\frac{\partial^2 u}{\partial\xi\,\partial\eta}=F_1\left(\xi,\eta,u,\frac{\partial u}{\partial\xi},\frac{\partial u}{\partial\eta}\right)\ . \tag{15}$$

This is the *canonical form of an equation of the hyperbolic type*.

If $A=C=0$, eq. (5) is of the hyperbolic type and is already in canonical form.

If eq. (5) is linear with respect to the first-order derivatives and to the function u itself, the transformed equation will also be linear:

$$\frac{\partial^2 u}{\partial\xi\,\partial\eta} + a(\xi,\eta)\frac{\partial u}{\partial\xi} + b(\xi,\eta)\frac{\partial u}{\partial\eta} + c(\xi,\eta)\,u = f(\xi,\eta)\ . \tag{16}$$

Setting

$$\xi = \mu + \nu\ , \qquad \eta = \mu - \nu\ ,$$

we reduce eq. (15) to the form

$$\frac{\partial^2 u}{\partial\mu^2} - \frac{\partial^2 u}{\partial\nu^2} = \Phi\left(\mu,\nu,u,\frac{\partial u}{\partial\nu},\frac{\partial u}{\partial\mu}\right)\ .$$

This is the *second canonical form* of an equation of the hyperbolic type.

CASE II: Suppose that $B^2 - AC = 0$ throughout the region in question. In this case, eq. (5) is of the *parabolic* type. We shall assume that throughout the region the coefficients in eq. (5) do not vanish simultaneously. The condition that $B^2 - AC = 0$ implies that at every point of this region one of the coefficients A and C is different from zero. Without loss of generality, we may assume that A is everywhere different from zero. Then, eqs. (12a) and (12b) are identical and take the form

$$A\frac{\partial\varphi}{\partial x} + B\frac{\partial\varphi}{\partial y} = 0\ . \tag{17}$$

It is easy to see that every solution of eq. (17), where $B^2 - AC = 0$, also satisfies the equation

$$B\frac{\partial\varphi}{\partial x} + C\frac{\partial\varphi}{\partial y} = 0\ . \tag{18}$$

We note that for an equation of the parabolic type the solutions (14) coincide, and we have only one family of real characteristics $\varphi_1(x,y) = \text{const.}$

Let us set

$$\xi = \varphi_1(x,y)\ ,$$

where $\varphi_1(x,y)$ is a solution of eq. (17). For $\varphi_2(x,y)$, let us take any function such that the Jacobian $\partial(\varphi_1,\varphi_2)/\partial(x,y) \neq 0$. Since A is different from zero and, consequently, $\partial\varphi_1/\partial y$ is different from zero, we may take $\varphi_2 = x$. Then, on the basis of (10), $\bar{A}$ is identically equal to zero in eq. (9) and the coefficient of $\partial^2 u/\partial\xi\,\partial\eta$ is of the following form:

$$\bar{B} = \left(A\frac{\partial\varphi_1}{\partial x} + B\frac{\partial\varphi_1}{\partial y}\right)\frac{\partial\varphi_2}{\partial x} + \left(B\frac{\partial\varphi_1}{\partial x} + C\frac{\partial\varphi_1}{\partial y}\right)\frac{\partial\varphi_2}{\partial y}\ .$$

According to (17) and (18), $\bar{B}$ is identically equal to zero in the region in question. The coefficient $\bar{C}$ in eq. (9) is transformed to the form

$$\bar{C} = \frac{1}{A}\left(A\frac{\partial\varphi_2}{\partial x} + B\frac{\partial\varphi_2}{\partial y}\right)\ ,$$

and hence $\bar{C} \neq 0$, because otherwise, on the basis of eq. (17), the Jacobian $\partial(\varphi_1,\varphi_2)/\partial(x,y)$ would vanish. Dividing eq. (9) by $\bar{C} \neq 0$, we reduce it to the form

$$\frac{\partial^2 u}{\partial \eta^2} = F_2\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) . \tag{19}$$

This is the *canonical form of an equation of the parabolic type*.

If eq. (5) is linear, eq. (19) will also be linear:

$$\frac{\partial^2 u}{\partial \eta^2} + a_1(\xi,\eta)\frac{\partial u}{\partial \xi} + b_1(\xi,\eta)\frac{\partial u}{\partial \eta} + c_1(\xi,\eta)\, u = f_1(\xi,\eta) . \tag{20}$$

CASE III. Suppose that $B^2 - AC < 0$ throughout the region in question. Eq. (5) is then said to be of the *elliptic* type *. It is easy to see that in this case the solutions (14) will be complex-conjugate and we shall not have real characteristics.

Let us set

$$\xi + i\eta = \varphi_1(x,y) , \qquad \xi - i\eta = \varphi_2(x,y) ,$$

where φ_1 and φ_2 are complex-conjugate functions satisfying eq. (12).

Making the substitution $\varphi_1(x,y) = \xi + i\eta$ in eq. (12), we obtain

$$A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2 - A\left(\frac{\partial \eta}{\partial x}\right)^2 - 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} - C\left(\frac{\partial \eta}{\partial y}\right)^2$$

$$+ 2i\left\{A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}\right\} = 0 .$$

Setting the real and imaginary parts of this identity equal to zero, we obtain

$$A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2 = A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C\left(\frac{\partial \eta}{\partial y}\right)^2 ,$$

$$A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} = 0 .$$

Hence, it follows on the basis of (10) that

$$\bar{A} = \bar{C} , \qquad \bar{B} = 0$$

and, after division by $\bar{A} \neq 0$, eq. (9) takes the form

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = F_3\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) . \tag{21}$$

This is the *canonical form of an equation of the elliptic type*.

If eq. (5) is linear, eq. (21) will also be linear:

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + a_2(\xi,\eta)\frac{\partial u}{\partial \xi} + b_2(\xi,\eta)\frac{\partial u}{\partial \eta} + c_2(\xi,\eta)u = f_2(\xi,\eta) . \tag{22}$$

Example. Let us consider the equation

$$x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} = 0 \qquad (x > 0,\ y > 0) . \tag{23}$$

* In the reduction of equations of the elliptic type to canonical form, we shall confine ourselves to analytic coefficients A, B, and C. Thus, we shall be able to find the solution to eqs. (12a) and (12b) in the form of an analytic function.

This equation is of the hyperbolic type because

$$B^2 - AC = x^2y^2 > 0 .$$

We set up the equation of the characteristics (13a) according to the general theory:

$$x^2 \, dy^2 - y^2 \, dx^2 = 0$$

or

$$x \, dy + y \, dx = 0 , \qquad x \, dy - y \, dx = 0 .$$

Integrating these equations, we obtain

$$xy = C_1 , \qquad y/x = C_2 .$$

Consequently, we need to introduce new variables ξ and η, defined by the formulae

$$\xi = xy , \qquad \eta = y/x .$$

Then, from (8), we obtain

$$\frac{\partial^2 u}{\partial x^2} = y^2 \frac{\partial^2 u}{\partial \xi^2} - 2 \frac{y^2}{x^2} \frac{\partial^2 u}{\partial \xi \, \partial \eta} + \frac{y^2}{x^4} \frac{\partial^2 u}{\partial \eta^2} + 2 \frac{y}{x^3} \frac{\partial u}{\partial \eta} ,$$

$$\frac{\partial^2 u}{\partial y^2} = x^2 \frac{\partial^2 u}{\partial \xi^2} + 2 \frac{\partial^2 u}{\partial \xi \, \partial \eta} + \frac{1}{x^2} \frac{\partial^2 u}{\partial \eta^2} .$$

Substituting the values of the second derivatives in eq. (23), we reduce it to the canonical form:

$$\frac{\partial^2 u}{\partial \xi \, \partial \eta} - \frac{1}{2\xi} \frac{\partial u}{\partial \eta} = 0 \qquad (\xi > 0, \ \eta > 0) .$$

3. *The reduction of mixed-type second-order equations to canonical form*

Let us again consider the equation

$$A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \, \partial y} + C \frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 . \tag{24}$$

If we consider eq. (24) not in the neighbourhood of a point but in the entire region D, the classification of second-order equations into three types that was made in section 2 is not exhaustive, since the expression $B^2 - AC$ may not, in the general case, retain its sign throughout the entire region. Therefore, the characteristics may be partially real and partially imaginary.

Thus, if $B^2 - AC$ changes sign in the region D, eq. (24) is said to be an equation of mixed type. The curve γ, determined by the equation $B^2 - AC = 0$, is said to be the *parabolic* curve of eq. (19), and the parts into which the region D is divided by the curve are said to be the elliptic and hyperbolic parts of the region, depending upon whether $B^2 - AC$ is less than or greater than zero (respectively) in that part.

Just as in section 2, we introduce, instead of x and y, the new independent variables ξ and η:

$$\xi = \xi(x, y) \ , \qquad \eta = \eta(x, y) \ . \tag{25}$$

Then, eq. (24), in the new independent variables ξ and η, becomes

$$\bar{A}(\xi,\eta) \frac{\partial^2 u}{\partial \xi^2} + 2\bar{B}(\xi,\eta) \frac{\partial^2 u}{\partial \xi \, \partial \eta} + \bar{C}(\xi,\eta) \frac{\partial^2 u}{\partial \eta^2} + F\left(\xi,\eta,u,\frac{\partial u}{\partial \xi},\frac{\partial u}{\partial \eta}\right) = 0 \ , \tag{26}$$

where

$$\begin{aligned} \bar{A}(\xi,\eta) &= A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B \frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2 \ , \\ \bar{B}(\xi,\eta) &= A \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + C \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} \ , \\ \bar{C}(\xi,\eta) &= A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B \frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C\left(\frac{\partial \eta}{\partial y}\right)^2 \end{aligned} \tag{27}$$

We may choose the two functions $\xi(x, y)$ and $\eta(x, y)$ so that the following conditions are satisfied:

$$A \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + C \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} = 0 \ , \tag{28}$$

$$A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B \frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C\left(\frac{\partial \eta}{\partial y}\right)^2 \neq 0 \ . \tag{29}$$

Since $AC - B^2 = 0$ on the parabolic curve γ, we may put it in the form

$$AC - B^2 = H^n(x, y) \ M(x, y) \ , \tag{30}$$

where $M(x, y)$ is different from zero throughout the region D and $H(x, y) = 0$ is the equation of the curve γ; here, $\partial H/\partial x$ and $\partial H/\partial y$ do not vanish simultaneously.

Let us examine two cases.

CASE I. The direction of the characteristic of eq. (24) on points of the parabolic curve γ does not coincide with the direction of the tangent to that curve; that is, along γ,

$$A\left(\frac{\partial H}{\partial x}\right)^2 + 2B \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} + C\left(\frac{\partial H}{\partial y}\right)^2 \neq 0 \ . \tag{31}$$

Let us set

$$\eta = H(x, y) \ . \tag{32}$$

For the function $\xi = \xi(x, y)$, we take a solution of the equation

$$\left(A \frac{\partial H}{\partial x} + B \frac{\partial H}{\partial y}\right) \frac{\partial \xi}{\partial x} + \left(B \frac{\partial H}{\partial x} + C \frac{\partial H}{\partial y}\right) \frac{\partial \xi}{\partial y} = 0 \ . \tag{33}$$

For such a choice of the functions $\xi = \xi(x, y)$ and $\eta = H(x, y)$, conditions (28) and (29) are satisfied.

Let us show that the Jacobian of these functions is not equal to zero in some neighbourhood of the curve γ. For if we set

$$\frac{\partial \xi}{\partial x} = \rho \left(B \frac{\partial H}{\partial x} + C \frac{\partial H}{\partial y}\right), \qquad \frac{\partial \xi}{\partial y} = -\rho \left(A \frac{\partial H}{\partial x} + B \frac{\partial H}{\partial y}\right), \tag{34}$$

where $\rho(x, y) \neq 0$ along γ, then

$$\begin{aligned}\frac{\partial(\xi,\eta)}{\partial(x,y)} &= \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x} \\ &= \rho \left\{\left(B \frac{\partial H}{\partial x} + C \frac{\partial H}{\partial y}\right) \frac{\partial H}{\partial y} + \left(A \frac{\partial H}{\partial x} + B \frac{\partial H}{\partial y}\right) \frac{\partial H}{\partial x}\right\} \\ &= \rho \left\{A \left(\frac{\partial H}{\partial x}\right)^2 + 2B \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} + C\left(\frac{\partial H}{\partial y}\right)^2\right\} \neq 0\end{aligned}$$

along γ.

Because of the continuity of the functions A, B, C, and H, the Jacobian will be different from zero in some neighbourhood of the curve γ. Therefore, in that neighbourhood, we may take

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y) = H(x, y)$$

in eqs. (25).

It then follows from (27) and (33) that $\overline{B} = 0$ in the left member of eq. (26). The coefficient $\overline{C}$ is different from zero in a neighbourhood of the curve γ. Dividing eq. (26) by it, we obtain

$$\frac{\overline{A}}{\overline{C}} \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = F_1 \left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right),$$

or, taking into consideration (27), (30), (32), and (34), we finally obtain

$$\eta^n K_1(\xi, \eta) \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = F_1 \left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right), \tag{35}$$

where $K_1(\xi, \eta)$ is different from zero in some neighbourhood of the curve γ.

CASE II. The parabolic curve γ is the characteristic or the envelope of the family of characteristics of eq. (24); that is, at all points of the curve γ,

$$A\left(\frac{\partial H}{\partial x}\right)^2 + 2B \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} + C\left(\frac{\partial H}{\partial y}\right)^2 = 0 . \tag{36}$$

Let us assume that $A \geqslant 0$ and $C \geqslant 0$ along γ; then, since $B^2 - AC = 0$, eq. (36) can be written in the form

$$\sqrt{A} \frac{\partial H}{\partial x} + \epsilon \sqrt{C} \frac{\partial H}{\partial y} = 0 , \tag{37}$$

where $\epsilon = \operatorname{sgn} B$. If $B = 0$, then either $A = 0$ or $C = 0$ along γ, and from (37), we have either $H_y = 0$ or $H_x = 0$.

For the function $\eta = \eta(x, y)$, we take a solution of the equation

$$n(x, y) \frac{\partial \eta}{\partial x} - m(x, y) \frac{\partial \eta}{\partial y} = 0 , \tag{38}$$

where the functions $n(x, y)$ and $m(x, y)$ satisfy the condition

$$Am^2 + 2Bmn + Cn^2 \neq 0 \tag{39}$$

in some neighbourhood of γ. For example, if either A or C is different from zero in the region D, we may take $\eta = x$ for $m = 1$ and $n = 0$ or $\eta = y$ for $m = 0$ and $n = 1$.

For the function $\xi = \xi(x, y)$, we take a solution of the equation

$$\left(A \frac{\partial \eta}{\partial x} + B \frac{\partial \eta}{\partial y}\right) \frac{\partial \xi}{\partial x} + \left(B \frac{\partial \eta}{\partial x} + C \frac{\partial \eta}{\partial y}\right) \frac{\partial \xi}{\partial y} = 0 , \tag{40}$$

where $\eta(x, y)$ is a solution of eq. (38).

With this choice of the functions $\xi(x, y)$ and $\eta(x, y)$, eqs. (28) and (29) are satisfied.

We note that it is always possible to choose a solution of eq. (40) such that $\xi(x, y) = 0$ at the point of intersection of the curves $\eta(x, y) = 0$ and γ.

Just as in Case I, it is easy to show that the Jacobian $\partial(\xi, \eta)/\partial(x, y)$ is different from zero along the curve γ. Then, because of the continuity of the functions A, B, C, m, and n, this Jacobian will also be different from zero in some neighbourhood of the curve γ.

Let us now set $H(x, y) = \overline{H}(\xi, \eta)$; then, we obtain

$$\begin{aligned} \frac{\partial \overline{H}}{\partial \xi} &= \frac{\partial H}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial H}{\partial y} \frac{\partial y}{\partial \xi} = \frac{H_x \eta_y - H_y \eta_x}{\xi_x \eta_y - \xi_y \eta_x} , \\ \frac{\partial \overline{H}}{\partial \eta} &= \frac{\partial H}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial H}{\partial y} \frac{\partial y}{\partial \eta} = - \frac{H_x \xi_y - H_y \xi_x}{\xi_x \eta_y - \xi_y \eta_x} \end{aligned} \tag{41}$$

Along γ, $\partial \overline{H}/\partial \xi$ will be different from zero because the curve η = constant is nowhere tangential to the curve γ. From eq. (37), it follows that along γ,

$$\begin{aligned} &\frac{\partial H}{\partial x} = - \sigma \epsilon \sqrt{C} , \quad \frac{\partial H}{\partial y} = \sigma \sqrt{A} \quad [\sigma(x, y) \neq 0] , \\ &\frac{\partial \overline{H}}{\partial \eta} = \frac{\sigma}{\xi_x \eta_y - \xi_y \eta_x} \left(\sqrt{A} \frac{\partial \xi}{\partial x} + \epsilon \sqrt{C} \frac{\partial \xi}{\partial y}\right) = 0 , \end{aligned} \tag{42}$$

since along γ eq. (40) is reduced to the form

$$\sqrt{A} \frac{\partial \xi}{\partial x} + \epsilon \sqrt{C} \frac{\partial \xi}{\partial y} = 0 ,$$

which follows from the condition that $B^2 - AC = 0$.

From the fact that $\overline{H}(\xi, \eta) = 0$, we have

$$\partial \xi / \partial \eta = - \overline{H}_\eta / \overline{H}_\xi ,$$

and from (42), it follows that ξ = constant along γ. But $\xi = 0$ at the point of intersection with the curve γ and, consequently, $\xi = 0$ along γ. Therefore, $\overline{H}(0, \eta) = 0$ and we may write

$$\overline{H}(\xi, \eta) = \xi \overline{H}_\xi(\theta(\xi, \eta)\xi, \eta) = \xi N(\xi, \eta) , \tag{43}$$

where

$$0 < \theta(\xi, \eta) < 1 \quad \text{and} \quad N(\xi, \eta) \neq 0 .$$

In the transformed equation (26), $\overline{B} = 0$ and $\overline{C} \neq 0$ in the neighbourhood

of the curve γ, which follows from (28) and (29). Dividing eq. (26) by the coefficient $\bar{C} \neq 0$, we obtain

$$\frac{\bar{A}}{\bar{C}} \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = F_2\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) ,$$

but

$$\frac{\bar{A}}{\bar{C}} = \frac{A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B \frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + C\left(\frac{\partial \xi}{\partial y}\right)^2}{A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B \frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C\left(\frac{\partial \eta}{\partial y}\right)^2} = \rho^2(AC - B^2)$$

$$= \rho^2 H^n(x,y)\, M(x,y) = \xi^n \rho^2 N^n M = \xi^n K_2(\xi,\eta) ,$$

and we finally have

$$\xi^n K_2(\xi,\eta) \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = F_2(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}) , \tag{44}$$

where $K_2(\xi,\eta)$ is different from zero in a neighbourhood of the curve γ.

Example. Let us examine the equation

$$(1 - x^2) \frac{\partial^2 u}{\partial x^2} - 2xy \frac{\partial^2 u}{\partial x\, \partial y} - (1 + y^2) \frac{\partial^2 u}{\partial y^2} - 2x \frac{\partial u}{\partial x} - 2y \frac{\partial u}{\partial y} = 0 . \tag{45}$$

This equation is of mixed type since

$$AC - B^2 = x^2 - y^2 - 1 = H(x,y) .$$

In the region $1 - x^2 + y^2 > 0$, the equation belongs to the hyperbolic type and in the region $1 - x^2 + y^2 < 0$, it belongs to the elliptic type. The curve $x^2 - y^2 = 1$ is the parabolic curve.

Since

$$A\left(\frac{\partial H}{\partial x}\right)^2 + 2B \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} + C\left(\frac{\partial H}{\partial y}\right)^2 = 4x^2(1-x^2) + 8x^2y^2 - 4y^2(1+y^2)$$

$$= 4(y^2 - x^2)(x^2 - y^2 - 1) = 0$$

along the curve γ, we have Case II. According to the general theory, we take for the functions $\xi(x,y)$ and $\eta(x,y)$ solutions of eqs. (38) and (40).

For example, we take $n = 1 + x$ and $m = -y$. Then, eq. (38) takes the form:

$$(1+x) \frac{\partial \eta}{\partial x} + y \frac{\partial \eta}{\partial y} = 0 .$$

Its partial solution is then

$$\eta = \frac{y}{1 + x} .$$

Substituting $\eta(x,y)$ into eq. (40), we obtain

$$y(1+x) \frac{\partial \xi}{\partial x} + (1 + x + y^2) \frac{\partial \xi}{\partial y} = 0 .$$

This equation has the partial solution

$$\xi(x, y) = \frac{x^2 - y^2 - 1}{(1+x)^2} .$$

Thus, we need to introduce the new variables ξ and η according to the formulae

$$\xi = \frac{x^2 - y^2 - 1}{4(1+x)^2} , \qquad \eta = \frac{y}{1+x} .$$

Then, from formulae (8), we have

$$\begin{aligned}
&\frac{\partial u}{\partial x} = \frac{1+x+y^2}{2(1+x)^3}\frac{\partial u}{\partial \xi} - \frac{y}{(1+x)^2}\frac{\partial u}{\partial \eta}, \qquad \frac{\partial u}{\partial y} = -\frac{y}{2(1+x)^2}\frac{\partial u}{\partial \xi} + \frac{1}{1+x}\frac{\partial u}{\partial \eta},\\
&\frac{\partial^2 u}{\partial x^2} = \frac{(1+x+y^2)^2}{4(1+x)^6}\frac{\partial^2 u}{\partial \xi^2} - \frac{y(1+x+y^2)}{(1+x)^5}\frac{\partial^2 u}{\partial \xi\,\partial \eta}\\
&\qquad + \frac{y^2}{(1+x)^4}\frac{\partial^2 u}{\partial \eta^2} - \frac{1+x+3y^2}{2(1+x)^4}\frac{\partial u}{\partial \xi} + \frac{2y}{(1+x)^3}\frac{\partial u}{\partial \eta},\\
&\frac{\partial^2 u}{\partial y^2} = \frac{y^2}{4(1+x)^4}\frac{\partial^2 u}{\partial \xi^2} - \frac{y}{(1+x)^3}\frac{\partial^2 u}{\partial \xi\,\partial \eta} + \frac{1}{(1+x)^2}\frac{\partial^2 u}{\partial \eta^2} - \frac{1}{2(1+x)^2}\frac{\partial u}{\partial \xi},\\
&\frac{\partial^2 u}{\partial x\,\partial y} = -\frac{y(1+x+y^2)}{4(1+x)^5}\frac{\partial^2 u}{\partial \xi^2} + \frac{1+x+2y^2}{2(1+x)^4}\frac{\partial^2 u}{\partial \xi\,\partial \eta}\\
&\qquad - \frac{y}{(1+x)^3}\frac{\partial^2 u}{\partial \eta^2} + \frac{y}{(1+x)^3}\frac{\partial u}{\partial \xi} - \frac{1}{(1+x)^2}\frac{\partial u}{\partial \eta} .
\end{aligned} \tag{46}$$

Substituting the expression (46) into eq. (45), we reduce it to canonical form:

$$\xi\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{1}{2}\frac{\partial u}{\partial \xi} = 0 .$$

4. *The classification of second-order equations with many independent variables*

Consider the second-order linear equation

$$\sum_{i,j=1}^{n} A_{ij}\frac{\partial^2 u}{\partial x_i\,\partial x_j} + \sum_{i=1}^{n} B_i\frac{\partial u}{\partial x_i} + Cu + F = 0 , \tag{47}$$

where the A_{ij}, the B_i, C, and F are real functions of the independent variables $x_1, x_2, \ldots, x_n$.

Instead of $x_1, x_2, \ldots, x_n$, let us introduce new independent variables $\xi_1, \xi_2, \xi_3, \ldots, \xi_n$.

Suppose that the

$$\xi_k = \xi_k(x_1, x_2, \ldots, x_n) \qquad (k = 1, 2, \ldots, n) \tag{48}$$

are twice continuously differentiable functions and that the Jacobian of the transformation is different from zero throughout the region in question. Then,

$$\frac{\partial u}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial u}{\partial \xi_k} \frac{\partial \xi_k}{\partial_i}, \tag{49}$$

$$\frac{\partial^2 u}{\partial x_i\, \partial x_j} = \sum_{k,l=1}^{n} \frac{\partial^2 u}{\partial \xi_k\, \partial \xi_l} \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_l}{\partial x_j} + \sum_{k=1}^{n} \frac{\partial u}{\partial \xi_k} \frac{\partial^2 \xi_k}{\partial x_i\, \partial x_j}. \tag{50}$$

Substituting the values of the derivatives in (49) and (50) into eq. (47), we obtain

$$\sum_{k,l=1}^{n} \bar{A}_{kl} \frac{\partial^2 u}{\partial \xi_k\, \partial \xi_l} + \sum_{k=1}^{n} \bar{B}_k \frac{\partial u}{\partial \xi_k} + Cu + F = 0, \tag{51}$$

where

$$\bar{A}_{kl} = \sum_{i,j=1}^{n} A_{ij} \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_l}{\partial x_j}, \qquad \bar{B}_k = \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 \xi_k}{\partial x_i\, \partial x_j} + \sum_{i=1}^{n} B_i \frac{\partial \xi_k}{\partial x_i}.$$

Let us consider some particular point $M(x_1^0, x_2^0, \ldots, x_n^0)$. Suppose that at this point,

$$\partial \xi_k / \partial x_i = \alpha_{ki}. \tag{52}$$

The transformation formulae

$$\bar{A}_{kl} = \sum_{i,j=1}^{n} A_{ij}\, \alpha_{ki}\, \alpha_{lj} \tag{53}$$

coincide with the formulae for the transformation of the coefficients of the quadratic form

$$\sum_{i,j=1}^{n} A_{ij}\, P_i\, P_j \tag{54}$$

if we make the change of variables

$$P_i = \sum_{k=1}^{n} \alpha_{ki}\, q_k, \tag{55}$$

reducing it to the form

$$\sum_{k,l=1}^{n} \bar{A}_{kl}\, q_k\, q_l. \tag{56}$$

Consequently, the coefficients A_{ij} in eq. (47) are transformed at the given point $(x_1^0, x_2^0, \ldots, x_n^0)$ in exactly the same way as are the coefficients of the quadratic form (54) by the linear transformation (55). The coefficients A_{ij} of the form (54) are assumed to be constant and equal to the values of the coefficients $A_{ij}(x_1, \ldots, x_n)$ in eq. (47) at the point $(x_1^0, x_2^0, \ldots, x_n^0)$.

In algebra, it is proved that a non-singular transformation (55) exists

that reduces the quadratic form (54) with real coefficients A_{ij} to the form

$$\sum_{i=1}^{m} \pm q_i^2 \quad (m \leqslant n) , \tag{57}$$

such that the number of terms with positive and negative signs in the form (57) is determined exclusively by the form (54) and does not depend on the choice of the transformation (55). This is the law of invariance of quadratic forms [1].

Let us now suppose that we have found a transformation (55) that reduces the form (54) to the form (57). Then, the transformation (48) satisfying the condition (52) reduces eq. (47) to the form (51):

$$\sum_{k,l=1}^{n} \bar{A}_{kl} \frac{\partial^2 u}{\partial \xi_k \, \partial \xi_l} + \sum_{k=1}^{n} \bar{B}_k \frac{\partial u}{\partial \xi_k} + Cu + F = 0 ,$$

where

$$\bar{A}_{kl}(x_1^0, \ldots, x_n^0) = \pm 1 \quad \text{if } k = l \leqslant m ,$$
$$\bar{A}_{kl}(x_1^0, \ldots, x_n^0) = 0 \quad \text{if } k \neq l \text{ and } k = l > m.$$

This form of eq. (47) is called its *canonical form at the point* $(x_1^0, x_2^0, \ldots, x_n^0)$.

Thus, for every point $(x_1^0, x_2^0, \ldots, x_n^0)$, it is possible to find a nonsingular transformation (48) of the independent variables that reduces eq. (47) to canonical form at that point.

We call eq. (47) at the point $(x_1^0, x_2^0, \ldots, x_n^0)$ an equation of *elliptic* type if, in eq. (51), all n coefficients $\bar{A}_{kk}(x_1^0, x_2^0, \ldots, x_n^0)$ are different from zero and have the same sign. It is of *hyperbolic* type if n - 1 of the coefficients $\bar{A}_{kk}(x_1^0, x_2^0, \ldots, x_n^0)$ have the same sign and the other coefficient has the opposite sign. It is of *ultrahyperbolic* type if there is more than one positive coefficient among the $\bar{A}_{kk}(x_1^0, x_2^0, \ldots, x_n^0)$ and more than one negative coefficient and $m = n$. It is of *parabolic type in the broad sense* if at least one of the coefficients $\bar{A}_{kk}(x_1^0, x_2^0, \ldots, x_n^0)$ is equal to zero. It is of *parabolic type in the narrow sense* (or simply it is of *parabolic* type) if exactly one of the coefficients is equal to zero and all other coefficients have the same sign.

Eq. (47) is said to be of the *elliptic* (*hyperbolic*, and so on) type *in the region D* if it is of the elliptic (hyperbolic, and so on) type at every point of the region D.

We note that it is generally impossible to obtain a canonical form of eq. (47) by means of the same transformation of the independent variables for the entire region in which the differential equation is of the type in question. For if we attempted, by means of transformation (48), to reduce eq. (47) to canonical form throughout a certain region, it would be necessary to impose on the n functions $\xi_i(x_1, \ldots, x_n)$ the $\frac{1}{2}n(n-1)$ conditions

$$\bar{A}_{kl} = \sum_{i,j=1}^{n} A_{ij} \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_l}{\partial x_j} = 0 \quad (k \neq l) .$$

If n exceeds 3, this system is not soluble in the general case, because

the number of equations exceeds the number of functions $\xi_i(x_1,\ldots,x_n)$ to be determined. For $n = 3$, a solution does exist for the system, but, in contrast with the case of $n = 2$, it is not possible in the general case to impose further conditions on the coefficients of the derivatives $\partial^2 u/\xi_i^2$.

We note also that if the coefficients A_{ij} in eq. (47) are constants, the equation may be reduced to canonical form simultaneously at all points of the region by means of a single linear transformation of the independent variables.

Problems

1. Reduce the following equations to canonical form:

a)
$$\frac{\partial^2 u}{\partial x^2} - 2\sin x\,\frac{\partial^2 u}{\partial x\,\partial y} - \cos^2 x\,\frac{\partial^2 u}{\partial y^2} - \cos x\,\frac{\partial u}{\partial y} = 0\ ,$$

b)
$$y^2\,\frac{\partial^2 u}{\partial x^2} + x^2\,\frac{\partial^2 u}{\partial y^2} = 0 \qquad (x > 0,\ y > 0)\ ,$$

c)
$$x^2\,\frac{\partial^2 u}{\partial x^2} + 2xy\,\frac{\partial^2 u}{\partial x\,\partial y} + y^2\,\frac{\partial^2 u}{\partial y^2} = 0 \qquad (x > 0)\ .$$

Answers:

a)
$$\frac{\partial^2 u}{\partial \xi\,\partial \eta} = 0\ , \qquad \xi = x + y - \cos x\ , \qquad \eta = x - y + \cos x\ .$$

b)
$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{1}{2\xi}\,\frac{\partial u}{\partial \xi} + \frac{1}{2\eta}\,\frac{\partial u}{\partial \eta} = 0\ , \qquad \xi = y^2\ , \qquad \eta = x^2\ .$$

c)
$$\frac{\partial^2 u}{\partial \eta^2} = 0\ , \qquad \xi = \frac{y}{x}\ , \qquad \eta = y\ .$$

2. Show that the equation

$$\frac{\partial}{\partial x}\left\{\left(1 - \frac{x}{h}\right)^2 \frac{\partial u}{\partial x}\right\} = \frac{1}{a^2}\left(1 - \frac{x}{h}\right)^2 \frac{\partial^2 u}{\partial t^2}$$

can be reduced to the form

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{a^2}\,\frac{\partial^2 v}{\partial t^2}\ .$$

Method of solution: Define a new function $v = (h - x)u$.

3. Reduce to canonical form:

$$b^4 \sin^4(2x + c)\,\frac{\partial^2 u}{\partial x^2} + 4b^4 \sin^4(2x + c)u = \frac{\partial^2 u}{\partial t^2}\ .$$

Answer:

$$\partial^2 v/\partial \xi\,\partial \eta = 0\ ,$$

where

$$\xi = t - \frac{1}{2b^2}\cot(2x+c), \qquad \eta = t + \frac{1}{2b^2}\cot(2x+c), \qquad u = b\sin(2x+c)v .$$

4. Show that the equation

$$(l-x)\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} = 0 \qquad (0 < x < l)$$

can be reduced to the form

$$\frac{\partial^2 w}{\partial \xi\, \partial \eta} + \frac{1}{4}\frac{w}{(\xi+\eta)^2} = 0 .$$

Method of solution: Set

$$\xi = \sqrt{l-x} + \tfrac{1}{2}y, \qquad \eta = \sqrt{l-x} - \tfrac{1}{2}y, \qquad u = \frac{w}{\sqrt{\xi+\eta}} .$$

5. Reduce to canonical form:

$$(l^2 - x^2)\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 2x\frac{\partial u}{\partial x} - \tfrac{1}{4}u = 0 \qquad (0 < x < l) .$$

Answer:

$$\frac{\partial^2 w}{\partial \xi\, \partial \eta} - \frac{1}{4}\frac{w}{\sin^2(\xi-\eta)},$$

where

$$\xi = \tfrac{1}{2}(y+\omega), \qquad \eta = \tfrac{1}{2}(y-\omega), \qquad \omega = \arccos\frac{x}{l}, \qquad u = \frac{w}{\sqrt{\sin(\xi-\eta)}} .$$

PART I

DIFFERENTIAL EQUATIONS OF THE HYPERBOLIC TYPE

Chapter I

METHODS OF FINDING THE GENERAL SOLUTION TO EQUATIONS OF THE HYPERBOLIC TYPE

1. *General remarks. Examples*

For an ordinary differential equation

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$$

the solution

$$y = \varphi(x, C_1, \dots, C_n) ,$$

containing n arbitrary constants, is called the general solution in the domain D of the variables $x, y, y', \dots, y^{(n-1)}$, if, for a suitable choice of constants $C_1, C_2, \dots, C_n$, it is possible to solve an arbitrary Cauchy problem in D.

For partial differential equations, the matter becomes more complicated. However, even here, we may seek the "general solution", now containing arbitrary functions, the number of which, generally, is equal to the order of the differential equation. For a second-order hyperbolic equation with two independent variables, we shall call a solution containing two arbitrary functions the general solution if, for a suitable choice of arbitrary functions, it is possible to solve the Cauchy problem with arbitrary initial conditions on a non-characteristic curve; that is, to find the solution to the second-order equation satisfying on the given curve the initial conditions

$$u|_l = \varphi , \qquad \left.\frac{\partial u}{\partial n}\right|_l = \psi ,$$

where $\partial/\partial n$ indicates differentiation with respect to the normal to the curve l.

Knowing the general solution makes it possible, in certain cases, to obtain solutions in closed form for some problems in mathematical physics.

Example 1. Let us consider the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{1}$$

and find its general solution.

To do this, we reduce eq. (1) to canonical form. First we set up the equation for its characteristics:

$$dx^2 - a^2\, dt^2 = 0 .$$

Since this equation has the two solutions

$$x - at = \text{constant} , \qquad x + at = \text{constant} ,$$

in accordance with the general theory, we set

$$\xi = x - at , \qquad \eta = x + at . \tag{2}$$

The second derivatives that appear in eq. (1) are expressed in terms of the derivatives with respect to ξ and η by means of the equations

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi\, \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} ,$$

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial \xi^2} - 2a^2 \frac{\partial^2 u}{\partial \xi\, \partial \eta} + a^2 \frac{\partial^2 u}{\partial \eta^2} .$$

Substituting these expressions in eq. (1) and performing the obvious transformations, we obtain

$$\frac{\partial^2 u}{\partial \xi\, \partial \eta} = 0 . \tag{3}$$

We rewrite eq. (3) in the form

$$\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right) = 0 ,$$

so that

$$\frac{\partial u}{\partial \eta} = \theta(\eta) ,$$

where $\theta(\eta)$ is an arbitrary function of η. Integrating this equation with respect to η, treating ξ as a constant, we find that

$$u = \int \theta(\eta)\, d\eta + \varphi(\xi) ,$$

where $\varphi(\xi)$ is an arbitrary function of ξ. If we now set

$$\int \theta(\eta)\, d\eta = \psi(\eta) ,$$

we obtain

$$u = \varphi(\xi) + \psi(\eta) ,$$

and, returning to the original variables x and t, we find the general solution of the wave equation (1)

$$u = \varphi(x - at) + \psi(x + at) , \tag{4}$$

where φ and ψ are arbitrary twice continuously differentiable functions.

This general solution of the wave equation (1) is called *d'Alembert's solution*.

Let us examine the Cauchy problem for eq. (1), that is, the problem of finding the solution to eq. (1) satisfying the following initial conditions

$$u\big|_{t=0} = f(x) , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x) , \tag{5}$$

where $f(x)$ and $F(x)$ are given functions.

In the general solution (4), we determine the functions φ and ψ in such a manner that the initial conditions (5) are satisfied:

$$\varphi(x) + \psi(x) = f(x) , \qquad - \varphi'(x) + \psi'(x) = \frac{1}{a} F(x) . \tag{6}$$

Integrating the second equation, we obtain

$$\frac{1}{a} \int_0^x F(x) \, dx = - \varphi(x) + \psi(x) + C , \tag{7}$$

where C is an arbitrary constant.

From eqs. (6) and (7), we determine the functions $\varphi(x)$ and $\psi(x)$:

$$\varphi(x) = \tfrac{1}{2} f(x) - \frac{1}{2a} \int_0^x F(z) \, dz + \tfrac{1}{2}C ,$$
$$\psi(x) = \tfrac{1}{2} f(x) + \frac{1}{2a} \int_0^x F(z) \, dz - \tfrac{1}{2}C .$$

Substituting the expressions that we have obtained into eq. (4), we find that

$$u(x, t) = \frac{f(x - at) + f(x + at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} F(z) \, dz . \tag{8}$$

It is easy to verify by direct differentiation that the expression obtained for $u(x, t)$ is, in fact, a solution to the wave equation (1) satisfying the initial conditions (5). For this, it is sufficient to assume that the function $f(x)$ has a first and a second derivative and that the function $F(x)$ has at least a first derivative.

The method of deriving eq. (8) shows the *uniqueness* of the solution to the Cauchy problem for eq. (1) with the initial conditions (5). It is easy to show that the solution to the problem expressed by (1) and (5) is a continuous function of the initial conditions. Specifically, for every $\epsilon > 0$ and any arbitrary finite time interval t, we can find an $\eta > 0$ such that if we replace $f(x)$ and $F(x)$ by $f_1(x)$ and $F_1(x)$ in such a way that

$$| f(x) - f_1(x) | < \eta , \qquad | F(x) - F_1(x) | < \eta ,$$

the difference between the new and the original solution will be less in absolute value than ϵ. This follows directly from eq. (8).

Example 2. Find the solution to the equation

$$x^2 \frac{\partial^2 u}{\partial x^2} - y^2 \frac{\partial^2 u}{\partial y^2} = 0 , \tag{9}$$

satisfying the initial conditions

$$u \big|_{y=1} = f(x) , \qquad \frac{\partial u}{\partial y}\bigg|_{y=1} = F(x) . \tag{10}$$

As was shown above (see the Introduction), eq. (9) can, by means of a change of variables

$$\xi = xy\ , \qquad \eta = y/x\ , \tag{11}$$

be reduced to the canonical form

$$\frac{\partial^2 u}{\partial\xi\,\partial\eta} - \frac{1}{2\xi}\frac{\partial u}{\partial\eta} = 0\ . \tag{12}$$

Setting

$$w = \partial u/\partial\eta\ , \tag{13}$$

we reduce eq. (12) to the linear equation

$$\frac{\partial w}{\partial\xi} - \frac{1}{2\xi}\,w = 0\ ,$$

the general solution of which is of the form

$$w = \sqrt{\xi}\ \psi_0(\eta)\ . \tag{14}$$

Substituting (14) into (13), we obtain the equation

$$\frac{\partial u}{\partial\eta} = \sqrt{\xi}\ \psi_0(\eta)\ ,$$

which can be integrated in quadrature:

$$u = \sqrt{\xi}\int\psi_0(\eta)\,d\eta + \varphi(\xi) = \sqrt{\xi}\ \psi(\eta) + \varphi(\xi)\ ,$$

where φ and ψ are arbitrary functions.

Returning to the initial variables x and y, we obtain the general solution to eq. (9):

$$u = \varphi(xy) + \sqrt{xy}\ \psi(y/x)\ . \tag{15}$$

Let us now show how we can choose the functions φ and ψ so as to obtain the solution satisfying the conditions (10).

First of all, we note that the general solution (15) and the conditions (10) imply the two equations

$$\varphi(x) + \sqrt{x}\ \psi(1/x) = f(x)\ , \tag{16}$$

$$x\varphi'(x) + \tfrac{1}{2}\sqrt{x}\ \psi(1/x) + \frac{1}{\sqrt{x}}\,\psi'(1/x) = F(x)\ . \tag{17}$$

Differentiating (16), we obtain

$$\varphi'(x) + \frac{1}{2\sqrt{x}}\,\psi\left(\frac{1}{x}\right) - \frac{1}{x\sqrt{x}}\,\psi'\left(\frac{1}{x}\right) = f'(x)\ . \tag{18}$$

Eliminating the functions $\varphi'(x)$ and $\psi(1/x)$ from eqs. (17) and (18), we obtain

$$\psi'(1/x) = -\tfrac{1}{2}x^{\frac{3}{2}}\ f'(x) + \tfrac{1}{2}\sqrt{x}\ F(x)\ ,$$

and hence,

$$\psi\left(\frac{1}{x}\right) = \frac{1}{2}\int_{x_0}^{x}\frac{f'(z)\,dz}{\sqrt{z}} - \frac{1}{2}\int_{x_0}^{x}\frac{F(z)\,dz}{\sqrt{z^3}} + C\ , \tag{19}$$

where C is a constant of integration.

Substituting eq. (19) into (16), we obtain

$$\varphi(x) = f(x) - \frac{\sqrt{x}}{2}\int_{x_0}^{x} \frac{f'(z)\,dz}{\sqrt{z}} + \frac{\sqrt{x}}{2}\int_{x_0}^{x} \frac{F(z)\,dz}{\sqrt{z^3}} - C\sqrt{x} \tag{20}$$

Using eqs. (15), (19), and (20), we easily obtain the following expression for the desired solution to eq. (9):

$$u(x, y) = f(xy) + \frac{\sqrt{xy}}{2}\int_{xy}^{x/y} \frac{f'(z)\,dz}{\sqrt{z^3}} - \frac{\sqrt{xy}}{2}\int_{xy}^{x/y} \frac{F(z)\,dz}{\sqrt{z^3}}$$

or, integrating the first integral by parts, we finally obtain

$$u(x, y) = \tfrac{1}{2}f(xy) + \tfrac{1}{2}yf\left(\frac{x}{y}\right) + \tfrac{1}{4}\sqrt{xy}\int_{xy}^{x/y} \frac{f(z)\,dz}{\sqrt{z}} - \tfrac{1}{2}\sqrt{xy}\int_{xy}^{x/y} \frac{F(z)\,dz}{\sqrt{z^3}}$$

Example 3. Consider the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{2}{2n+1}\, x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \quad (n = 0, 1, 2, \ldots)\ . \tag{21}$$

In place of x, we introduce the new variable

$$y = 2(2n+1)x\ , \tag{22}$$

so that

$$\frac{1}{4}\frac{\partial^2 u}{\partial t^2} = y\frac{\partial^2 u}{\partial y^2} + \frac{2n+1}{2}\frac{\partial u}{\partial y}\ . \tag{23}$$

We denote by u_n the function satisfying eq. (23) for given n. Then, for the function u_0, we obtain

$$\frac{1}{4}\frac{\partial^2 u_0}{\partial t^2} = y\frac{\partial^2 u_0}{\partial y^2} + \frac{1}{2}\frac{\partial u_0}{\partial y}\ .$$

Instead of y, we introduce the variable $\xi = \sqrt{y}$ and obtain the wave equation:

$$\frac{\partial^2 u_0}{\partial t^2} = \frac{\partial^2 u_0}{\partial \xi^2}\ ,$$

the general solution to which is

$$u_0 = f_1(\xi - t) + f_2(\xi + t)\ ,$$

where f_1 and f_2 are arbitrary functions. Thus,

$$u_0 = f_1(\sqrt{y} - t) + f_2(\sqrt{y} + t)\ . \tag{24}$$

We now show that if the function u_n is known, the function u_{n+1} can be obtained by a simple differentiation. For when we differentiate eq. (23) with respect to y, we obtain for $\partial u_n/\partial y$ the equation

$$\frac{1}{4}\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_n}{\partial y}\right) = y\frac{\partial^2}{\partial y^2}\left(\frac{\partial u_n}{\partial y}\right) + \frac{2(n+1)+1}{2}\frac{\partial}{\partial y}\left(\frac{\partial u_n}{\partial y}\right),$$

which coincides with eq. (23) for the function u_{n+1}. Thus,

$$u_{n+1} = \partial u_n/\partial y \ .$$

Applying this formula n times to the function u_0 and returning to the original variable x, we obtain the desired general solution to eq. (21):

$$u(x,t) = \frac{\partial^n}{\partial x^n}\,[f_1(\sqrt{2(2n+1)x} - t) + f_2(\sqrt{2(2n+1)x} + t)] \ . \tag{25}$$

2. *The Euler-Darboux equation*

1. In investigating boundary value problems for mixed-type equations, we often encounter the Euler-Darboux equation

$$E(\alpha,\beta) \equiv \frac{\partial^2 u}{\partial x\,\partial y} - \frac{\beta}{x-y}\frac{\partial u}{\partial x} + \frac{\alpha}{x-y}\frac{\partial u}{\partial y} = 0 \ , \tag{26}$$

where α and β are constants. We introduce the new function $v(x,y)$ and set

$$u(x,y) = (x-y)^{1-\alpha-\beta}\, v(x,y) \ . \tag{27}$$

Then eq. (26) is reduced to the form

$$\frac{\partial^2 v}{\partial x\,\partial y} - \frac{1-\alpha}{x-y}\frac{\partial v}{\partial x} + \frac{1-\beta}{x-y}\frac{\partial v}{\partial y} = 0 \ . \tag{28}$$

We denote by $Z(\alpha,\beta)$ an arbitrary solution to the equation $E(\alpha,\beta) = 0$. Then, on the basis of eq. (27), we obtain

$$Z(\alpha,\beta) = (x-y)^{1-\alpha-\beta}\, Z(1-\beta, 1-\alpha) \ . \tag{29}$$

Let us find the particular solutions to eq. (26). Let us set

$$t = y/x \ , \qquad u = x^{\lambda}\varphi(t) \ , \tag{30}$$

where λ is some constant.

Substituting eq. (30) in eq. (26), and making some simple transformations, we obtain the Gauss equation

$$t(1-t)\,\varphi''(t) + [1-\lambda-\alpha-(1-\lambda+\beta)t]\,\varphi'(t) + \lambda\beta\varphi(t) = 0 \ . \tag{31}$$

It is known [1)] that eq. (31) has two linearly independent solutions in a neighbourhood of the point $t = 0$:

$$\varphi_1(t) = F(-\lambda,\beta,1-\lambda-\alpha;t) \ , \qquad \varphi_2(t) = t^{\lambda+\alpha}\,F(\alpha,\alpha+\beta+\lambda,1+\alpha+\lambda;t) \ ,$$

where $f(a,b,c;t)$ is the hypergeometric series

$$F(a,b,c;t) = 1 + \frac{ab}{1!\,c}\,t + \frac{a(a+1)\,b(b+1)}{2!\,c(c+1)}\,t^2 \ \dots$$
$$+ \frac{a(a+1)\ \dots\ (a+n-1)\ b(b+1)\ \dots\ (b+n-1)}{n!\,c(c+1)\ \dots\ (c+n-1)}\,t^n + \dots \ .$$

Then, on the basis of eq. (30), eq. (26) has particular solutions of the form

$$\begin{aligned} u_1(x,y) &= x^{\lambda}\, F\left(-\lambda, \beta, 1-\lambda-\alpha; \frac{y}{x}\right), \\ u_2(x,y) &= x^{-\alpha}\, y^{\lambda+\alpha}\, F\left(\alpha, \alpha+\beta+\lambda, 1+\alpha+\lambda; \frac{y}{x}\right). \end{aligned} \tag{32}$$

We note that if λ is a positive integer, the first solution is a homogeneous polynomial of degree λ.

2. Let us find the general solution to eq. (26) when α and β are positive integers. To do this, we rewrite eq. (26) in the form

$$(x-y)\frac{\partial^2 u}{\partial x\,\partial y} - \beta\frac{\partial u}{\partial x} + \alpha\frac{\partial u}{\partial y} = 0 . \tag{33}$$

Differentiating eq. (33) once with respect to x, we obtain

$$(x-y)\frac{\partial^3 u}{\partial x^2\,\partial y} - \beta\frac{\partial^2 u}{\partial x^2} + (1+\alpha)\frac{\partial^2 u}{\partial x\,\partial y} = 0 ,$$

or

$$(x-y)\frac{\partial^2}{\partial x\,\partial y}\left(\frac{\partial u}{\partial x}\right) - \beta\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + (1+\alpha)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = 0 ,$$

from which it is clear that $\partial u/\partial x$ satisfies the equation

$$E(1+\alpha, \beta) = 0 .$$

Consequently,

$$\frac{\partial Z(\alpha,\beta)}{\partial x} = Z(1+\alpha, \beta) .$$

Analogously, differentiating eq. (33) with respect to y, we obtain

$$\frac{\partial Z(\alpha,\beta)}{\partial y} = Z(\alpha, 1+\beta)$$

and, in general,

$$Z(\alpha+m-1, \beta+n-1) = \frac{\partial^{m+n-2} Z(\alpha,\beta)}{\partial x^{m-1}\,\partial y^{n-1}} . \tag{34}$$

From formula (34), for $\alpha = \beta = 1$, we have

$$Z(m,n) = \frac{\partial^{m+n-2} Z(1,1)}{\partial x^{m-1}\,\partial y^{n-1}} , \tag{35}$$

where $Z(1,1)$ is a solution of the equation

$$E(1,1) \equiv \frac{\partial^2 u}{\partial x\,\partial y} - \frac{1}{x-y}\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 ,$$

the general solution of which is

$$Z(1,1) = \frac{\Phi(x) - \Psi(y)}{x-y} ,$$

where $\Phi(x)$ and $\Psi(x)$ are arbitrary functions. Consequently, the general solution to the equation

$$E(m,n) \equiv \frac{\partial^2 u}{\partial x\, \partial y} - \frac{n}{x-y}\frac{\partial u}{\partial x} + \frac{m}{x-y}\frac{\partial u}{\partial y} = 0 \tag{36}$$

is given by the formula

$$u(x,y) = \frac{\partial^{m+n-2}}{\partial x^{m-1}\, \partial x^{n-1}} \left\{\frac{\Phi(x) - \Psi(x)}{x-y}\right\} . \tag{37}$$

3. Let us suppose now that α and β are negative integers. On the basis of eq. (29), eq. (34) can be rewritten in the form

$$(x-y)^{3-m-n-\alpha-\beta}\, Z(2-\beta-n, 2-\alpha-m) = \frac{\partial^{m+n-2}}{\partial x^{m-1}\, \partial y^{n-1}} \left\{\frac{Z(1-\beta, 1-\alpha)}{(x-y)^{\alpha+\beta-1}}\right\} .$$

If we now replace α, β, $m-1$, and $n-1$ by $1-\beta$, $1-\alpha$, n, and m, respectively, we obtain

$$Z(\alpha-m, \beta-n) = (x-y)^{m+n+1-\alpha-\beta} \frac{\partial^{m+n}}{\partial x^n\, \partial y^m} \left\{\frac{Z(\alpha,\beta)}{(x-y)^{1-\alpha-\beta}}\right\} .$$

Setting $\alpha = \beta = 0$, we obtain

$$Z(-m,-n) = (x-y)^{m+n+1} \frac{\partial^{m+n}}{\partial x^n\, \partial y^m} \left\{\frac{\Phi(x) - \Psi(x)}{x-y}\right\} . \tag{38}$$

This is the general solution to the equation

$$E(-m,-n) \equiv \frac{\partial^2 u}{\partial x\, \partial y} + \frac{n}{x-y}\frac{\partial u}{\partial x} - \frac{m}{x-y}\frac{\partial u}{\partial y} = 0 . \tag{39}$$

4. Let us find the general solution to eq. (26) when α and β are not integers. Let us seek particular solutions to eq. (26) in the form

$$u = X(x)\; Y(y) . \tag{40}$$

Substituting this into eq. (26), we obtain

$$(x-y)\, X'(x)\, Y'(y) - \beta\, X'(x)\, Y(y) + \alpha\, X(x)\, Y'(y) = 0 ,$$

or

$$x + \alpha \frac{X(x)}{X'(x)} = y + \beta \frac{Y(y)}{Y'(y)} .$$

The left side of this equation is a function of x only and the right side is a function of y only; thus, equality of the two is possible only if neither the left nor the right side is, in fact, dependent on x or y (that is, if both sides represent one and the same constant). We denote this constant by a:

$$x + \alpha \frac{X(x)}{X'(x)} = y + \beta \frac{Y(y)}{Y'(y)} = a ,$$

from which we obtain the two equations

$$\frac{X'(x)}{X(x)} = -\frac{\alpha}{x-a} , \qquad \frac{Y'(y)}{Y(y)} = -\frac{\beta}{y-a} .$$

Particular solutions of these equations are

$$X(x) = (x-a)^{-\alpha}\,, \qquad Y(y) = (y-a)^{-\beta}\,.$$

On the basis of eq. (40), eq. (26) has particular solutions of the form

$$(x-a)^{-\alpha}\,(y-a)^{-\beta}\,.$$

By direct substitution we find that the function

$$u_1(x,y) = \int_x^y \varphi(\xi)\,(\xi-x)^{-\alpha}\,(y-\xi)^{-\beta}\,d\xi\,,$$

where $\varphi(\xi)$ is an arbitrary function, is also a solution to eq. (26).

Taking eq. (29) into account, we see that eq. (26) has particular solutions of the form

$$(x-y)^{1-\alpha-\beta}\,(x-a)^{\beta-1}\,(y-a)^{\alpha-1}$$

and, analogously, the function

$$u_2(x,y) = (y-x)^{1-\alpha-\beta}\int_x^y \psi(\xi)\,(\xi-x)^{\beta-1}\,(y-\xi)^{\alpha-1}\,d\xi\,,$$

where $\psi(\xi)$ is an arbitrary function, is a solution to eq. (26).

Consequently, the general solution to eq. (26) is

$$u(x,y) = \int_x^y \varphi(\xi)\,(\xi-x)^{-\alpha}(y-\xi)^{-\beta}d\xi + (y-x)^{1-\alpha-\beta}\int_x^y \psi(\xi)\,(\xi-x)^{\beta-1}(y-\xi)^{\alpha-1}d\xi\,.$$

Setting

$$\xi = x(1-t) + yt\,,$$

we finally obtain

$$u(x,y) = (y-x)^{1-\alpha-\beta}\int_0^1 \varphi[x+(y-x)t]\;t^{-\alpha}(1-t)^{-\beta}\,dt + \int_0^1 \psi[x+(y-x)t]\;t^{\beta-1}(1-t)^{\alpha-1}\,dt\,, \tag{41}$$

where φ and ψ are arbitrary functions, $0 \leq \alpha,\ \beta < 1$, and $\alpha+\beta \neq 1$.

In the case $\alpha+\beta = 1$, the general solution to eq. (26) is given by the formula

$$u(x,y) = \int_0^1 \varphi[x+(y-x)t]\;t^{-\alpha}(1-t)^{\alpha-1}\,dt + \int_0^1 \psi[x+(y-x)t]\;t^{-\alpha}(1-t)^{\alpha-1}\ln[t(1-t)(y-x)]\,dt\,. \tag{42}$$

We note that the general solution to eq. (26) can be obtained for other values of α and β if we use the formulae (29), (34), (41), and (42).

5. As an example illustrating the use of the general solution (41), let us consider the following problem: Find the solution to the Euler-Darboux equation

$$\frac{\partial^2 u}{\partial x\,\partial y} - \frac{\beta}{x-y}\frac{\partial u}{\partial x} + \frac{\alpha}{x-y}\frac{\partial u}{\partial y} = 0 \qquad (0 < \alpha+\beta < 1;\ \alpha \geqslant 0,\ \beta \geqslant 0)\ ,$$

satisfying the conditions

$$u\big|_{y=x} = f(x)\ , \qquad (y-x)^{\alpha+\beta}\left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\right)\bigg|_{y=x} F(x)\ . \tag{43}$$

Setting $y = x$ in the general solution (41) and taking (43) into account, we obtain

$$f(x) = \psi(x)\int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\,dt = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,\psi(x)\ ,$$

from which

$$\psi(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\,f(x)\ , \tag{44}$$

where

$$\Gamma(s) = \int_0^\infty e^{-x}\,x^{s-1}\,dx\ .$$

Differentiating (41) with respect to y and with respect to x, we obtain

$$\frac{\partial u}{\partial y} = (1-\alpha-\beta)(y-x)^{-\alpha-\beta}\int_0^1 \varphi[x+(y-x)t]\ t^{-\alpha}(1-t)^{-\beta}\,dt$$

$$+ (y-x)^{1-\alpha-\beta}\int_0^1 \varphi'[x+(y-x)t]\,t^{1-\alpha}(1-t)^{-\beta}\,dt + \int_0^1 \psi'[x+(y-x)t]\,t^{\beta}(1-t)^{\alpha-1}\,dt\ ,$$

$$\frac{\partial u}{\partial x} = -(1-\alpha-\beta)(y-x)^{-\alpha-\beta}\int_0^1 \varphi[x+(y-x)t]\ t^{-\alpha}(1-t)^{-\beta}\,dt$$

$$+ (y-x)^{1-\alpha-\beta}\int_0^1 \varphi'[x+(y-x)t]\,t^{-\alpha}(1-t)^{1-\beta}\,dt + \int_0^1 \psi'[x+(y-x)t]\,t^{\beta-1}(1-t)^{\alpha}\,dt\ .$$

Multiplying these equations by $(y-x)^{\alpha+\beta}$ and subtracting the second from the first, we obtain

$$(y-x)^{\alpha+\beta}\left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\right) = 2(1-\alpha-\beta)\int_0^1 \varphi[x+(y-x)t]\ t^{-\alpha}(1-t)^{-\beta}\,dt$$

$$+ (y-x)\int_0^1 \varphi'[x+(y-x)t]\ t^{-\alpha}(1-t)^{-\beta}\,(2t-1)\,dt$$

$$+ (y-x)^{\alpha+\beta}\int_0^1 \psi'[x+(y-x)t]\ t^{\beta-1}(1-t)^{\alpha-1}\,(2t-1)\,dt\ .$$

Setting $y = x$ in the last expression and taking (43) into account, we obtain

$$F(x) = 2(1-\alpha-\beta)\,\varphi(x)\int_0^1 t^{-\alpha}(1-t)^{-\beta}\,dt = 2(1-\alpha-\beta)\,\frac{\Gamma(1-\alpha)\,\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta)}\,\varphi(x)\;.$$

Therefore,

$$\varphi(x) = \frac{\Gamma(2-\alpha-\beta)}{2(1-\alpha-\beta)\,\Gamma(1-\alpha)\,\Gamma(1-\beta)}\,F(x)\;. \tag{45}$$

Substituting the functions $\varphi(x)$ and $\psi(x)$ found above into the general solution (41), we obtain the solution to the problem:

$$u(x,y) = \frac{\Gamma(2-\alpha-\beta)\,(y-x)^{1-\alpha-\beta}}{2(1-\alpha-\beta)\,\Gamma(1-\alpha)\,\Gamma(1-\beta)}\int_0^1 F[x+(y-x)t]\;t^{-\alpha}(1-t)^{-\beta}\,dt$$
$$+\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\int_0^1 f[x+(y-x)t]\;t^{\beta-1}(1-t)^{\alpha-1}\,dt\;. \tag{46}$$

Remark: The equation

$$\frac{\partial^2 u}{\partial x\,\partial y} - \frac{a}{x-y}\frac{\partial u}{\partial x} + \frac{b}{x-y}\frac{\partial u}{\partial y} - \frac{c}{(x-y)^2}\,u = 0\;,$$

where a, b, and c are constants, can be reduced, by means of the substitution

$$u = (x-y)^{\gamma}\,v(x,y)\;,$$

to the Euler-Darboux equation

$$\frac{\partial^2 v}{\partial x\,\partial y} - \frac{\beta}{x-y}\frac{\partial v}{\partial x} + \frac{\alpha}{x-y}\frac{\partial v}{\partial y} = 0\;,$$

where $\alpha = b+\gamma$, $\beta = a+\gamma$, and γ is a root of the equation

$$\gamma^2 + (a+b-1)\gamma + c = 0\;.$$

Problems

1. Find the general solution of the equation

$$x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} - 2y\frac{\partial u}{\partial y} = 0\;.$$

Answer:

$$u(x,y) = \sqrt{\frac{x}{y}}\,\varphi(xy) + \psi\!\left(\frac{y}{x}\right)\;.$$

2. Show that the general solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{2(n+1)}{x}\,\frac{\partial u}{\partial x}\right)$$

is of the form

$$u = \left(\frac{\partial}{x\,\partial x}\right)^n \left\{\frac{\varphi(x-at)+\psi(x+at)}{x}\right\}.$$

3. Find the solution of the equation

$$4y^2\frac{\partial^2 u}{\partial x^2} + 2(1-y^2)\frac{\partial^2 u}{\partial x\,\partial y} - \frac{\partial^2 u}{\partial y^2} - \frac{2y}{1+y^2}\left(2\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = 0,$$

satisfying the initial conditions

$$u\big|_{y=0} = f(x), \qquad \frac{\partial u}{\partial y}\bigg|_{y=0} = F(x).$$

Answer:

$$u(x, y) = f(x - \tfrac{2}{3}y^3) + \tfrac{1}{2}\int_{x-\frac{2}{3}y^3}^{x+2y} F(z)\,\mathrm{d}z.$$

4. Show that the function

$$u(x, t) = \frac{(h-x+at)\,f(x-at) + (h-x-at)\,f(x+at)}{2(h-x)} + \frac{1}{2a}\int_{x-at}^{x+at}\frac{h-z}{h-x}F(z)\,\mathrm{d}z$$

is a solution of the equation

$$\frac{\partial}{\partial x}\left\{\left(1-\frac{x}{h}\right)^2\frac{\partial u}{\partial x}\right\} = \frac{1}{a^2}\left(1-\frac{x}{h}\right)^2\frac{\partial^2 u}{\partial t^2},$$

satisfying the initial conditions

$$u\big|_{t=0} = f(x), \qquad \frac{\partial u}{\partial t}\bigg|_{t=0} = F(x).$$

Method of solution: Use the result of problem 2 of the Introduction.

5. Find the solution to the equation

$$y^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\frac{\partial u}{\partial x} = 0,$$

satisfying the conditions

$$u\big|_{y=0} = f(x), \qquad \frac{\partial u}{\partial y}\bigg|_{y=0} = F(x).$$

Answer:

$$u(x, y) = \tfrac{1}{2}\sqrt{\pi}\,\frac{y}{\Gamma(\frac{7}{8})\,\Gamma(\frac{5}{8})}\int_0^1 F[x+y^2(t-\tfrac{1}{2})]\,t^{-\frac{1}{8}}(1-t)^{-\frac{3}{8}}\,\mathrm{d}t$$
$$+\frac{\sqrt{\pi}}{\Gamma(\frac{3}{8})\,\Gamma(\frac{5}{8})}\int_0^1 f[x+y^2(t-\tfrac{1}{2})]\,t^{-\frac{5}{8}}(1-t)^{-\frac{7}{8}}\,\mathrm{d}t.$$

Method of solution: Reduce the equation to canonical form and then use the solution to the Darboux equation in the form of definite integrals, setting $\alpha = \frac{1}{8}$ and $\beta = \frac{3}{8}$.

Chapter II

THE CAUCHY PROBLEM ON A PLANE

1. *The Cauchy problem and its solution by the Riemann method*

Consider the equation

$$L(u) \equiv \frac{\partial^2 u}{\partial x\, \partial y} + a(x, y)\, \frac{\partial u}{\partial x} + b(x, y)\, \frac{\partial u}{\partial y} + c(x, y)u = f(x, y)\ . \tag{1}$$

As we have seen, any arbitrary linear hyperbolic equation with two independent variables can be reduced to this form. Eq. (13a) of section 2 of the Introduction, which defines the characteristics, takes the form $dx\,dy = 0$ for eq. (1) above. Thus, the lines x = constant and y = constant, parallel to the coordinate axes, are the characteristics of eq. (1).

Let us suppose that in the xy-plane a curve AB is given, and that this curve does not intersect any of the straight lines parallel to the coordinate axes at more than one point.

Suppose that two functions φ and ψ are defined on the curve AB.

The Cauchy problem consists in the following: find the solution to eq. (1) satisfying the conditions

$$u\big|_{\mathrm{AB}} = \varphi\ , \qquad \frac{\partial u}{\partial n}\bigg|_{\mathrm{AB}} = \psi \tag{2}$$

on the curve AB, where n denotes differentiation with respect to the normal to the curve AB. In what follows, we shall assume that a solution exists to the Cauchy problem.

Together with the equation $L(u) = 0$, let us examine the conjugate equation

$$L^*(v) = \frac{\partial^2 v}{\partial x\, \partial y} - \frac{\partial (av)}{\partial x} - \frac{\partial (bv)}{\partial y} + cv = 0\ ,$$

where we assume that the coefficients a and b have continuous second-order derivatives.

By direct differentiation, we can easily verify the identity

$$vL(u) - uL^*(v) = \frac{1}{2}\, \frac{\partial}{\partial x}\left(v\, \frac{\partial u}{\partial y} - u\, \frac{\partial v}{\partial y} + 2auv\right) + \frac{1}{2}\, \frac{\partial}{\partial y}\left(v\, \frac{\partial u}{\partial x} - u\, \frac{\partial v}{\partial x} + 2buv\right)\ . \tag{3}$$

We now take an arbitrary point $\mathrm{M}(x_0, y_0)$ and pass the characteristics $x = x_0$ and $y = y_0$ through it, intersecting the curve AB at the points P and Q, respectively (fig. 1). We denote by Ω the region bounded by these straight lines and the arc PQ.

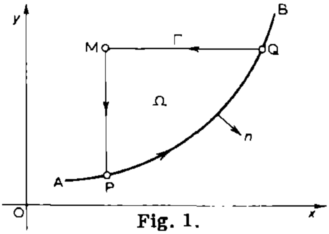

Fig. 1.

Integrating both sides of the identity (3) over the region Ω and using the well-known Green's formula, we obtain

$$\iint_{\Omega} [vL(u) - uL^*(v)]\, \mathrm{d}x\, \mathrm{d}y = \tfrac{1}{2} \int_{\Gamma} - \left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} + 2buv\right) \mathrm{d}x + \left(v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y} + 2auv\right) \mathrm{d}y \,, \tag{4}$$

where the curve Γ is the boundary of the region Ω and consists of three parts: the characteristics QM and MP and the arc PQ.

Let us now examine the integrals over the characteristics QM and MP. Since only x varies along QM, when we integrate over QM, we obtain the integral

$$- \tfrac{1}{2} \int_{\mathrm{QM}} \left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} + 2buv\right) \mathrm{d}x \,.$$

The integrand can be rewritten in the form

$$v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} + 2buv = \frac{\partial (uv)}{\partial x} + 2u \left(bv - \frac{\partial v}{\partial x}\right)$$

and consequently, we obtain

$$- \tfrac{1}{2} \int_{\mathrm{QM}} \left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} + 2buv\right) \mathrm{d}x = \tfrac{1}{2} (uv)_{\mathrm{Q}} - \tfrac{1}{2} (uv)_{\mathrm{M}} - \int_{\mathrm{QM}} u \left(bv - \frac{\partial v}{\partial x}\right) \mathrm{d}x \,, \tag{5}$$

where, for example, $(uv)_{\mathrm{M}}$ indicates the value of the product uv at the point M. In exactly the same manner, we obtain

$$\tfrac{1}{2} \int_{\mathrm{MP}} \left(v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y} + 2auv\right) \mathrm{d}y = \tfrac{1}{2} (uv)_{\mathrm{P}} - \tfrac{1}{2} (uv)_{\mathrm{M}} + \int_{\mathrm{MP}} u \left(av - \frac{\partial v}{\partial y}\right) \mathrm{d}y \,. \tag{6}$$

Substituting the expressions in eqs. (5) and (6) into eq. (4), we obtain

$$(uv)_{\mathrm{M}} = \frac{(uv)_{\mathrm{P}} + (uv)_{\mathrm{Q}}}{2} + \tfrac{1}{2} \int_{\mathrm{PQ}} - \left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} + 2buv\right) \mathrm{d}x + \left(v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y} + 2auv\right) \mathrm{d}y$$

$$- \int_{\mathrm{QM}} u \left(bv - \frac{\partial v}{\partial x}\right) \mathrm{d}x + \int_{\mathrm{MP}} u \left(av - \frac{\partial v}{\partial y}\right) \mathrm{d}y - \iint_{\Omega} [vL(u) - uM(v)]\, \mathrm{d}x\, \mathrm{d}y \,. \tag{7}$$

Suppose now that u is a solution to eq. (1) satisfying condition (2), and suppose that v is any solution to the homogeneous conjugate equation

$$L^*(v) = 0 . \tag{8}$$

Then, eq. (7) can be rewritten as follows:

$$(uv)_M = \frac{(uv)_P + (uv)_Q}{2} + \frac{1}{2} \int_{PQ} - \left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} + 2buv\right) dx + \left(v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y} + 2auv\right) dy - \int_{QM} u\left(bv - \frac{\partial v}{\partial x}\right) dx + \int_{MP} u\left(av - \frac{\partial v}{\partial y}\right) dy - \int\int_{\Omega} vf \, dx \, dy . \tag{9}$$

Examining the right side of eq. (9), we see that unknown values of u appear in the integrals

$$\int_{QM} u\left(bv - \frac{\partial v}{\partial x}\right) dx , \quad \int_{MP} u\left(av - \frac{\partial v}{\partial y}\right) dy \tag{10}$$

since we do not know the solution u on the characteristics QM and MP.

Following Riemann's idea, we eliminate these unknown terms from eq. (9) by a special choice of the solution v of the conjugate equation. Specifically, we choose the solution of eq. (8) that satisfied the following three conditions:

$$\begin{aligned} &1)\ \frac{\partial v}{\partial x} - bv = 0 \text{ on the characteristic QM,}\\ &2)\ \frac{\partial v}{\partial y} - av = 0 \text{ on the characteristic MP,}\\ &3)\ v = 1 \text{ at the point M.} \end{aligned} \tag{11}$$

It is easy to see that, in such a case, the integrals (10) will be equal to zero and eq. (9) will be transformed into the *Riemann equation*

$$u(M) = \frac{(uv)_P + (uv)_Q}{2} + \frac{1}{2} \int_{QP} \left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} + 2buv\right) dx - \left(v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y} + 2auv\right) dy - \int\int_{\Omega} vf(x, y) \, dx \, dy \tag{12}$$

This equation is a solution to the Cauchy problem, since the expressions that are being integrated along QP contain functions that are known on the curve AB. (The function v was defined above and the functions u, $\partial u/\partial x$, and $\partial u/\partial y$ are also defined on the curve AB on the basis of the conditions (2):

$$\left.\frac{\partial u}{\partial x}\right|_{AB} = \left.\frac{\partial u}{\partial s} \cos (s, x) + \frac{\partial u}{\partial n} \cos (nx)\right|_{AB} = \frac{\partial \varphi}{\partial s} \cos (s, x) + \psi(s) \cos (nx) ,$$

$$\left.\frac{\partial u}{\partial y}\right|_{AB} = \left.\frac{\partial u}{\partial s} \cos (s, y) + \frac{\partial u}{\partial n} \cos (ny)\right|_{AB} = \frac{\partial \varphi}{\partial s} \cos (s, y) + \psi(s) \cos (ny) ,$$

where $\partial/\partial s$ is the directional derivative along the tangent to the curve AB.)

Let us now investigate more closely the nature of the solution v of the conjugate equation (8) satisfying the conditions (11). This solution is a function of two pairs of variables: the moving coordinates x and y, and the fixed coordinates x_0 and y_0 of the point M. Consequently, if we introduce the notation

$$v = v(x, y; x_0, y_0) ,$$

the conditions (11) can be rewritten as follows:

1) $$\frac{\partial v(x, y_0; x_0, y_0)}{\partial x} = b(x, y_0)\, v(x, y_0; x_0, y_0)$$

on the characteristic QM;

2) $$\frac{\partial v(x_0, y; x_0, y_0)}{\partial y} = a(x_0, y)\, v(x_0, y; x_0, y_0)$$

on the characteristic MP;

3) $$v(x_0, y_0; x_0, y_0) = 1 .$$

By integrating, we then obtain

$$\begin{aligned} v(x, y_0; x_0, y_0) &= \exp\left[\int_{x_0}^{x} b(x, y_0)\, dx\right], \\ v(x_0, y; x_0, y_0) &= \exp\left[\int_{y_0}^{y} a(x_0, y)\, dy\right]. \end{aligned} \tag{13}$$

The solution $v(x, y, x_0, y_0)$ of the homogeneous conjugate equation (8) satisfying the conditions (13) is called the *Riemann function*. This function depends neither on the given Cauchy conditions (2) for the curve AB nor on the shape of this curve,

The Riemann method described above reduces the solving of the Cauchy problem to a matter of finding the Riemann function $v(x, y, x_0, y_0)$. It is possible to prove the existence and uniqueness of the Riemann function, though we shall not stop to do this. We note once again that the Riemann equation (12) was obtained under the assumption that the Cauchy problem has a solution. Thus, if a solution exists to the Cauchy problem, it must be expressed by eq. (12); this proves the *uniqueness* of the solution to the Cauchy problem.

It follows directly from eq. (12) that for a sufficiently small change in the given Cauchy conditions on the curve AB, the solution to the problem changes by an arbitrarily small amount; that is, the solution to the Cauchy problem is a continuous function of the initial conditions. It also follows from eq. (12) that the value of the solution u at the point M depends only on the initial conditions along the arc PQ, which is cut from the curve AB by the characteristics passing through the point M. If we change the given Cauchy conditions on the curve AB outside the arc PQ, maintaining continuity at the points P and Q, the solution will vary only outside the curvilinear triangle MPQ. Thus, each characteristic separates the region in which the

solution remains invariant from the region in which it varies. Consequently, beyond each characteristic line, the solutions of the equation can not be extended uniquely.

The assumption made above that the straight lines parallel to the axes (that is, the characteristics) intersect the curve AB at not more than one point is essential. If this condition is not satisfied, the Cauchy problem is, generally speaking, insoluble. Suppose, for example, that the curve AB has the shape shown in fig. 2.

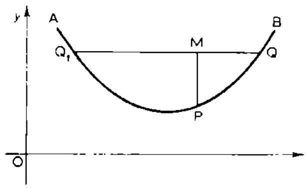

Fig. 2.

When we apply Riemann's method, we can determine the value of the unknown function $u(x, y)$ at the point M by using either the curvilinear triangle PQM or the curvilinear triangle Q_1PM. The two formulae obtained will generally give different solutions at the point M and, hence, the Cauchy problem is insoluble.

2. *Examples of applications of Riemann's method*

Example 1. Let us solve, by Riemann's method, the example given at the end of Chapter I; that is, let us find the solution to the equation

$$x^2 \frac{\partial^2 u}{\partial x^2} - y^2 \frac{\partial^2 u}{\partial y^2} = 0 , \tag{14}$$

satisfying the conditions

$$u\big|_{y=1} = f(x) , \qquad \frac{\partial u}{\partial y}\bigg|_{y=1} = F(x) . \tag{15}$$

We know that if we make the change of variables

$$\xi = xy , \qquad \eta = y/x , \tag{16}$$

eq. (14) is reduced to canonical form:

$$\frac{\partial^2 u}{\partial \xi\, \partial \eta} - \frac{1}{2\xi} \frac{\partial u}{\partial \eta} = 0 . \tag{17}$$

The line $y = 1$, in the new variables, will take the form of the rectangular hyperbola (fig. 3)

$$\xi\eta = 1 . \tag{18}$$

Further, it is clear from the relations

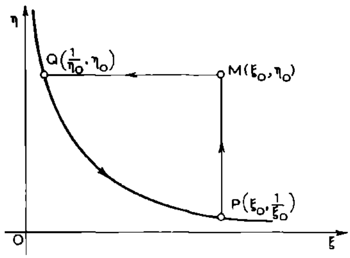

Fig. 3.

$$x = \sqrt{\xi/\eta}\,, \qquad y = \sqrt{\xi\eta}$$

that

$$\left.\frac{\partial u}{\partial \xi}\right|_{\xi\eta=1} = \frac{1}{2}\frac{\partial u}{\partial x} + \left.\frac{1}{2\xi}\frac{\partial u}{\partial y}\right|_{\xi\eta=1}$$

$$\left.\frac{\partial u}{\partial \eta}\right|_{\xi\eta=1} = -\frac{\xi^2}{2}\frac{\partial u}{\partial x} + \left.\frac{\xi}{2}\frac{\partial u}{\partial y}\right|_{\xi\eta=1}$$

Consequently, on the basis of the conditions (15), we have

$$\left.\frac{\partial u}{\partial \xi}\right|_{\xi\eta=1} = \tfrac{1}{2}f'(\xi) + \frac{1}{2\xi}F(\xi)\,, \qquad \left.\frac{\partial u}{\partial \eta}\right|_{\xi\eta=1} = -\tfrac{1}{2}\xi^2 f'(\xi) + \tfrac{1}{2}\xi\, F(\xi)\,, \tag{19}$$

and

$$u|_{\xi\eta=1} = f(\xi)\,. \tag{20}$$

Setting $a = 0$, $b = -1/2\xi$, and $f = 0$ in Riemann's formula, we obtain

$$u(\xi_0,\eta_0) = \frac{(uv)_P + (uv)_Q}{2} + \frac{1}{2}\int_{QP}\left(v\frac{\partial u}{\partial \xi} - u\frac{\partial v}{\partial \xi} - \frac{uv}{\xi}\right)d\xi - \left(v\frac{\partial u}{\partial \eta} - u\frac{\partial v}{\partial \eta}\right)d\eta \tag{21}$$

We now turn to finding the Riemann function $v(\xi,\eta;\xi_0,\eta_0)$. According to the general theory, it must satisfy the conjugate equation

$$\frac{\partial^2 v}{\partial\xi\,\partial\eta} + \frac{1}{2\xi}\frac{\partial v}{\partial \eta} = 0 \tag{22}$$

and, on the characteristics, it must satisfy the following conditions:

$$\begin{aligned} v(\xi,\eta_0;\xi_0,\eta_0) &= \exp\left[-\int_{\xi_0}^{\xi}\frac{d\xi}{2\xi}\right] = \sqrt{\frac{\xi_0}{\xi}} \qquad \text{(on MQ)}\,, \\ v(\xi_0,\eta;\xi_0,\eta_0) &= \exp\left[\int_{\eta_0}^{\eta} 0\cdot d\eta\right] = 1 \qquad \text{(on MP)}\,. \end{aligned} \tag{23}$$

It is easy to show that the function

$$v(\xi,\eta;\xi_0,\eta_0) = \sqrt{\xi_0/\xi} \tag{24}$$

satisfies both eq. (22) and the two conditions (23); consequently, it is the desired Riemann function. Substituting eqs. (19), (20), and (24) in formula (21) and remembering that

$$u(\mathrm{P}) = f(\xi_0)\ , \qquad u(\mathrm{Q}) = f\Big(\frac{1}{\eta_0}\Big)\ ,$$

$$v(\mathrm{P}) = v\Big(\xi_0, \frac{1}{\xi_0}; \xi_0, \eta_0\Big) = 1\ , \qquad v(\mathrm{Q}) = v\Big(\frac{1}{\eta_0}, \eta_0; \xi_0, \eta_0\Big) = \sqrt{\xi_0\eta_0}\ ,$$

we obtain

$$u(\xi_0,\eta_0) = \tfrac{1}{2}f(\xi_0) + \tfrac{1}{2}\sqrt{\xi_0\eta_0}\ f\Big(\frac{1}{\eta_0}\Big) + \tfrac{1}{4}\sqrt{\xi_0}\int_{\xi_0}^{1/\eta_0} \frac{f(\xi)}{\xi^{\frac{3}{2}}}\,\mathrm{d}\xi - \tfrac{1}{2}\sqrt{\xi_0}\int_{\xi_0}^{1/\eta_0} \frac{F(\xi)}{\xi^{\frac{3}{2}}}\,\mathrm{d}\xi.$$

Returning now to the original variables x and y, we obtain the solution to the Cauchy problem in the form found above, namely,

$$u(x,y) = \tfrac{1}{2}f(xy) + \tfrac{1}{2}y\, f\Big(\frac{x}{y}\Big) + \tfrac{1}{4}\sqrt{xy}\int_{xy}^{x/y} \frac{f(z)\,\mathrm{d}z}{z^{\frac{3}{2}}} - \tfrac{1}{2}\sqrt{xy}\int_{xy}^{x/y} \frac{F(z)\,\mathrm{d}z}{z^{\frac{3}{2}}}\ .$$

Example 2. Find the solution of the equation

$$x\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = 0 \qquad (x > 0)\ , \tag{25}$$

satisfying the conditions

$$u\big|_{y=0} = f(x)\ , \qquad \frac{\partial u}{\partial y}\Big|_{y=0} = F(x)\ . \tag{26}$$

To solve this problem by Riemann's method, we reduce eq. (25) to canonical form. First, we set up the equation for the characteristics

$$x\,\mathrm{d}y^2 - \mathrm{d}x^2 = 0\ .$$

This equation has two distinct solutions

$$\tfrac{1}{2}y + \sqrt{x} = C_1\ , \qquad \tfrac{1}{2}y - \sqrt{x} = C_2\ ,$$

and, consequently, we need to introduce new variables ξ and η defined by

$$\xi = \tfrac{1}{2}y + \sqrt{x}\ , \qquad \eta = \tfrac{1}{2}y - \sqrt{x} \qquad (x > 0)\ . \tag{27}$$

To these equations we add another relation

$$w = u\sqrt{\xi - \eta}\ , \tag{28}$$

and then eq. (25) can be transformed to the following canonical form:

$$\frac{\partial^2 w}{\partial\xi\,\partial\eta} - \frac{1}{4}\frac{w}{(\xi-\eta)^2} = 0\ . \tag{29}$$

Let us now turn to conditions (26) and eq. (27). It is clear from these that, in applying Riemann's method, we must choose the line $\eta = -\xi$ as the *curve* AB (fig. 4).

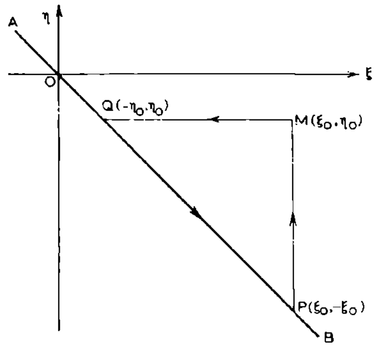

Fig. 4.

Next, we must find a particular solution to the conjugate equation

$$\frac{\partial^2 v}{\partial\xi\,\partial\eta} - \frac{1}{4}\,\frac{v}{(\xi-\eta)^2} = 0\ , \tag{31}$$

that satisfies the following conditions on the characteristics

$$\begin{aligned} v(\xi_0,\eta;\xi_0,\eta_0) &= 1 \qquad \text{(on MP)},\\ v(\xi,\eta_0;\xi_0,\eta_0) &= 1 \qquad \text{(on MQ)}. \end{aligned} \tag{32}$$

Let us seek a solution to eq. (31) of the form

$$v = G(\sigma)\ , \tag{33}$$

where

$$\sigma = \frac{(\xi-\xi_0)\,(\eta-\eta_0)}{(\xi_0-\eta_0)\,(\xi-\eta)}\ . \tag{34}$$

Then, for $G(\sigma)$, we obtain the following equation:

$$\sigma(1-\sigma)\,G''(\sigma) + (1-2\sigma)\,G'(\sigma) - \tfrac{1}{2}G(\sigma) = 0\ . \tag{35}$$

It is easy to see that this equation is a particular case of the hypergeometric equation of Gauss:

$$\sigma(1-\sigma)\,y'' + [\gamma-(1+\alpha+\beta)\sigma]\,y' - \alpha\beta y = 0 \tag{36}$$

for

$$\alpha = \beta = \tfrac{1}{2}\ , \qquad \gamma = 1\ .$$

Gauss' equation has a particular solution in the form of the hypergeometric series

$$F(\alpha,\beta,\gamma;\sigma) = 1 + \frac{\alpha\beta}{1!\,\gamma}\,\sigma + \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{2!\,\gamma(\gamma+1)}\,\sigma^2 + \dots\ , \tag{37}$$

which converges absolutely for $|\sigma| < 1$.

Thus, it is clear that if we take

$$v = G(\sigma) = F(\tfrac{1}{2}, \tfrac{1}{2}, 1; \sigma) = 1 + (\tfrac{1}{2})^2 \sigma + (\tfrac{1}{2} \cdot \tfrac{3}{4})^2 \sigma^2 + \dots , \tag{38}$$

we shall satisfy eq. (31) and the conditions (32). Consequently, the function

$$v = G\left(\frac{(\xi - \xi_0)(\eta - \eta_0)}{(\xi_0 - \eta_0)(\xi - \eta)}\right) \tag{39}$$

is the desired Riemann function.

Turning now to the problem of finding the solution to eq. (25) with the conditions (26), we set

$$a = b = 0 , \qquad f = 0$$

in the Riemann equation (12). Then, we obtain

$$w(\xi_0, \eta_0) = \frac{w(\mathrm{P}) + w(\mathrm{Q})}{2} + \frac{1}{2} \int_{\mathrm{QP}} \left(v \frac{\partial w}{\partial \xi} - w \frac{\partial v}{\partial \xi}\right) d\xi - \left(v \frac{\partial w}{\partial \eta} - w \frac{\partial v}{\partial \eta}\right) d\eta ,$$

where the function v is determined by eq. (39) or, remembering eq. (30), we have

$$w(\xi_0, \eta_0) = \frac{w(\mathrm{P}) + w(\mathrm{Q})}{2} + \frac{1}{2} \int_{-\eta_0}^{\xi_0} v\left(\frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial \eta}\right) d\xi - \frac{1}{2} \int_{-\eta_0}^{\xi_0} w\left(\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}\right) d\xi . \tag{40}$$

Let us evaluate the derivatives appearing in eq. (40). From the equations

$$x = \tfrac{1}{4}(\xi - \eta)^2 , \qquad y = \xi + \eta ,$$

it is clear that

$$\left.\frac{\partial u}{\partial \xi}\right|_{\eta=-\xi} = \xi \frac{\partial u}{\partial x} + \left.\frac{\partial u}{\partial y}\right|_{y=0} , \qquad \left.\frac{\partial u}{\partial \eta}\right|_{\eta=-\xi} = -\xi \frac{\partial u}{\partial x} + \left.\frac{\partial u}{\partial y}\right|_{y=0} .$$

Consequently, on the basis of the conditions (26), we have

$$\frac{\partial u}{\partial \xi} + \left.\frac{\partial u}{\partial \eta}\right|_{\eta=-\xi} = 2 \left.\frac{\partial u}{\partial y}\right|_{y=0} = 2F(\xi^2) . \tag{41}$$

Differentiating eq. (28) with respect to ξ and η and then setting $\eta = -\xi$, we obtain

$$\left.\frac{\partial w}{\partial \xi}\right|_{\eta=-\xi} = \sqrt{2\xi} \frac{\partial u}{\partial \xi} + \frac{u}{2\sqrt{2\xi}} , \qquad \left.\frac{\partial w}{\partial \eta}\right|_{\eta=-\xi} = \sqrt{2\xi} \frac{\partial u}{\partial \eta} - \frac{u}{2\sqrt{2\xi}} .$$

Hence, from (41), we obtain

$$\frac{\partial w}{\partial \xi} + \left.\frac{\partial w}{\partial \eta}\right|_{\eta=-\xi} = \sqrt{2\xi} \left.\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\right|_{\eta=-\xi} = 2\sqrt{2\xi}\, F(\xi^2) . \tag{42}$$

Furthermore, from the equations

$$\left.\frac{\partial v}{\partial \xi}\right|_{\eta=-\xi} = \frac{dG}{d\sigma}\left.\frac{\partial \sigma}{\partial \xi}\right|_{\eta=-\xi} = -\frac{1}{4}\frac{(\xi+\eta_0)\,(\xi+\xi_0)}{(\xi_0-\eta_0)\,\xi^2}\left(\frac{dG}{d\sigma}\right)_{\eta=-\xi},$$

$$\left.\frac{\partial v}{\partial \eta}\right|_{\eta=-\xi} = \frac{dG}{d\sigma}\left.\frac{\partial \sigma}{\partial \eta}\right|_{\eta=-\xi} = \frac{1}{4}\frac{(\xi-\eta_0)\,(\xi-\xi_0)}{(\xi_0-\eta_0)\,\xi^2}\left(\frac{dG}{d\sigma}\right)_{\eta=-\xi},$$

it follows that

$$\left.\frac{\partial v}{\partial \xi}+\frac{\partial v}{\partial \eta}\right|_{\eta=-\xi} = -\frac{\xi_0+\eta_0}{2(\xi_0-\eta_0)\xi}\left(\frac{dG}{d\sigma}\right)_{\eta=-\xi}. \tag{43}$$

To use eq. (40), we still need to find the values of the function w on the line $\eta = -\xi$ and at the points P and Q. It is easy to see that

$$w|_{\eta=-\xi} = w(\xi,-\xi) = \sqrt{2\xi}\,u(x,0) = \sqrt{2\xi}\,f(\xi^2). \tag{44}$$

From this, we easily obtain

$$w(\mathrm{P}) = w(\xi_0\,,-\xi_0) = \sqrt{2\xi_0}\,f(\xi_0^2), \qquad w(\mathrm{Q}) = w(-\eta_0,\eta_0) = \sqrt{-2\eta_0}\,f(\eta_0^2). \tag{45}$$

Remembering that

$$u(x_0,y_0) = \frac{w(\xi_0,\eta_0)}{\sqrt{2}\,\sqrt[4]{x_0}},$$

we find from eqs. (40) and (42) - (45) that

$$u(x_0,y_0) = \frac{\sqrt{\xi_0}\,f(\xi_0^2)+\sqrt{-\eta_0}\,f(\eta_0^2)}{2\sqrt[4]{x_0}} + \frac{1}{\sqrt[4]{x_0}}\int_{-\eta_0}^{\xi} G\left(\frac{(\xi_0-\xi)\,(\xi+\eta_0)}{2\xi(\xi_0-\eta_0)}\right) F(\xi^2)\sqrt{\xi}\,d\xi$$
$$+\frac{\xi_0+\eta_0}{4(\xi_0-\eta_0)\sqrt[4]{x_0}}\int_{-\eta_0}^{\xi_0}\left(\frac{dG}{d\sigma}\right)_{\eta=-\xi} f(\xi^2)\,\frac{d\xi}{\sqrt{\xi}}.$$

Returning now to the original variables x and y, and omitting the subscripts from these letters, we obtain the solution of the Cauchy problem for eq. (25):

$$u(x,y) = \frac{\sqrt{\sqrt{x}+\frac{1}{2}y}\,f(x+\sqrt{x}y+\frac{1}{4}y^2)+\sqrt{\sqrt{x}-\frac{1}{2}y}\,f(x-\sqrt{x}y+\frac{1}{4}y^2)}{2\sqrt[4]{x}}$$
$$+\frac{1}{\sqrt[4]{x}}\int_{\sqrt{x}-\frac{1}{2}y}^{\sqrt{x}+\frac{1}{2}y}\Phi(x,y,z)\,dz,$$

where

$$\Phi(x,y,z) = \sqrt{z}\,F(z^2)\,G\left(\frac{\frac{1}{4}y^2-(z-\sqrt{x})^2}{4z\sqrt{x}}\right)+\frac{yf(z^2)}{8\sqrt{xz}}\,G'\left(\frac{\frac{1}{4}y^2-(z-\sqrt{x})^2}{4z\sqrt{x}}\right).$$

Problems

1. Find the Riemann function for the Euler-Darboux equation

$$\frac{\partial^2 u}{\partial x\,\partial y} - \frac{\beta}{x-y}\frac{\partial u}{\partial x} + \frac{\alpha}{x-y}\frac{\partial u}{\partial y} = 0 .$$

Answer:

$$v(x,y;x_0,y_0) = (y_0 - x)^{-\beta}\,(y-x_0)^{-\alpha}\,(y-x)^{\alpha+\beta}\,F(\alpha,\beta,1;\sigma) ,$$

where

$$\sigma = \frac{(x-x_0)\,(y-y_0)}{(x-y_0)\,(y-x_0)} .$$

2. Integrate, by the Riemann method, the equation

$$(l^2 - x^2)\,\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 2x\,\frac{\partial u}{\partial x} - \tfrac{1}{4}u = 0 \qquad (0 < x < l)$$

with the conditions

$$u\Big|_{y=0} = f(x) , \qquad \frac{\partial u}{\partial y}\Big|_{y=0} = F(x) .$$

Answer:

$$u(x,y) = \frac{\sqrt{\sin(\omega - y)}\; f(l\cos(\omega - y)) + \sqrt{\sin(\omega + y)}\; f(l\cos(\omega + y))}{2\sqrt{\sin\omega}}$$

$$+ \frac{1}{2\sqrt{\sin\omega}}\int_{\omega-y}^{\omega+y}\Phi(\omega,y,z)\,dz , \qquad \omega = \arccos\frac{x}{l} ,$$

where

$$\Phi(\omega,y,z) = \sqrt{\sin z}\; G\Big(\frac{\cos(\omega - z) - \cos y}{2\sin\omega\,\sin z}\Big)\, F(l\cos z)$$

$$+ \frac{1}{2}\,\frac{\sin y}{\sin\omega\,\sqrt{\sin z}}\; G'\Big(\frac{\cos(\omega - z) - \cos y}{2\sin\omega\,\sin z}\Big)\, f(l\cos z) .$$

Method of solution: first reduce the given equation (see problem 5 of the Introduction) to the canonical form

$$\frac{\partial^2 w}{\partial\xi\,\partial\eta} - \frac{1}{4}\,\frac{w}{\sin^2(\xi - \eta)} = 0 .$$

Then show that the Riemann function is of the form

$$v(\xi,\eta;\xi_0,\eta_0) = G\Big(\frac{\sin(\xi - \xi_0)\,\sin(\eta - \eta_0)}{\sin(\xi_0 - \eta_0)\,\sin(\xi - \eta)}\Big) ,$$

where the function $G(\sigma)$ is defined by the series (38).

Chapter III

THE APPLICATION OF THE METHOD OF CHARACTERISTICS TO THE STUDY OF LOW-AMPLITUDE VIBRATIONS OF A STRING

1. *Derivation of the equation for the vibrations of a string*

Let us consider the problem of a stretched string fixed at both ends. By a "string", we mean a thin thread that can be freely bent, that is, one which offers no resistance to such changes in its shape as can be made without changes in length. The tension T_0 acting on the string is assumed to be large, so that we may neglect the force of gravity. Suppose that, at its equilibrium position, the string lies along the x-axis.

We shall consider only *transverse vibrations* of the string, assuming that motion takes place only in a single plane and that all points of the string move in a direction perpendicular to the x-axis.

Let us denote by $u = u(x, t)$ the displacement of points of the string from the equilibrium position at the instant of time t. At every fixed value of t, the graph of the function $u = u(x, t)$ clearly represents the shape of the string at that instant (fig. 5).

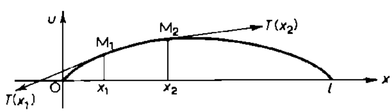

Fig. 5.

We shall also consider only *small-amplitude vibrations*, so that we may neglect the square of the derivative $\partial u/\partial x$ as being small in comparison with unity.

Let us pick out an arbitrary segment (x_1, x_2) of the string which is deformed into the segment M_1M_2 as the vibrations take place. The arc length of this segment at the instant t is equal to

$$S' = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}\, dx \approx x_2 - x_1 = S ,$$

so that we may assume that the vibrations do not lengthen the segments of the string. It then follows from Hooke's law that the value of the tension at every point of the string remains constant with time. Let us show that we may also consider the tension T as being independent of x, that is, that

$T \approx T_0$. Tension (directed along the tangent to the string at the points M_1 and M_2), external forces, and inertial forces act on the segment M_1M_2. The sum of the projections of all these forces on the x-axis must be equal to zero. Since we are considering only transverse vibrations, the inertial forces and the external forces are directed parallel to the x-axis; therefore

$$T(x_1) \cos \alpha(x_1) - T(x_2) \cos \alpha(x_2) = 0 ,$$

where $\alpha(x)$ is the angle between the tangent to the string (at the point whose abscissa is x at the time t) and the positive x-axis.

Because the vibrations of the string are of small amplitude,

$$\cos \alpha(x) = \frac{1}{\sqrt{1 + \tan^2\alpha(x)}} = \frac{1}{\sqrt{1 + (\partial u/\partial x)^2}} \approx 1 ,$$

and we have

$$T(x_1) \approx T(x_2) .$$

Since x_1 and x_2 are arbitrary points, it follows that the tension is independent of x. Therefore, we may assume that $T \approx T_0$ for all values of x and t.

We now proceed to derive the equation for the vibrations of the string. For this, we employ the principle of kinetic equilibrium (d'Alembert's principle), which states that the sum of all the forces, including the inertial forces, acting on any particular segment of the string must be zero.

Let us consider an arbitrary segment M_1M_2 of the string (fig. 5), and let us set up the condition for the vanishing of the sum of the projections onto the u-axis of all the forces acting on that segment: the tension forces (equal in magnitude and directed along the tangents to the string at the points M_1 and M_2), the external force, the resistance force of the medium (directed parallel to the u-axis), and the inertial force. The sum of the projections onto the u-axis of the tension acting at the points M_1 and M_2 is equal to

$$Y = T_0[\sin \alpha(x_2) - \sin \alpha(x_1)] ,$$

but, because of our assumptions,

$$\sin \alpha(x) = \frac{\tan \alpha(x)}{\sqrt{1 + \tan^2\alpha(x)}} = \frac{\partial u/\partial x}{\sqrt{1 + (\partial u/\partial x)^2}} \approx \frac{\partial u}{\partial x}$$

and, consequently,

$$Y = T_0 \left\{\left(\frac{\partial u}{\partial x}\right)_{x=x_2} - \left(\frac{\partial u}{\partial x}\right)_{x=x_1}\right\} .$$

We now note that

$$\left(\frac{\partial u}{\partial x}\right)_{x=x_2} - \left(\frac{\partial u}{\partial x}\right)_{x=x_1} = \int_{x_1}^{x_2} \frac{\partial^2 u}{\partial x^2}\, dx ,$$

and we finally obtain

$$Y = T_0 \int_{x_1}^{x_2} \frac{\partial^2 u}{\partial x^2}\, dx \,. \tag{1}$$

We denote by $p(x, t)$ the external force per unit of length acting on the string in a direction parallel to the u-axis. Then, the projection onto the u-axis of this force, acting on the segment M_1M_2 of the string, will be equal to

$$\int_{x_1}^{x_2} p(x, t)\, dx \,. \tag{2}$$

To find the resistance force of the medium, we need to consider both the nature of the medium and the speed at which the string is vibrating. If the string is vibrating in air and the speed of vibration is not very great, then we can assume that the resistance force of the medium is proportional to the first power of the speed. This leads to the following expression for the projection onto the u-axis of the resistance force acting on the segment M_1M_2:

$$-\int_{x_1}^{x_2} 2k \frac{\partial u}{\partial t}\, dx \,, \tag{3}$$

where k is a positive constant.

Let $\rho(x)$ denote the linear density of the string. Then, the inertial force acting on the segment M_1M_2 will be equal to

$$-\int_{x_1}^{x_2} \rho(x) \frac{\partial^2 u}{\partial t^2}\, dx \,. \tag{4}$$

The sum of all the forces (1) - (4) must be equal to zero; that is,

$$\int_{x_1}^{x_2} \left\{T_0 \frac{\partial^2 u}{\partial x^2} + p(x, t) - 2k \frac{\partial u}{\partial t} - \rho(x) \frac{\partial^2 u}{\partial t^2}\right\} dx = 0 \,. \tag{5}$$

Since x_1 and x_2 are arbitrary points, it follows that the integrand must be equal to zero at every point of the string, at an arbitrary instant of time t:

$$\rho(x) \frac{\partial^2 u}{\partial t^2} + 2k \frac{\partial u}{\partial t} = T_0 \frac{\partial^2 u}{\partial x^2} + p(x, t) \,. \tag{6}$$

This is the desired equation for the vibrations of the string.

If ρ = constant, that is, if the string is homogeneous, eq. (6) is ordinarily written in the form

$$\frac{\partial^2 u}{\partial t^2} + 2h \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + g(x, t) \,, \tag{7}$$

where

$$a = \sqrt{\frac{T_0}{\rho}}\,, \qquad h = \frac{k}{\rho}\,, \qquad g(x, t) = \frac{1}{\rho}\, p(x, t) \,.$$

It is easy to see that if we neglect the resistance, then, in the absence of an external force, eq. (7) becomes

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{8}$$

which is called the equation for a freely vibrating string.

Eq. (6) alone, as we know, does not completely determine the motion of the string: we must also know the position and velocity of all points of the string at the initial time ($t = 0$):

$$u|_{t=0} = f(x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x), \tag{9}$$

where $f(x)$ and $F(x)$ are given for $0 < x < l$. The conditions (9) are called *initial conditions*. Furthermore, since the ends of the string are fixed, the distance from the x-axis $u(x, t)$, at the points $x = 0$ and $x = l$ must be equal to zero for all t; that is,

$$u|_{x=0} = 0, \qquad u|_{x=l} = 0. \tag{10}$$

The conditions (10) are called *boundary conditions*.

Thus, the physical problem of the vibrations of a string is reduced to the *mathematical problem* of finding the solution to eq. (6) that satisfies the initial conditions (9) and the boundary conditions (10).

The problem of finding the solution to eq. (6) with initial conditions (9) and boundary conditions (10) is called a *mixed problem*.

We may also consider an infinite string. Such a problem arises if the string is so long that we may neglect the effect of its ends. In such a case, the initial conditions alone are sufficient to determine the unique solution to eq. (8).

2. *Vibrations of a homogeneous infinite string*

Suppose that we are dealing with a string that is so long that we may consider it as extending infinitely far in both directions. Let us examine the free vibrations of a homogeneous infinitely long string. As was shown in section 1, this problem is reduced to solving the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (a = \sqrt{T/\rho}) \tag{8}$$

with initial conditions

$$u|_{t=0} = f(x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x), \tag{9}$$

where the functions $f(x)$ and $F(x)$ are given in the interval $(-\infty, \infty)$.

As we know (see section 1, Chapter I), the unique solution to eq. (8) satisfying the initial conditions (9) is of the form

$$u(x,t) = \frac{f(x-at) + f(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} F(z)\,dz\ . \tag{11}$$

This solution can be written as

$$u = \varphi(x-at) + \psi(x+at)\ , \tag{12}$$

where

$$\varphi(x) = \tfrac{1}{2}f(x) - \frac{1}{2a}\int_0^x F(z)\,dz\ , \qquad \psi(x) = \tfrac{1}{2}f(x) + \frac{1}{2a}\int_0^x F(z)\,dz\ .$$

The displacement $u(x,t)$ is thus the sum of the two terms

$$u_1 = \varphi(x-at) \tag{13}$$

and

$$u_2 = \psi(x+at)\ . \tag{14}$$

Let us first examine the particular case where $\psi = 0$, that is, when the displacement of the string is determined by eq. (13). Let us assume that the independent variables vary in such a way that the difference remains constant, that is, that

$$x - at = c\ .$$

In this case,

$$dx - a\ dt = 0\ , \qquad \text{i.e.,} \qquad dx/dt = a\ .$$

Thus, we may conclude the following: if the point x moves with constant velocity a in the positive direction (from left to right), then the displacement u_1 of the string at that point will at all times be equal to $\varphi(c)$, thus remaining constant (fig. 6). This displacing motion $\varphi(c)$ along the string is called the *forward wave*. It is characterized by the particular solution $u_1 = \varphi(x-at)$ of the wave equation (8).

Similarly, a displacement $\psi(c)$, analogous to the preceding one, but taking place in the opposite direction (fig. 7), corresponds to the particular solution $u_2 = \psi(x+at)$. Here, we are dealing with the *backward wave*. The constant

$$a = \sqrt{T_0/\rho}$$

is the velocity of propagation of the waves along the string.

In the general case, expressed by eq. (12), the actual displacement of the string is obtained by adding the displacements u_1 and u_2 at every given instant of time t. This leads to the following graphical method of constructing the displaced points of the string.

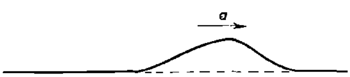

Fig. 6.

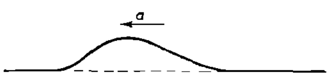

Fig. 7.

Let us draw the curves

$$y_1 = \varphi(x)\ , \qquad y_2 = \psi(x)\ ,$$

representing the forward and backward waves at the initial instant of time, then, without changing the shape of these waves, let us displace them simultaneously with velocity a in the two directions: $y_1 = \varphi(x)$, to the right; and $y_2 = \psi(x)$, to the left. Now, to obtain the graph of the curve, we need only take the algebraic sums of the ordinates of the displaced curves.

Eq. (11) gives the complete solution to the problem under consideration. Let us examine this formula in greater detail for the two most interesting cases (namely, when there are no initial velocities and when there are no initial displacements).

CASE I. The initial velocities are equal to zero. In this case, $F(x) = 0$ and from eq. (11), we obtain:

$$u(x,t) = \frac{f(x-at) + f(x+at)}{2}\ , \tag{15}$$

from which it is easy to compute the displacement of any point on the string from the equilibrium position. This formula may be used to draw the graph of the string at an arbitrary instant of time t (we need only proceed as above).

Suppose, for example, that at the initial instant the string has the shape shown in fig. 8a, that is, that the function $f(x)$ is different from zero only in a finite interval $(-\alpha,\alpha)$. Let us first draw the graphs of the forward and backward waves (figs. 8b, c). In this case, they will be the same and

$$u = \tfrac{1}{2}f(x)\ ,$$

which follows from eq. (15). If we displace these graphs over a distance $\frac{1}{2}\alpha$ in the directions indicated by the arrows, and then take the sums of the or-

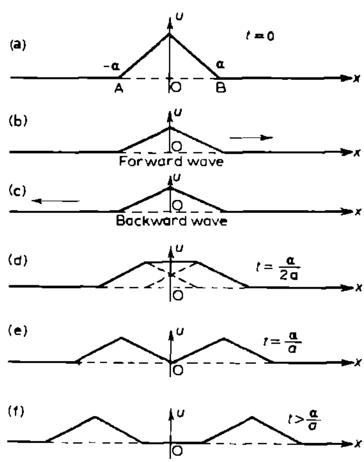

Fig. 8.

dinates of the displaced graphs, we shall obtain the shape of the string at the time $t = \alpha/2a$ (fig. 8d).

Let us now again displace the graphs of the waves over a distance $\frac{1}{2}\alpha$. Then, by construction, we obtain the shape of the curve at the instant of time $t = \alpha/a$ (fig. 8e). If we displace the curves representing the string still further, we shall have the shape shown in fig. 8f. Here, the forward and backward waves are being propagated in opposite directions and the displacement of the string is only one half as great as the corresponding displacement at the segment AB (fig. 8a). After both waves have passed a given point (or after only one wave, in the case of points lying outside the region of the initial disturbance), that point will remain at rest.

CASE II. There are no initial displacements. In this case, $f(x) = 0$ and the displacement of the string is expressed by the equation

$$u = \frac{1}{2a}\int_{x-at}^{x+at} F(z)\,dz\ . \tag{16}$$

If we set

$$\frac{1}{2a}\int_{0}^{x} F(z)\,dz = \psi(x)\ ,$$

this displacement can be represented in the form

$$u = \psi(x+at) - \psi(x-at)\ ,$$

from which it is clear that, in this case also, we are dealing with the propagation of a forward and a backward wave. The function $F(x)$, expressing the initial velocity, can be given in different ways.

As an example, let us consider the following representation of this function:

$$\begin{aligned} F(x) &= 0 \quad \text{outside the interval } (-\alpha,\alpha)\ ,\\ F(x) &> 0 \quad \text{within the interval } (-\alpha,\alpha)\ , \end{aligned} \tag{17}$$

where α is some given number.

Let us partition the interval $(-\infty,\infty)$ in which x varies into five subintervals:

$$\mathrm{I}(-\infty,-at-\alpha),\ \mathrm{II}(-at-\alpha,-at+\alpha),\ \mathrm{III}(-at+\alpha,at-\alpha),\ \mathrm{IV}(at-\alpha,at+\alpha),\ \mathrm{V}(at+\alpha,+\infty)$$

and let us examine the oscillation of the string beginning at some time $t > \alpha/a$.

When x varies in the first interval, the upper limit of the integral shown in eq. (16) always remains less than $-\alpha$ and, consequently, the entire interval of integration lies outside the interval $(-\alpha,\alpha)$. Hence, on the basis of (17), we conclude that $u = 0$ for all values of x lying within the first interval.

The same may be said with regard to the fifth interval. When x varies in the second interval, we obviously obtain the inequality

$$-\alpha < x + at < \alpha\ . \tag{18}$$

On the other hand, it follows from the inequality $t > \alpha/a$ that

$$-2at < -2\alpha \ . \tag{19}$$

Adding inequalities (18) and (19), we find that

$$x - at < -\alpha \ .$$

Therefore, it follows on the basis of (17) that

$$u = \frac{1}{2a} \int_{-\alpha}^{x+at} F(z) \, dz \ .$$

Similarly, when x varies in the fourth interval, the integral (16) is reduced to

$$u = \frac{1}{2a} \int_{x-at}^{\alpha} F(z) \, dz \ .$$

Now suppose that x varies within the third interval. Then,

$$x - at < -\alpha \quad \text{and} \quad x + at > \alpha \ ;$$

that is, the interval $(-\alpha, \alpha)$ is entirely contained within the interval $(x - at, x + at)$. Taking (17) into account, we find that

$$u = \frac{1}{2a} \int_{-\alpha}^{\alpha} F(z) \, dz \ ; \tag{20}$$

that is, u will be constant in this case.

Thus, for time $t > \alpha/a$, the string has the shape shown in fig. 9. Section III of the string is represented by a segment of the straight line whose distance from the x-axis is numerically equal to the right side of eq. (20). The forward and backward waves have already passed through this portion of the string. At the instant in question, the forward wave is passing through section IV and the backward wave through section II. Sections I and V are still at rest because the waves have not yet reached them.

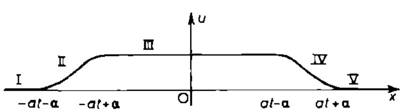

Fig. 9.

Here we observe the interesting phenomenon of the *residual effect.* Waves that have passed through section III leave a trace of their passage. The points of this section remain displaced at the same elevation; this constant elevation, of course, depends on the magnitude of the initial impulse.

3. *Vibrations of a string fixed at both ends*

Suppose that we have a string of finite length l fixed at both ends. The problem of the vibration of such a string is reduced to finding the solution to the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{8}$$

with initial conditions

$$u|_{t=0} = f(x) , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x) \qquad (0 < x < l) \tag{9}$$

and boundary conditions

$$u|_{x=0} = 0 , \qquad u|_{x=l} = 0 . \tag{10}$$

As we know, the solution to eq. (8) is of the form

$$u = \varphi(x - at) + \psi(x + at) , \tag{21}$$

where the functions $\varphi(x)$ and $\psi(x)$ are determined by the formulae

$$\varphi(x) = \tfrac{1}{2} f(x) - \frac{1}{2a} \int_0^x F(z)\, dz , \qquad \psi(x) = \tfrac{1}{2} f(x) + \frac{1}{2a} \int_0^x F(z)\, dz . \tag{22}$$

If we compare the present problem with the case of the infinitely long string, we find one significant difference: in the case of the fixed string, the functions $f(x)$ and $F(x)$ appearing in the expressions for the initial conditions (9) are known only within the interval $(0, l)$; however, with the passage of time, the arguments of these functions pass beyond the limits of the above interval. Therefore, we cannot determine the displacement of the string by formulae (21) and (22). Thus, in applying the method of characteristics, we need to extend the functions $f(x)$ and $F(x)$ outside the interval $(0, l)$.

From a physical standpoint, this extension is reduced to a determination of an initial disturbance of an *infinite* string such that its motion in the interval $(0, l)$ would be the same as if it were fixed at the ends (of that interval). The motion of the remaining portion of the infinite string can be ignored.

The functions $f(x)$ and $F(x)$ can be extended outside the interval $(0, l)$ without difficulty, if we use the boundary conditions (10). In fact, these conditions and solution (21) give the formulae

$$\varphi(-at) + \psi(at) = 0 , \qquad \varphi(l - at) + \psi(l + at) = 0 ,$$

from which, after we replace at by x, we obtain

$$\varphi(-x) = -\psi(x) , \qquad \psi(l + x) = -\varphi(l - x) . \tag{23}$$

The first of the formulae (23) determines the function $\varphi(x)$ over the interval $(-l, 0)$, and the second determines the function $\psi(x)$ over the interval $(l, 2l)$; consequently, both the functions $\varphi(x)$ and $\psi(x)$ are completely determined over an interval of length $2l$.

Further, it follows from eqs. (23) that

$$\varphi(x+2l) = \varphi(x) , \qquad \psi(x+2l) = \psi(x) , \tag{24}$$

that is, that the functions $\varphi(x)$ and $\psi(x)$ are periodic functions with period $2l$.

If we now recall that

$$f(x) = \varphi(x) + \psi(x) , \qquad F(x) = a[\psi'(x) - \varphi'(x)] ,$$

we obtain the following formulae:

$$\begin{aligned} f(-x) &= -f(x) , & F(-x) &= -F(x) , \\ f(x+2l) &= f(x) , & F(x+2l) &= F(x) . \end{aligned} \tag{25}$$

These formulae show that the functions $f(x)$ and $F(x)$ can be extended from the interval $(0, l)$ to the interval $(-l, 0)$ (since they are odd functions), and then to the rest of the interval $(-\infty, \infty)$ (with period $2l$).

For the obtained solution to have continuous derivatives up to the second order inclusive, it is necessary (in addition to the requirement that the functions $f(x)$ and $F(x)$ be differentiable) that the following conditions be satisfied:

$$f(0) = f(l) = 0 , \qquad f''(0) = f''(l) = 0 , \qquad F(0) = F(l) = 0 .$$

We will now illustrate the use of formulae (23) for determining the displacement of a fixed string at different instants of time. Let us suppose that a homogeneous string, fixed at the ends $x = 0$ and $x = l$ (fig. 10), has at the initial time, the shape of a parabola whose axis of symmetry is perpendicular to the x-axis and bisects the segment AB. We are to determine the position of the string at the instants

$$t_1 = l/2a \quad \text{and} \quad t_2 = l/a ,$$

under the assumption that there are no initial velocities.

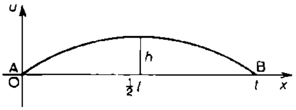

Fig. 10.

In this case, the initial conditions obviously are of the form

$$f(x) = \frac{4hx(l-x)}{l^2} , \qquad F(x) = 0 \qquad (0 < x < l) , \tag{26}$$

where h is the initial displacement of the string at the point $x = \frac{1}{2}l$.

We now use d'Alembert's solution

$$u = \varphi(x - at) + \psi(x + at) , \tag{27}$$

where, because of the initial conditions (26), the functions $\varphi(x)$ and $\psi(x)$ are determined in the interval $(0, l)$ by the formula

$$\varphi(x) = \psi(x) = \tfrac{1}{2} f(x) . \tag{28}$$

If we set $t = l/2a$ in eq. (27), we obtain

$$u = \varphi(x - \tfrac{1}{2}l) + \psi(x + \tfrac{1}{2}l) \ . \tag{29}$$

Let us now partition the interval $(0, l)$ into two sub-intervals:

$$(0, \tfrac{1}{2}l) \quad \text{and} \quad (\tfrac{1}{2}l, l) \ .$$

When x varies in the first interval, the argument of the function $\psi(x + \frac{1}{2}l)$ in eq. (29) varies within the interval $(\frac{1}{2}l, l)$ and, consequently, this function is determined by formula (28). The argument of the function $\varphi(x - \frac{1}{2}l)$, which appears in the same expression, varies within the interval $(-\frac{1}{2}l, 0)$, as a result of which we may not use formula (28). However, recalling the first of eqs. (23), we may set

$$\varphi(x - \tfrac{1}{2}l) = -\psi(\tfrac{1}{2}l - x) \ ,$$

so that we may then consider the function $\psi(\frac{1}{2}l - x)$, whose argument varies within the interval $(0, \frac{1}{2}l)$.

Thus, taking

$$u = \psi(x + \tfrac{1}{2}l) - \psi(\tfrac{1}{2}l - x) \qquad (0 < x < \tfrac{1}{2}l) \ ,$$

we obtain functions both of whose arguments vary within the interval $(0, l)$; therefore, by using eq. (28), we obtain

$$u = \frac{f(x + \frac{1}{2}l) - f(\frac{1}{2}l - x)}{2} \qquad (0 < x < \tfrac{1}{2}l) \ .$$

If, in this equation, we replace $f(x)$ by the expression given for it in eq. (26), we obtain

$$u = 0 \ , \qquad 0 < x < \tfrac{1}{2}l \ . \tag{30}$$

Let us now consider the second interval $(\frac{1}{2}l, l)$. For this interval, the function $\psi(u + \frac{1}{2}l)$ is unknown in eq. (29). But, by using the second of eqs. (23), we obtain

$$\psi(x + \tfrac{1}{2}l) = -\varphi(\tfrac{3}{2}l - x) \ .$$

Consequently, taking

$$u = \varphi(x - \tfrac{1}{2}l) - \varphi(\tfrac{3}{2}l - x) \qquad (\tfrac{1}{2}l < x < l) \ ,$$

we again obtain functions with arguments that vary within the interval $(0, l)$. It follows from what has been said that

$$u = \frac{f(x - \frac{1}{2}l) - f(\frac{3}{2}l - x)}{2} \qquad (\tfrac{1}{2}l < x < l) \ ;$$

then, replacing $f(x)$ by the expression given for it in eq. (26), we obtain

$$u = 0 \ , \qquad \tfrac{1}{2}l < x < l \ . \tag{31}$$

It follows from (30) and (31) that the displacement of the string at the instant $t_1 = l/2a$ is equal to zero; in other words, the points of the string are situated on the x-axis.

Let us now find the displacement of points of the string at the instant $t_2 = l/a$. From formula (27), we have

$$u = \varphi(x - l) + \psi(x + l) ,$$

but both arguments appearing in this expression for the functions vary outside the interval $(0, l)$; therefore, we need to use the formulae (23), whereby we obtain

$$u = -\psi(l - x) - \varphi(l - x) = -\tfrac{1}{2}f(l - x) - \tfrac{1}{2}f(l - x) = -f(x) ,$$

or

$$u = -\frac{4hx(l - x)}{l^2}$$

Thus, it follows that at the time $t_2 = l/a$ the string is symmetric about the x-axis with respect to its initial position. The same result can be obtained by a different method, if we use the graphical construction shown at the beginning of section 2 of the present chapter.

4. *A property of the characteristics*

Let us take some point M_0 of the string (with abscissa x_0), and let us observe it from some instant of time which we shall take as $t = 0$. If we displace the string from its equilibrium position, waves will begin to be propagated along it as a consequence of the created disturbance. At the time t_0, a forward and a backward wave will approach the observed point of the string, the first from the point $x_0 - at_0$ and the second from the point $x_0 + at_0$.

Let us try to determine these initial points by the graphical method. With this in mind, we look at the "phase plane" xOt (fig. 11), where the x-axis corresponds to the position of the string at the initial time $t = 0$. At the time t_0, the point $M_0(x_0, 0)$ will occupy the position $M_1(x_0, t_0)$ in the figure.

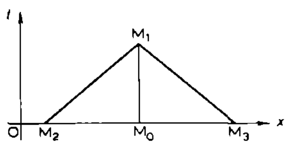

Fig. 11.

We now draw two straight lines through the point M_1 with slopes $\pm 1/a$; the equations of these lines are

$$x - at = x_0 - at_0 , \qquad x + at = x_0 + at_0 . \tag{32}$$

It is easy to see that these straight lines are the characteristics of eq. (8). Obviously, they intersect the x-axis at the desired points

$$M_2(x_0 - at_0, 0) \quad \text{and} \quad M_3(x_0 + at_0, 0) .$$

Thus, to find those points on the string at which the forward and backward waves are approaching the point of the string with abscissa x_0 at the

given instant of time t_0, we need to pass the characteristics (32) through the point $M_1(x_0, t_0)$. Their intersection with the x-axis will give the points that we are seeking.

5. *Wave reflection in a fastened string*

Let us use the construction of the preceding section to see what will happen in the case of a wave that reaches one of the ends of the fastened string.

We take the half-plane xOt (fig. 12) and draw the straight line $x = l$. This straight line, together with the t-axis, cuts a strip $tOlL$ of finite width from the original half-plane. Let us partition this strip into regions I, II, III, and so on, in the following fashion. First, let us draw through the points $x = 0$ and $x = l$ the characteristics $x - at = 0$ and $x + at = l$, extending them until they intersect the boundaries of the strip. Then let us draw the straight lines P_1P_4 and P_2P_3, and so on, parallel to these characteristics.

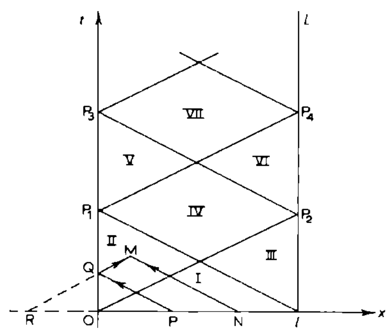

Fig. 12.

Let us examine successively the regions I, II, . . . , which correspond to the points of the string at different instants of time.

We begin with region I. Let us take an arbitrary point within it and let us draw the characteristics through that point, extending them until they intersect the x-axis. Then, recalling the remarks of the preceding section, we can easily see that region I corresponds to those points of the string reached by the forward and backward waves *immediately* after they leave the initially disturbed portion of the string.

Let us now take any point $M(x_0, t_0)$ lying within region II and let us pass the characteristics MN and MR through it. The equations of these lines will be

$$x + at = x_0 + at_0 , \qquad x - at = x_0 - at_0 .$$

The first characteristic intersects the x-axis at the point $N(x_0 + at_0, 0)$ lying *on* the string, and the second intersects it at the point $R(x_0 - at_0, 0)$ *not* on the string. Consequently, the backward wave, having left the point N at

the initial instant, approaches the point M. It is clear that the wave approaching the point M from the other side could not have left the point R, since that point does not lie on the string. Actually, this wave came from point P lying on the string; initially, it travelled in the form of a backward wave, and only after being *reflected* from the end $x=0$ (point Q in the figure) did it approach the point M as a forward wave. Let us prove this in the following manner. Let us take the point $P(-x_0+at_0, 0)$, symmetric with respect to the point R about the origin, and let us draw the characteristic, which will intersect the t-axis at the point $Q(0, (at_0-x_0)/a)$.

On the basis of eqs. (23), we know that

$$\varphi(x_0 - at_0) = -\psi(-x_0+at_0) \ . \tag{33}$$

From this, it is clear that we may replace the forward wave $\varphi(x_0 - at_0)$ which travels along the segment RQ, with the backward wave $-\psi(-x_0+at_0)$, which is propagated along the characteristic PQ. The latter wave is reflected from the end of the string at time $(at_0 - x_0)/a$, and then approaches the point M. Let us note that the reflected wave changes not only its direction, but also the sign of its displacement, as is clear from eq. (33).

Thus, we see that region II corresponds to those points of the string that are reached by the following two waves: the backward wave and the forward wave that is reflected from the end.

Clearly, just the opposite will be observed in region III; specifically, region III corresponds to those points of the string reached by the forward wave and the backward wave that is reflected from the end.

Regions IV, V and VI correspond to those parts of the string reached by waves reflected either once or twice from both ends of the string. All the points in the remaining regions will be reached by waves reflected several times from both ends.

6. *The concept of generalized solutions*

Let us again consider the Cauchy problem for the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{8}$$

with the initial conditions

$$u|_{t=0} = f(x) \ , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x) \ . \tag{9}$$

As has been shown, the solution to this problem will be the function

$$u(x, t) = \frac{f(x - at) + f(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} F(z)\, dz \ . \tag{11}$$

This formula gives the usual (classical) solution to eq. (8) only under the assumption that $f(x)$ has continuous derivatives up to the second order inclusive, and that $F(x)$ has at least a first-order continuous derivative.

In specific physical problems, the functions $f(x)$ and $F(x)$ may not satisfy these conditions. Then, we may not assert that the Cauchy problem has a solution. In this case, we introduce the so-called "generalized solutions" to the Cauchy problem.

Suppose that with the initial conditions (9) the function $u(x,t)$ is the limit of a uniformly convergent sequence of solutions $u_n(x, t)$ to eq. (8) with the initial conditions

$$u_n|_{t=0} = f_n(x) \ , \qquad \left.\frac{\partial u_n}{\partial t}\right|_{t=0} = F_n(x) \ ,$$

where the sequence of functions $f_n(x)$, with continuous second derivatives, converges uniformly to $f(x)$, and the sequence of functions $F_n(x)$, with continuous first derivatives, converges uniformly to $F(x)$. In this case, we shall call the function $u(x,t)$ the *generalized solution to the Cauchy problem.*

It is easy to show the existence and uniqueness of the generalized solution to the Cauchy problem for eq. (8) in the case of arbitrary continuous functions $f(x)$ and $F(x)$. This generalized solution is also given by eq. (11).

The introduction of the generalized solutions of eq. (8) is natural in two respects. First, to insure the existence of the ordinary solution to the Cauchy problem, we would need to impose very stringent smoothness conditions on the given functions $f(x)$ and $F(x)$, whereas such smoothness conditions are not required for the existence of the generalized solutions. Second, in specific physical problems, the functions $f(x)$ and $F(x)$ are known only approximately. Therefore, the corresponding function $u(x, t)$, given by eq. (11), is also only an approximation to the exact solution of the problem that is posed.

Consequently, it is not at all important whether this approximation is the ordinary or the generalized solution to the Cauchy problem. What is important is that it differ only slightly from the true solution when the deviations of the functions $f(x)$ and $F(x)$ from the true initial values of $u(x, 0)$ and $\partial u(x, 0)/\partial t$ are uniformly small.

Problems

1. Show that the equations for the transverse vibrations of a string in a medium whose resistance is proportional to the first power of the velocity can be written in the form

$$\frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial x^2} + h^2 v \qquad (a = \sqrt{T_0/\rho}) \ ,$$

where $v(x, t)$ is a function of the displacement $u(x, t)$, defined by the equation

$$u = e^{-ht} v \qquad (h > 0) \ .$$

2. The ends $x = \pm l$ of a homogeneous string are supported by means of elastic forces directed parallel to the u-axis. Show that the boundary conditions for the vibrations of the string take the form

$$\frac{\partial u}{\partial x} - h_1 u = 0 \quad \text{for } x = -l,$$

$$\frac{\partial u}{\partial x} + h_2 u = 0 \quad \text{for } x = l \;,$$

where h_1 and h_2 are positive constants.

3. A semi-infinite string, fixed at $x = 0$, undergoes transverse vibrations. Find the formula for the vibration of the string, if the initial conditions are of the form

$$u|_{t=0} = f(x) , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x) \qquad (x > 0) .$$

Answer:

$$u(x, t) = \begin{cases} \dfrac{f(x+at) - f(at-x)}{2} + \dfrac{1}{2a}\displaystyle\int_{at-x}^{x+at} F(z)\,\mathrm{d}z & \text{for } x < at , \\[2ex] \dfrac{f(x+at) + f(x-at)}{2} + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at} F(z)\,\mathrm{d}z & \text{for } x > at . \end{cases}$$

4. Investigate the oscillations of an infinitely long string that is under the influence of an external force $F(x, t)$ with initial conditions

$$u|_{t=0} = \varphi(x) , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x) .$$

Answer:

$$u(x, t) = \frac{\varphi(x-at) + \varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(z)\,\mathrm{d}z + \frac{a}{2T_0}\iint_D F(z,\tau)\,\mathrm{d}z\,\mathrm{d}\tau ,$$

where the integration is performed over the region D shown in fig. 13. Method of solution: introduce new independent variables ξ and η by setting

$$\xi = x - at , \qquad \eta = x + at .$$

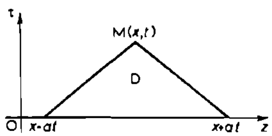

Fig. 13.

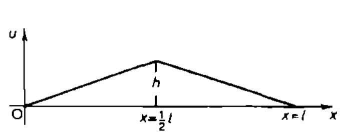

Fig. 14.

5. A segment of a string of length l that is fastened at the ends $x = 0$ and $x = l$ is displaced at the point $x = \frac{1}{2}l$ to an elevation h and then released (fig. 14). Determine the shape of the string at the time $t_1 = l/2a$ and $t_2 = l/a$ by the analytic method.

6. An infinitely long string whose equilibrium position is a straight line is

hit, at the initial time ($t = 0$), with a hammer of mass M. This hammer strikes the string at the point $x = 0$ with an initial velocity v_0.
Show that at any subsequent instant the disturbed string will have the form shown in fig. 15, where u_1 is the forward wave:

$$u_1 = \frac{Mav_0}{2T}\left\{1 - \exp\left[\frac{2T_0}{Ma^2}(x - at)\right]\right\} \quad \text{for } x - at < 0,$$

$$u_1 = 0 \quad \text{for } x - at > 0,$$

and $u_2(x, t)$ is the backward wave:

$$u_2 = \frac{Mav_0}{2T_0}\left\{1 - \exp\left[-\frac{2T_0}{Ma^2}(x + at)\right]\right\} \quad \text{for } x + at > 0,$$

$$u_2 = 0 \quad \text{for } x + at < 0.$$

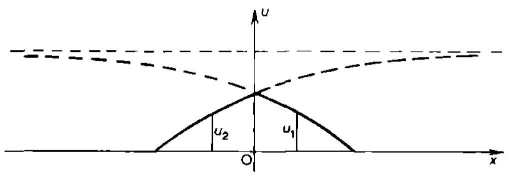

Fig. 15.

Method of solution: integrate the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2},$$

and use the conditions

$$M \left.\frac{\partial^2 u_1}{\partial t^2}\right|_{x=0} = M \left.\frac{\partial^2 u_2}{\partial t^2}\right|_{x=0} = -T_0 \frac{\partial u_2}{\partial x} + T_0 \left.\frac{\partial u_1}{\partial x}\right|_{x=0}.$$

Chapter IV

LONGITUDINAL VIBRATIONS OF A ROD

1. *The differential equation for longitudinal vibrations of a homogeneous rod of constant cross section. The initial and boundary conditions*

Let us examine a homogeneous rod of length l, that is, a cylindrical or otherwise-shaped body whose elongation or bending requires the application of a force. Thus, even a very thin rod differs from a string, which, as we know, bends freely.

In the present chapter, we shall deal with the application of the method of characteristics to the study of the longitudinal vibrations of a rod. We shall also confine ourselves to vibrations in which the cross sections pq remain plane and parallel to each other during their displacement along the axis of the rod (fig. 16). This condition is justified if the lateral dimensions of the rod are small in comparison with its length.

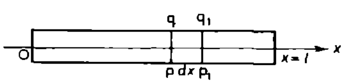

Fig. 16.

If the rod is somewhat stretched or compressed along its longitudinal axis and then released, it will start vibrating. Let us direct the x-axis along the axis of the rod, and let us assume that, in a state of rest, the ends of the rod are at the points $x = 0$ and $x = l$. Suppose that x is the abscissa of an arbitrary cross section of the rod when at rest. Let us denote by $u(x, t)$ the displacement of this section at the time t. Then, the displacement of the section whose abscissa is $x + dx$ will be equal to

$$u + \frac{\partial u}{\partial x}\, dx .$$

Thus, it is clear that the relative lengthening of the rod at the cross section whose abscissa is x is given by the derivative

$$\partial u(x, t)/\partial x .$$

Recalling that the rod undergoes only small oscillations, we can compute the tension T in this cross section. From Hooke's law, we have

$$T = ES\, \frac{\partial u}{\partial x} , \tag{1}$$

where E is the modulus of elasticity of the material of which the rod is composed, and S is the cross sectional area. Let us consider the element of the rod included between two cross sections whose abscissas, when the rod is at rest, are x and $x + \mathrm{d}x$. This element is acted on by the forces of tension T_x and $T_{x+\mathrm{d}x}$, which are directed along the x-axis. The value of the resultant of these forces is

$$T_{x+\mathrm{d}x} - T_x = ES \left.\frac{\partial u}{\partial x}\right|_{x+\mathrm{d}x} - ES \left.\frac{\partial u}{\partial x}\right|_x \approx ES \frac{\partial^2 u}{\partial x^2} \mathrm{d}x \tag{2}$$

and is also directed along the x-axis. On the other hand, the acceleration of the element is equal to $\partial^2 u/\partial t^2$, so that we obtain the equation

$$\rho S \, \mathrm{d}x \frac{\partial^2 u}{\partial t^2} = ES \frac{\partial^2 u}{\partial x^2} \mathrm{d}x , \tag{3}$$

where ρ is the density of the rod. Setting

$$a = \sqrt{E/\rho} \tag{4}$$

and dividing by $S\mathrm{d}x$, we obtain the differential equation of longitudinal vibrations of a homogeneous rod:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} . \tag{5}$$

The form of this equation shows that the longitudinal vibrations of a rod are of a wave nature; here, the velocity a of propagation of the longitudinal waves is determined by eq. (4).

If an external force $F(x, t)$, calculated per unit of volume, acts on the rod, we obtain instead of (3)

$$\rho S \, \mathrm{d}x \frac{\partial^2 u}{\partial t^2} = ES \frac{\partial^2 u}{\partial x^2} \mathrm{d}x + F(x, t) \, S \, \mathrm{d}x ,$$

so that

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{\rho} F(x, t) . \tag{6}$$

This is the equation of *forced* longitudinal vibrations of a rod.

As is usually the case in dynamics, a single equation of motion (6) is not sufficient for complete determination of the motion of the rod. We must know the initial conditions; that is, the displacements of the cross sections of the rod and their velocities $\partial u(x, t)/\partial t$ at the initial instant of time:

$$u|_{t=0} = f(x) , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x) , \tag{7}$$

where $f(x)$ and $F(x)$ are given functions defined over the interval $(0, l)$.

In addition, the boundary conditions at the ends of the rod must also be given. Thus, for ecample:

(1) The rod may be fixed at both ends. In this case,

$$u(0, t) = 0 , \qquad u(l, t) = 0 \tag{8}$$

for all instants of time t.

(2) One ends of the rod may be fixed and the other free; that is,

$$u(0,t) = 0 \ , \qquad \left.\frac{\partial u}{\partial x}\right|_{x=l} = 0 \tag{9}$$

for all instants of time t. At the free end ($x = l$), the tension $T = ES\,\partial u/\partial x$ will be equal to zero (no external forces) and, consequently,

$$\left.\frac{\partial u}{\partial x}\right|_{x=l} = 0 \ .$$

(3) Both ends of the rod may be free; that is,

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \ , \qquad \left.\frac{\partial u}{\partial x}\right|_{x=l} = 0 \tag{10}$$

for all instants of time t.

Thus, the problem of the longitudinal vibrations of a homogeneous rod of finite length reduces to finding the solution of eq. (6) satisfying the initial conditions (7) and one of the boundary conditions (8), (9), (10), and so on.

2. *The vibrations of a rod with one end fixed*

As an example, let us solve the following problem. An elastic cylindrical rod of length l in its unstretched (natural) state is fixed at the end $x = 0$ and is then stretched to a length l_1; the free end is then released so that the rod is set into longitudinal vibration. We are to determine the velocity of vibration of an arbitrary cross section of the disturbed rod. To solve this problem, we must find the solution to eq. (5) that satisfies the boundary conditions (9) and the initial conditions (7). Let us determine the functions $f(x)$ and $F(x)$ that appear in the initial conditions (7), remembering that at the initial instant the displacement of a cross section with abscissa x is proportional to this abscissa. We set

$$u|_{t=0} = f(x) = rx \qquad (0 < x < l) \ , \tag{11}$$

where r is a proportionality constant. This constant can be easily determined, since, at the initial instant, the displacement at the free end of the rod is equal to $l_1 - l$, that is,

$$l_1 - l = rl$$

or

$$r = \frac{l_1 - l}{l}$$

Furthermore, since the velocities of all the intermediate cross sections of the shaft are equal to zero at the initial instant, we have

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \qquad (0 < x < l) \tag{12}$$

Thus, the initial conditions are of the form shown in eqs. (11) and (12).

We know that the general solution to eq. (5) is of the form

$$u = \varphi(at - x) + \psi(at + x) \; . \tag{13}$$

We now determine the functions φ and ψ so that eq. (13) satisfies the boundary conditions (9) and the initial conditions (11) and (12). From the first of the boundary conditions (9), it follows that

$$u|_{x=0} = \varphi(at) + \psi(at) = 0$$

or

$$\psi(z) = -\varphi(z) \qquad (z = at) \; ,$$

so that eq. (13) takes the form

$$u = \varphi(at - x) - \varphi(at + x) \; . \tag{14}$$

Differentiating this equation with respect to x and then setting $x = l$, we obtain, because of the boundary condition (9), the following result:

$$0 = -\varphi'(at - l) - \varphi'(at + l)$$

or, denoting the argument $at + l$ of the function by z, we obtain the equation

$$\varphi'(z) = -\varphi'(z - 2l) \; , \tag{15}$$

Using this equation, it is easy to find the expression for the function $\varphi'(z)$ for all values of z.

In fact, on the basis of the initial conditions (11) and (12), we have

$$rx = \varphi(-x) - \varphi(x) \tag{16}$$

$$(0 < x < l) \; .$$

$$0 = \varphi'(-x) - \varphi'(x) \tag{17}$$

Differentiating eq. (16) with respect to x and solving the equation obtained and eq. (17) simultaneously, we obtain the following expression for the function $\varphi'(z)$:

$$\varphi'(z) = -\tfrac{1}{2}r \; , \tag{18}$$

which is valid for all values of z within the interval

$$-l < z < l \; . \tag{19}$$

Then, it follows from eq. (15) that

$$\varphi'(z) = \tfrac{1}{2}r \tag{20}$$

for all values of z satisfying the inequality

$$l < z < 3l \; . \tag{21}$$

Now, we should note that, on the basis of eq. (15), the function $\varphi'(z)$ has period $4l$. Then, it is clear from formulae (18) - (21) that the function $\varphi'(z)$ is defined for all values of z.

Let us use these results to obtain a picture of the propagation of the waves in the disturbed rod. We denote by v the velocity of the cross section of the rod whose abscissa is x. This velocity is found from eq. (14), on the basis of which

$$\frac{v}{a} = \varphi'(at - x) - \varphi'(at + x) \ . \tag{22}$$

Using this equation, it is easy to determine (for any arbitrary instant of time) which waves are approaching the cross section P whose abscissa is x.

In fact, since this abscissa lies within the interval $(0, l)$, both arguments of the functions on the right side of eq. (22) remain within the interval $(-l, l)$ provided $0 \leqslant t \leqslant (l-x)/a$. Therefore, on the basis of eqs. (18) and (22), it follows that

$$\frac{v}{a} = -\tfrac{1}{2}r + \tfrac{1}{2}r = 0 \ .$$

In other words, from the instant at which the oscillations begin until the time $t = (l-x)/a$, the cross section P remains at rest. It begins to vibrate at the time $(l-x)/a$, when the backward wave (which originated at the free end at the initial time) approaches it.

Let us determine the velocity of the cross section P. In the time interval from $t = (l-x)/a$ to $t = (l+x)/a$, the argument of the function $\varphi'(at - x)$ will vary in the interval $(-l, l)$ and the argument of the function $\varphi'(at + x)$ will vary in the interval $(l, 3l)$. Applying formulae (18) - (22), we see that during the period of time

$$t = \frac{l+x}{a} - \frac{l-x}{a} = \frac{2x}{a}$$

the cross section P will have a velocity determined by the equation

$$\frac{v}{a} = -\tfrac{1}{2}r - \tfrac{1}{2}r = -r \ .$$

Now let us see what will happen in the rod after the time $t = (l+x)/a$. At this instant, the forward wave, which arose from the backward wave that was reflected from the fixed end $x = 0$ at the time $t = l/a$, approaches the cross section P.

It is easy to show that from $t = (l+x)/a$ to $t = (3l-x)/a$, the cross section P will be in a state of rest. In fact, during this time, the arguments of both the functions in eq. (22) lie within the interval $(l, 3l)$. Therefore, it follows from eq. (20) that

$$\frac{v}{a} = \tfrac{1}{2}r - \tfrac{1}{2}r = 0 \ .$$

At the time $t = (3l-x)/a$, the backward wave, which arose from the forward wave after the latter was reflected from the free end $x = l$ at the time $t = 2l/a$, again reaches the cross section P. This wave will exert its influence on the cross section P until the time $t = (3l+x)/a$. When t changes from $(3l-x)/a$ to $(3l+x)/a$, the argument of the function $\varphi'(at - x)$ lies within the interval $(l, 3l)$, and the argument of the function $\varphi'(at + x)$ lies within the interval $(3l, 5l)$. Therefore,

$$\frac{v}{a} = \tfrac{1}{2}r + \tfrac{1}{2}r = r \ .$$

Finally, we consider the interval of time from $t=(3l+x)/a$ to $t=(5l-x)/a$. During this period of time, the cross section P is once more at rest. For, at the instant $t=(3l+x)/a$, the forward wave that arises from the backward wave (after the latter is reflected from the fixed end at the instant $t=3l/a$) approaches this cross section. The effect of this wave on the cross section P is the following: since both functions on the right side of eq. (22) have their arguments in the interval $(3l, 5l)$, when t lies in the interval $((3l+x)/a, (5l-x)/a)$,

$$\frac{v}{a} = -\tfrac{1}{2}r + \tfrac{1}{2}r = 0$$

Thus, during the period of time $t=2(l-x)/a$, the cross section P will be at rest.

After this time, the entire picture of the propagation of waves will be repeated, since (as was noted above) the function $\varphi'(z)$ has period $4l$.

3. *Axial impact on a rod*

Let us consider a cylindrical rod, one end $(x=0)$ of which is fixed and the other $(x=l)$ is free. At the initial time $t=0$, the free end is hit by a mass M, which is moving with velocity v in the direction of the axis of the rod. Let us study the resulting longitudinal vibrations of the rod.

We know that the equation of longitudinal vibrations of a homogeneous rod is of the form

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (a=\sqrt{E/\rho}) \ . \tag{23}$$

The boundary condition at the left end $(x=0)$ is obviously

$$u(0,t) = 0 \tag{24}$$

Furthermore, the equation of motion of the mass under the action of the reactive force of the rod (equal in magnitude but opposite in direction to the force at the end $x=l$ of the rod) is of the form

$$M\frac{\partial^2 u}{\partial t^2}\bigg|_{x=l} = -\,ES\frac{\partial u}{\partial x}\bigg|_{x=l} \tag{25}$$

This is the boundary condition at the end $x=l$. Eq. (25) can be written in the form

$$ml\frac{\partial^2 u}{\partial t^2}\bigg|_{x=l} = -\,a^2\frac{\partial u}{\partial x}\bigg|_{x=l}, \tag{26}$$

if we denote by $m=M/(\rho Sl)$ the ratio of the incident mass to the mass of the shaft. The solution $u(x,t)$ that we are seeking must also satisfy the initial conditions

$$u\big|_{t=0} = 0 \quad \text{for} \quad 0 \leqslant x \leqslant l\ , \tag{27}$$

$$\frac{\partial u}{\partial t}\bigg|_{t=0} = 0 \quad \text{for} \quad 0 \leqslant x < l, \qquad \frac{\partial u}{\partial t} = -v \quad \text{for} \quad t=0 \text{ and } x=l.$$

The second of these initial conditions means that, at the instant of collision of the moving mass and the rod, all the cross sections of the rod have velocity equal to zero except that at the very end of the shaft, which has a velocity equal to that of the incident mass.

We know that the general solution of eq. (23) is of the form

$$u = \varphi(at - x) + \psi(at + x) \ , \tag{28}$$

where φ and ψ are arbitrary functions. Let us define the functions φ and ψ so that the solution (28) satisfies the boundary conditions (24) and (25) and the initial conditions (27).

It follows from the boundary condition (24) that $\psi = -\varphi$; the solution then takes the form

$$u = \varphi(at - x) - \varphi(at + x) \ . \tag{29}$$

From the initial conditions (27), we have

$$\varphi(-z) - \varphi(z) = 0 \ , \qquad \varphi'(-z) - \varphi'(z) = 0 \qquad (0 \leqslant z \leqslant l) \ .$$

It then follows that $\varphi'(z) = 0$ when $-l < z < l$; that is, in this interval, $\varphi(z)$ is a constant, which we may take as equal to zero. Consequently, we have

$$\varphi(z) = 0 \qquad (-l < z < l) \ . \tag{30}$$

Let us now determine the function $\varphi(z)$ outside the interval $(-l, l)$. To do this, we use the boundary condition (26). Substituting (29) into (26), we obtain

$$ml[\varphi''(at - l) - \varphi''(at + l)] = \varphi'(at - l) + \varphi'(at + l)$$

or, setting $z = at + l$,

$$\varphi''(z) + \frac{1}{ml}\varphi'(z) = \varphi''(z - 2l) - \frac{1}{ml}\varphi'(z - 2l) \ . \tag{31}$$

This equation makes possible the extension of the function $\varphi(z)$ beyond the end points of the interval $(-l,l)$. From eq. (31), we first determine $\varphi'(z)$ outside the interval $(-l, l)$.

When $l < z < 3l$, the right side of eq. (31) is equal to zero and we have

$$\varphi''(z) + \frac{1}{ml}\varphi'(z) = 0 \ ,$$

and hence

$$\varphi'(z) = C\, e^{-z/ml} \ ,$$

where C is an arbitrary constant.

The initial condition (27) gives

$$a[\varphi'(-l + 0) - \varphi'(l + 0)] = -v$$

or, on the basis of (30),

$$\varphi'(l + 0) = \frac{v}{a} \ .$$

Consequently, $v/a = C \exp[-1/m]$, so that $C = (v/a)\exp[1/m]$ and

$$\varphi'(z) = \frac{v}{a}\exp\left[-\frac{z-l}{ml}\right] \quad (l < z < 3l) . \tag{32}$$

We note that $\varphi'(z)$ has a discontinuity at the point $z = l$.

When $3l < z < 5l$, eq. (31) takes the form

$$\varphi''(z) + \frac{1}{ml}\varphi'(z) = -\frac{2v}{aml}\exp\left[-\frac{z-3l}{ml}\right],$$

so that

$$\varphi'(z) = C\exp\left[-\frac{z}{ml}\right] - \frac{2v}{aml}(z-3l)\exp\left[-\frac{z-3l}{ml}\right], \tag{33}$$

where C is an arbitrary constant.

This arbitrary constant C can be found from the continuity of the velocity $\partial u/\partial t$ at the cross section $x = l$ when $t > 0$, in particular, when $t = 2l/a$. This gives

$$\varphi'(l-0) - \varphi'(3l-0) = \varphi'(l+0) - \varphi'(3l+0)$$

or, on the basis of (30), (32), and (33),

$$-\frac{v}{a}e^{-2/m} = \frac{v}{a} - C e^{-3/m} ,$$

so that

$$C = \frac{v}{a}(e^{1/m} + e^{3/m}) . \tag{34}$$

Substituting eq. (34) into eq. (33), we obtain

$$\varphi'(z) = \frac{v}{a}\exp\left[-\frac{z-l}{ml}\right] + \frac{v}{a}\left[1 - \frac{2}{ml}(z-3l)\right]\exp\left[-\frac{z-3l}{ml}\right] \quad (3l < z < 5l) . \tag{35}$$

In the same manner, we can find $\varphi'(z)$ in the intervals $(5l, 7l)$, $(7l, 9l)$, and so on.

The function $\varphi(z)$ is determined by integrating $\varphi'(z)$. The constant of integration is determined from the continuity of the function $x(u,t)$ at the point $x = l$. If we set t successively equal to $0, 2l/a, \ldots$, this condition gives the equations

$$0 = \varphi(-l+0) - \varphi(l+0) , \quad \varphi(l-0) - \varphi(3l-0) = \varphi(l+0) - \varphi(3l+0), \ \ldots$$

from which, on the basis of eq. (30), we obtain

$$\varphi(l+0) = \varphi(-l+0) = 0 , \quad \varphi(3l+0) = \varphi(3l-0) , \ \ldots .$$

Thus, we have

$$\varphi(z) = \frac{mlv}{a}\left(1 - \exp\left[-\frac{z-l}{ml}\right]\right) \quad (l < z < 3l) ,$$

$$\varphi(z) = -\frac{mlv}{a}\exp\left[-\frac{z-l}{ml}\right] + \frac{mlv}{a}\left[1 + \frac{2}{ml}(z-3l)\right]\exp\left[-\frac{z-3l}{ml}\right] \tag{36}$$

$$(3l < z < 5l) , \ \ldots .$$

From the above solutions (29), (30), and (36), it follows, on the basis of eq. (30), that for $0 < t < l/a$, we have $\varphi(at-x) = 0$ and, from eq. (29), we have

$$u(x,t) = -\varphi(at+x) ;$$

that is, only the backward wave (travelling from the end $x = l$ which received the impact) is propagated along the rod. At $t = l/a$, it reaches the fixed end, and when $l/a < t < 2l/a$, the reflected wave $\varphi(at-x)$ is added to it. Thus, the solution is of the form

$$u(x,t) = \varphi(at-x) - \varphi(at+x) .$$

At $t = 2l/a$, the wave $\varphi(at-x)$ will be reflected from the end $x = l$, so that the term $\varphi(at+x)$ in the solution (29) will have a different form in the interval $2l/a < t < 3l/a$. Thus, $u(x,t)$ takes different forms in the intervals

$$0 < t < \frac{l}{a}, \ \frac{l}{a} < t < \frac{2l}{a}, \ \ldots, \ n\frac{l}{a} < t < (n+1)\frac{l}{a} .$$

In the above exposition, we have been assuming that the rod behaves as if it were united with the striking body, so that condition (25) is satisfied for an arbitrary instant of time $t > 0$. However, if the body is separated from the rod, the solution that we have obtained is applicable only for that interval of time during which $\partial u(l,t)/\partial x < 0$. When $\partial u/\partial x$, in this solution, becomes positive at the point $x = l$, the collision is over.

For $0 < t < 2l/a$,

$$\frac{\partial u(l,t)}{\partial x} = -\frac{v}{a} e^{-at/ml} < 0$$

and the collision cannot have ended.

For $2l/a < t < 4l/a$,

$$\frac{\partial u(l,t)}{\partial x} = -\frac{v}{a} e^{-at/ml} \left[1 + 2 e^{-2/m} \left(1 - \frac{at-2l}{ml}\right)\right]$$

and $\partial u(l,t)/\partial x$ becomes positive when

$$\frac{2at}{ml} = \frac{4}{m} + 2 + e^{-\frac{1}{2}m} ;$$

this last equation can have a root in the interval $2l/a < t < 4l/a$ only if

$$2 + e^{-\frac{1}{2}m} < \frac{4}{m} .$$

The equation

$$2 + e^{-\frac{1}{2}m} = \frac{4}{m}$$

has root $m = 1.73\ldots$.

If m is less than $1.73\ldots$, the collision ends at the instant t in the interval $(2l/a, 4l/a)$ that is determined by the equation

$$t = \frac{l}{a}\left(2 + m + \tfrac{1}{2}m\ e^{-2/m}\right)$$

If m is greater than 1.73. . . , then we may continue the same procedure to determine whether the collision ends at some time t in the interval $(4l/a, 6l/a)$.

Problems

1. A cylindrical rod is sufficiently long that we may consider it as extending infinitely in one direction. At the end $x = 0$, a disturbing harmonic force $A \sin \omega t$ is applied. Show that the relative displacement of the cross section of the rod whose abscissa is x is expressed by the formula

$$u(x,t) = \begin{cases} 0 & \text{for } at \leqslant x, \\ A \sin \dfrac{\omega}{a}(at - x) & \text{for } at \geqslant x\,. \end{cases}$$

Method of solution: Use the method of characteristics to integrate the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

with boundary condition $u|_{x=0} = A \sin \omega t$ and with the initial conditions $u|_{t=0} = 0$ and $\partial u/\partial t|_{t=0} = 0$ for positive values of x.

2. A semi-infinite rod, fixed at the end $x \to \infty$ and free of forces at the end $x = 0$, undergoes longitudinal vibrations. Find the formula for the vibrations of the rod if the initial conditions are of the form

$$u|_{t=0} = f(x)\,, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \quad [f(x) \to 0 \text{ as } x \to \infty]\,.$$

Answer:

$$u(x,t) = \begin{cases} \dfrac{f(x+at) + f(at-x)}{2} & \text{for } 0 < x < at\,, \\ \dfrac{f(x-at) + f(x+at)}{2} & \text{for } x > at \end{cases}$$

Method of solution: The initial conditions must be extended (by treating $f(x)$ as an even function) to the negative half of the x-axis. The solution to the Cauchy problem for the infinitely long rod may then be used.

3. Derive the differential equation for longitudinal vibrations of a conical rod (fig. 17).
Answer:

$$\frac{\partial}{\partial x}\left[\left(1 - \frac{x}{h}\right)^2 \frac{\partial u}{\partial x}\right] = \frac{1}{a^2}\left(1 - \frac{x}{h}\right)^2 \frac{\partial^2 u}{\partial t^2} \quad (a^2 = E/\rho)\,, \tag{37}$$

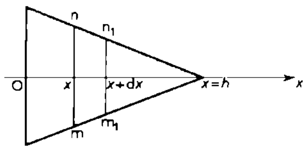

Fig. 17.

where h is the altitude of the entire cone of which the conical rod is a part.

4. One end ($x = 0$) of a rod shaped like a truncated cone is fixed and the other ($x = l$) is free. At the initial instant of time ($t = 0$), the free end is hit by a mass M moving with velocity v in the direction of the axis of the rod. Find the longitudinal vibrations of the conical rod.
Answer:

$$u(x, t) = \frac{\varphi(at - x) - \varphi(at + x)}{h - x} .$$

The function $\varphi(z)$ is defined as follows

$$\varphi(z) = 0 \qquad (-l < z < l) ,$$

$$\varphi(z) = \frac{v(h - l)}{a} \frac{\exp[k_1(z - l)] - \exp[k_2(z - l)]}{k_1 - k_2} \qquad (l < z < 3l)$$

and so on, where k_1 and k_2 are the roots of the equation

$$k^2 + \frac{k}{m} + \frac{1}{m(h - l)} = 0 , \qquad m = \frac{M}{\rho S l} .$$

The equation for extension of the function $\varphi(z)$ outside the interval $(-l, l)$ is of the form

$$m\varphi''(z) + \varphi'(z) + \frac{\varphi(z)}{h - l} = m\varphi''(z - 2l) - \varphi'(z - 2l) + \frac{\varphi(z - 2l)}{h - l} .$$

Method of solution: The problem is reduced to finding the solution of eq. (37) with boundary conditions

$$u|_{x=0} = 0 , \qquad m \left.\frac{\partial^2 u}{\partial t^2}\right|_{x=l} = - a^2 \left.\frac{\partial u}{\partial t}\right|_{x=l}$$

and initial conditions

$$u|_{t=0} = 0 \qquad \text{for } 0 \leqslant x \leqslant l ,$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \quad \text{for } 0 \leqslant x < l , \qquad \frac{\partial u}{\partial t} = -v \quad \text{for } t = 0 \text{ and } x = l .$$

Chapter V

APPLICATION OF THE METHOD OF CHARACTERISTICS TO THE STUDY OF ELECTRICAL VIBRATIONS IN CONDUCTORS

1. *Differential equations for free electrical oscillations*

When an electric current is passed through a conductor, an electromagnetic field is formed which causes changes in the magnitudes of both the current and the potential. We now consider the oscillation which takes place in the conductor as a result of these changes.

Let the x-axis lie along the axis of the conductor with the coordinate origin at one of its ends. We denote the length of the conductor by l. The current i and the potential v at any point of the conductor will be functions of the abscissa x and the time t. The quantities i and v are related by certain first-order partial differential equations. In deriving these equations, we shall assume that the capacitance, resistance, self-inductance, and leakage are continuously and uniformly distributed along the conductor and that the constants C, R, L, and G, characterizing them, are measured per unit of length of the conductor.

Let us examine that portion of the conductor contained between two sections $x = x_1$ and $x = x_2$. Applying Ohm's law to this portion of the conductor, we obtain

$$v(x_1, t) - v(x_2, t) = R \int_{x_1}^{x_2} i(x, t)\, \mathrm{d}x + L \int_{x_1}^{x_2} \frac{\partial i(x, t)}{\partial t}\, \mathrm{d}x . \tag{1}$$

Since, on the other hand,

$$v(x_1, t) - v(x_2, t) = - \int_{x_1}^{x_2} \frac{\partial v(x, t)}{\partial x}\, \mathrm{d}x ,$$

we have the equation

$$\int_{x_1}^{x_2} \left(\frac{\partial v}{\partial x} + L \frac{\partial i}{\partial t} + Ri\right) \mathrm{d}x = 0 ,$$

from which (since x_1 and x_2 are arbitrary), we obtain

$$\frac{\partial v}{\partial x} + L \frac{\partial i}{\partial t} + Ri = 0 . \tag{2}$$

The amount of electricity flowing through the section (x_1, x_2) of the conductor in a unit of time

$$i(x_1, t) - i(x_2, t) = -\int_{x_1}^{x_2} \frac{\partial i}{\partial x}\,dx ,$$

is equal to the sum of the electricity necessary for charging this section of the conductor and the electricity that is being lost as a result of imperfect insulation:

$$C\int_{x_1}^{x_2} \frac{\partial v}{\partial t}\,dx + G\int_{x_1}^{x_2} v\,dx .$$

Thus,

$$\int_{x_1}^{x_2} \left(\frac{\partial i}{\partial x} + C\frac{\partial v}{\partial t} + Gv\right) dx = 0 ,$$

so that

$$\frac{\partial i}{\partial x} + C\frac{\partial v}{\partial t} + Gv = 0 . \tag{3}$$

2. *The telegraph equation*

If we differentiate eq. (2) (derived in the preceding section) with respect to x and eq. (3) with respect to t, and then eliminate $\partial^2 i/\partial x\,\partial t$ from the resulting equations, we obtain the following second-order differential equation for v:

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} + (RC + GL)\frac{\partial v}{\partial t} + GRv . \tag{4}$$

In an analogous way, we derive the differential equation

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (RC + GL)\frac{\partial i}{\partial t} + GRi \tag{5}$$

for the current.

From these equations, we see that the potential v and the current i satisfy the same differential equation

$$\frac{\partial^2 w}{\partial x^2} = a_0\frac{\partial^2 w}{\partial t^2} + 2b_0\frac{\partial w}{\partial t} + c_0 w , \tag{6}$$

where, for brevity, we use the notation

$$a_0 = LC , \quad 2b_0 = RC + GL , \quad c_0 = GR . \tag{7}$$

This equation is called the *telegraph equation*.

If we introduce a new function $u(x, t)$ by setting

$$w = \exp\left[-\frac{b_0}{a_0}t\right] u , \tag{8}$$

eq. (6) takes the simpler form

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + b^2 u \,, \tag{9}$$

where

$$a = \frac{1}{\sqrt{a_0}} \,, \qquad b = \frac{\sqrt{b_0^2 - a_0 c_0}}{a_0} \,. \tag{10}$$

3. *Integration of the telegraph equation by the Riemann method*

We use the Riemann method to find the solution to eq. (9) satisfying the initial conditions

$$u|_{t=0} = f(x) \,, \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = F(x) \,. \tag{11}$$

First of all, we convert this equation into canonical form, introducing the new independent variables ξ and η defined by

$$\xi = \frac{b}{a}(x + at) \,, \qquad \eta = \frac{b}{a}(x - at) \,. \tag{12}$$

Then, eq. (9) acquires the form

$$L(u) = \frac{\partial^2 u}{\partial \xi \, \partial \eta} + \tfrac{1}{4} u = 0 \,. \tag{13}$$

The straight line $t = 0$ will, in the new variables, be the line (fig. 18):

$$\xi = \eta \,. \tag{14}$$

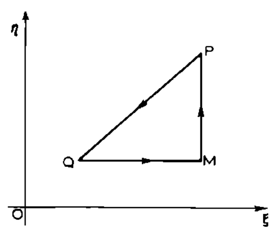

Fig. 18.

Also, from eqs. (12), it follows that

$$x = \frac{a}{b}\frac{\xi + \eta}{2} \,, \qquad t = \frac{1}{b}\frac{\xi - \eta}{2} \,,$$

and hence that

$$\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} = \frac{1}{b}\frac{\partial u}{\partial t}$$

or, on the basis of the initial conditions (11), we have

$$\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\Big|_{\eta=\xi} = \frac{1}{b}\frac{\partial u}{\partial t}\Big|_{t=0} = \frac{1}{b}F(x) = \frac{1}{b}F\left(\frac{a}{b}\xi\right), \tag{15}$$

and also

$$u|_{\eta=\xi} = f\left(\frac{a}{b}\xi\right). \tag{16}$$

If we set $a = 0$, $b = 0$, and $c = 0$ in Riemann's formula (eq. (12) of Chapter II) and if we take into consideration eq. (14) of this chapter, we obtain

$$u(\xi_0,\eta_0) = \frac{(uv)_P + (uv)_Q}{2} + \tfrac{1}{2}\int\limits_{QP} v\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right) d\xi - \tfrac{1}{2}\int\limits_{QP} u\left(\frac{\partial v}{\partial \xi} - \frac{\partial v}{\partial \eta}\right) d\xi . \tag{17}$$

Let us now find the Riemann function $v(\xi,\eta;\xi_0,\eta_0)$. It must satisfy the conjugate equation

$$\frac{\partial^2 v}{\partial \xi\, \partial \eta} + \tfrac{1}{4}v = 0 \tag{18}$$

and it must be equal to unity on the characteristics MP and MQ.

Let us seek a solution to eq. (18) in the form

$$v = G(\sqrt{(\xi-\xi_0)(\eta-\eta_0)}) .$$

Substituting this equation into eq. (18) and denoting by λ the root $\sqrt{(\xi-\xi_0)(\eta-\eta_0)}$, we see that the function v satisfies the ordinary differential equation

$$G''(\lambda) + \frac{1}{\lambda}G'(\lambda) + G(\lambda) = 0 . \tag{19}$$

The so-called *Bessel function of order zero* is a particular solution of this equation:

$$J_0(\lambda) = 1 - \frac{\lambda^2}{2^2} - \frac{\lambda^4}{(2\cdot 4)^2} - \frac{\lambda^6}{(2\cdot 4\cdot 6)^2} + \dots . \tag{20}$$

It is clear from this expansion that if we take

$$v = J_0(\lambda) ,$$

we shall obtain a solution to eq. (18) that is equal to unity on the characteristics $\xi = \xi_0$ and $\eta = \eta_0$ (since on these characteristics $\lambda = 0$).

Thus, we have found the Riemann function; it is

$$v(\xi,\eta;\xi_0,\eta_0) = J\ (\sqrt{(\xi-\xi_0)(\eta-\eta_0)}) . \tag{21}$$

From this, we easily obtain

$$\frac{\partial v}{\partial \xi}\Big|_{\eta=\xi} = \frac{dJ_0}{d\lambda}\frac{\partial \lambda}{\partial \xi}\Big|_{\eta=\xi} = \frac{1}{2}\frac{\xi-\eta_0}{\sqrt{(\xi-\xi_0)(\xi-\eta_0)}} J_0'(\lambda)|_{\eta=\xi} ,$$

$$\frac{\partial v}{\partial \eta}\Big|_{\eta=\xi} = \frac{dJ_0}{d\lambda}\frac{\partial \lambda}{\partial \eta}\Big|_{\eta=\xi} = \frac{1}{2}\frac{\xi-\xi_0}{\sqrt{(\xi-\xi_0)(\xi-\eta_0)}} J_0'(\lambda)|_{\eta=\xi} ,$$

and, consequently,

$$\frac{\partial v}{\partial \xi} - \frac{\partial v}{\partial \eta}\Big|_{\eta=\xi} = \frac{\xi_0 - \eta_0}{2\sqrt{(\xi - \xi_0)(\xi - \eta_0)}} J_0'(\lambda)\big|_{\eta=\xi} . \tag{22}$$

Now substituting (15), (16), and (22) into eq. (17) and remembering that:

$$u(\mathrm{P}) = f\left(\frac{a}{b}\,\xi_0\right) , \qquad u(\mathrm{Q}) = f\left(\frac{a}{b}\,\eta_0\right) ,$$

we obtain

$$u(\xi_0, \eta_0) = \frac{f\left(\frac{a}{b}\,\xi_0\right) + f\left(\frac{a}{b}\,\eta_0\right)}{2} + \frac{1}{2b}\int_{\eta_0}^{\xi_0} J_0(\sqrt{(\xi - \xi_0)(\xi - \eta_0)})\, F\left(\frac{a}{b}\,\xi\right) \mathrm{d}\xi$$

$$- \frac{\xi_0 - \eta_0}{4}\int_{\eta_0}^{\xi} f\left(\frac{a}{b}\,\xi\right) \frac{J_0'(\sqrt{(\xi - \xi_0)(\xi - \eta_0)})}{\sqrt{(\xi - \xi_0)(\xi - \eta_0)}}\, \mathrm{d}\xi .$$

Returning now to the old variables x and t (dropping the subscript zero) and introducing the new integration variable $z = a\xi/b$, we obtain

$$u(x, t) = \frac{f(x - at) + f(x + at)}{2} + \tfrac{1}{2}\int_{x-at}^{x+at} \Phi(x, t, z)\, \mathrm{d}z , \tag{23}$$

where

$$\Phi(x, t, z) = \frac{1}{a} F(z)\, J_0\left(\frac{b}{a}\sqrt{(z - x)^2 - a^2t^2}\right) + bt\, f(z)\, \frac{J_0'\left(\frac{b}{a}\sqrt{(z - x)^2 - a^2t^2}\right)}{\sqrt{(z - x)^2 - a^2t^2}} . \tag{24}$$

4. *Electrical oscillations in an infinite conductor*

Suppose that we are dealing with a conductor that is so long that we may consider it as extending infinitely far in both directions. In this case, both of the functions $f(x)$ and $F(x)$ that appear in the initial conditions must be known for the entire interval $(-\infty, \infty)$. Then, using eq. (23), we may compute the value of the function $u(x, t)$ at every point of the conductor at any instant of time. Knowing $u(x, t)$, we may also compute the value of the potential $v(x, t)$, since

$$v(x, t) = \mathrm{e}^{-\mu t}\, u(x, t) \qquad (\mu = b_0/a_0) . \tag{25}$$

Let us investigate more closely the physical meaning of eq. (25). With this in mind, we set $a_0 = b_0 = 1$ for simplicity, and we assume that at the initial instant the electrical disturbances are being propagated only through some segment $(0, \alpha)$ of the conductor. Consequently, the functions $f(x)$ and $F(x)$ will be equal to zero outside this interval.

Let us take a point on the conductor, with abscissa $x = \zeta > \alpha$, and let us observe it for a certain period of time (fig. 19). At the time τ, this point will occupy the position $\mathrm{M}(\zeta, \tau)$. Let us draw, through the point M, the characteristics

$$x - t = \zeta - \tau\ , \qquad x + t = \zeta + \tau\ ,$$

which intersect the x-axis at points with abscissae $\zeta_1 = \zeta - \tau$ and $\zeta_2 = \zeta + \tau$ Now consider the interval of time from $t = 0$ to $t = \tau$, where

$$\tau < \zeta - \alpha\ ; \tag{26}$$

during this time, the point $(\zeta, 0)$ is displaced to the position $M(\zeta, \tau)$, and the characteristic $x - t = \zeta - \tau$ intersects the x-axis at the point ζ_1, which lies to the right of the point α (that is, outside the segment of the initial vibrations; this is obvious from the inequality (26)). It is easy to see that the electrical oscillations do not reach the point under observation during the time interval $t = 0$ to $t = \tau$. Indeed, the interval of integration $(\zeta - \tau, \zeta + \tau)$ in eq. (23) does not contain the interval $(0, \alpha)$, as can be seen directly in fig. 19. Remembering that the functions $f(x)$ and $F(x)$ are equal to zero outside the interval $(0, \alpha)$, we can see that not only the functions $f(\zeta - \tau)$ and $f(\zeta + \tau)$, but also the function $\Phi(\zeta, \tau, z)$ is equal to zero in the interval $(\zeta - \tau, \zeta + \tau)$. From this, it is clear, on the basis of eq. (23), that

$$u = 0 \qquad (0 < \tau < \zeta - \alpha)\ .$$

Consequently, in the above interval of time $(0, \zeta - \alpha)$,

$$v = 0\ ,$$

which confirms the proposition stated above.

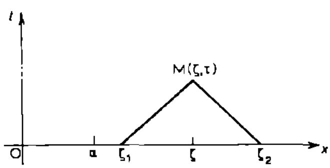

Fig. 19.

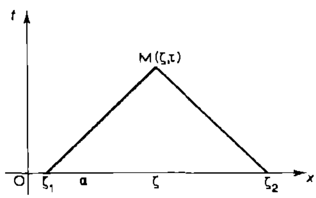

Fig. 20.

Let us now consider the interval of time from $\tau = \zeta - \alpha$ to $\tau = \zeta$. In this case, we have

$$0 < \zeta - \tau < \alpha$$

and consequently, the characteristic $x - t = \zeta - \tau$ intersects the x-axis at the point ζ_1, which lies between 0 and α (fig. 20). From the figure, we can see that the interval of integration $(\zeta - \tau, \zeta + \tau)$ can be partitioned into two intervals:

$$(\zeta - \tau, \alpha) \qquad \text{and} \qquad (\alpha, \zeta + \tau)\ .$$

In the second of these intervals, the function $\Phi(\zeta, \tau, z)$ is equal to zero and, consequently, formula (23) gives the following equation:

$$u = \frac{f(\zeta - \tau)}{2} + \frac{1}{2}\int_{\zeta - \tau}^{\alpha} \Phi(\zeta, \tau, z)\, dz \qquad (\zeta - \alpha < \tau < \zeta)\ , \tag{27}$$

which shows that in the above interval of time the electrical oscillations reach the point under observation. The potential v can then be computed, at that point, from the formula

$$v = e^{-\mu\tau} u ,$$

where u is determined by eq. (27).

Now let us see what happens at the observed point for instants of time τ after ζ.

Since, in this case

$$\zeta - \tau < 0 ,$$

the characteristic $x - t = \zeta - \tau$ intersects the x-axis at a point ζ_1, which lies to the left of the point O (that is, outside the segment of the initial oscillations). However, it is easy to see that the potential at the point under observation will no longer be equal to zero (as it was for time $\tau < \zeta - \alpha$), since it is clear from fig. 21 that the interval $(0,\alpha)$ is entirely contained in the interval $(\zeta - \tau, \zeta + \tau)$. Thus, eq. (23) gives

$$u = \tfrac{1}{2} \int_0^{\alpha} \Phi(\zeta, \tau, z)\, dz \qquad (\zeta < \tau) ,$$

and hence

$$v = \frac{e^{-\mu\tau}}{2} \int_0^{\alpha} \Phi(\zeta, \tau, z)\, dz . \tag{28}$$

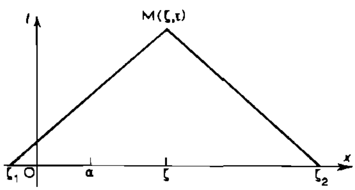

Fig. 21.

This last formula indicates that the electrical oscillations which passed through the point ζ during the time τ (where $\zeta - \alpha < \tau < \zeta$) have left behind a *residual disturbance*, expressed by eq. (28). In fact, the presence of such a residual effect in the conductor was confirmed by Fizeau's experiments.

We note, in conclusion, that the integral in eq. (28) remains finite as τ approaches $+\infty$. Therefore, it follows that the potential v in an infinite conductor decreases to zero with the passage of time.

5. *Oscillations in a line that is free of distortion*

This name was given by Heaviside to those lines for which the constants G, C, L, and R are related by the equation

$$\frac{G}{C} = \frac{R}{L} \,. \tag{29}$$

For lines of such a type, the telegraph equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \, b^2 u$$

takes the form of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (a = 1/\sqrt{LC}) \,, \tag{30}$$

since, in this case, $b = 0$. In fact, it follows from eq. (7) that

$$b = \frac{\sqrt{b_0{}^2 - a_0 c_0}}{a_0} = \frac{RC - GL}{2LC} \,.$$

Thus, it is clear that when eq. (29) is satisfied, the coefficient b will be equal to zero.

Recalling the general solution to eq. (30), we find, on the basis of the relationship

$$v = e^{-(R/L)t}\, u \,,$$

that the value of the potential in the line that we are considering is determined by the formula

$$v = e^{-(R/L)t}\, [\varphi(x - at) + \psi(x + at)] \,, \tag{31}$$

where φ and ψ are arbitrary functions.

To find the current, we take the equation

$$-\frac{\partial i}{\partial x} = Gv + C\frac{\partial v}{\partial t}$$

and substitute, on the right side, the expressions for v and $\partial v/\partial t$ as given by eq. (31); we then obtain

$$\frac{\partial i}{\partial x} = \sqrt{\frac{C}{L}}\, e^{-(R/L)t}\, [\varphi'(x - at) - \psi'(x + at)]$$

Integrating this expression with respect to x, we obtain

$$i = \sqrt{\frac{C}{L}}\, e^{-(R/L)t}\, [\varphi(x - at) - \psi(x + at) + \varkappa(t)] \,, \tag{32}$$

where $\varkappa(t)$ is an arbitrary function.

Substituting (31) and (32) into the equation

$$\frac{\partial v}{\partial x} + L\frac{\partial i}{\partial t} + Ri = 0 \,,$$

we obtain

$$\varkappa'(t) = 0 \,,$$

and hence,

$$\varkappa(t) = K = \text{constant} \,.$$

With no loss of generality, we may consider the constant K as being equal to zero. To show this, let us assume that $K \neq 0$; then, if we replace the functions $\varphi(x - at)$ and $\psi(x+at)$ in (31) and (32) by the functions $\varphi(x - at) - \frac{1}{2}K$ and $\psi(x+at) + \frac{1}{2}K$, we see that the constant K vanishes in these formulae.

Thus,

$$i = \sqrt{\frac{C}{L}}\, e^{-(R/L)t}\, [\varphi(x - at) - \psi(x+at)] \,. \tag{33}$$

Eqs. (31) and (33) show that the process of propagation of electrical disturbances is of a wave nature. The velocity of propagation of these waves is computed from the formula

$$a = 1/\sqrt{LC} \,. \tag{34}$$

The factor $\exp[-(R/L)t]$ on the right sides of formulae (31) and (33) shows that the oscillation that occurs in the conductor when an electric current is passed through it is damped out with the passage of time.

As regards the functions φ and ψ, on which the shape of the wave depends, they are determined from the initial conditions

$$v|_{t=0} = f(x) \,, \qquad i|_{t=0} = \sqrt{\frac{C}{L}}\, F(x) \,, \tag{35}$$

where $f(x)$ and $F(x)$ are given functions. Specifically, if we set $t = 0$ in formulae (31) and (33), we find, on the basis of the conditions (35), that

$$\varphi(x) + \psi(x) = f(x) \,, \qquad \varphi(x) - \psi(x) = F(x) \,,$$

and hence

$$\varphi(x) = \frac{f(x) + F(x)}{2} \,, \qquad \psi(x) = \frac{f(x) - F(x)}{2} \,. \tag{36}$$

If we are dealing with a conductor that is so long that we may consider it as extending infinitely far in both directions, the functions $f(x)$ and $F(x)$ must be known over the entire interval $(-\infty, \infty)$. Then, we may determine the current and the potential at any point of the circuit at any instant of time from eqs. (31), (33), and (36).

6. *Boundary conditions for a conductor of finite length*

If a conductor is of finite length l, we do not encounter the same situation as prevailed with the vibration of a finite string. Specifically, the functions $f(x)$ and $F(x)$ that appear in the expression for the initial conditions are known only in the interval $(0, l)$, whereas the application of eqs. (31) and (33) demands the knowledge of these functions for any arbitrary value of their arguments. It is, therefore, necessary to find a rule for extending the functions $f(x)$ and $F(x)$ beyond the end points of the interval $(0,l)$. Methods for such continuation can be found from the boundary conditions that must be satisfied at each end of the conductor.

We give some examples of the more frequently encountered boundary conditions, for which we use the following formyla, known from electrical theory:

$$v = Ri + L\frac{di}{dt} + \frac{1}{C}\int L\,dt\ ,$$

where R is the resistance, L is the self-inductance of a coil, and C is the capacitance of a condenser in the circuit.

(1) At one end of the line, a battery of constant emf E is included; the other end of the line is grounded. The boundary conditions are

$$v|_{x=0} = E\ , \qquad v|_{x=l} = 0\ .$$

(2) One end of the line is subjected to a sinusoidal potential with frequency ω; the other end of the line is insulated. The boundary conditions are

$$v|_{x=0} = E\sin\omega t\ , \qquad i|_{x=l} = 0\ .$$

(3) Pickups with ohmic resistances R_0 and R_l and self-inductances L_0 and L_l are put at each end of the line. The boundary conditions are

$$v|_{x=0} = E - R_0 i_0 - L_0\frac{di_0}{dt}\ , \qquad v|_{x=l} = R_l i_l + L_l\frac{di_l}{dt}\ ,$$

where E is the emf of the battery and i_0 and i_l are the values of the current at the two ends of the conductor.

(4) Separate capacitors with capacitances C_0 and C_l are inserted at each end of the conductor. The boundary conditions are

$$v|_{x=0} = E - \frac{1}{C_0}\int i_0\,dt\ , \qquad i|_{x=l} = C_l\frac{dv_l}{dt}\ ,$$

where v_l is the potential at the end of the conductor.

Chapter VI

THE WAVE EQUATION

1. *The differential equation for transverse vibrations of a membrane*

A stretched film that can be freely bent is called a membrane. Suppose that a membrane (in its equilibrium position) is situated in the xy-plane and that it occupies some region D bounded by a closed curve L. Let us further suppose that the membrane is subject to a uniform tension T that is applied at its edges. This means that if we draw a line on the membrane, in any arbitrary direction, the force between the two parts of the membrane that are separated by a given element of the line will be proportional to the length of the element and directed perpendicularly to it. The magnitude of the force acing on the element ds of the line will be equal to $T\,ds$.

Let us consider only transverse oscillations, in which each point of the membrane moves in a direction perpendicular to the xy-plane and parallel to the u-axis. Then, the displacement u of a point (x, y) of the membrane will be a function of x, y, and t.

Let us now derive the equation for transverse oscillations of the membrane. To do this, we take an arbitrary section (σ) of the membrane that, in a state of rest, is bounded by a curve l. When the membrane is displaced from the equilibrium position, this section moves into a new position represented by the section (S) of the surface of the membrane, bounded by the space curve l'. Here,

$$\sigma = S \cos \gamma \ ,$$

where γ is the angle between the u-axis and the normal to (S) (fig. 22), and σ and S are the areas of the sections (σ) and (S), respectively.

If we confine ourselves to the study of *small* amplitude vibrations (so that we may neglect the squares of the first derivatives $\partial u/\partial x$ and $\partial u/\partial y$), it follows from the formula

$$\cos \gamma = \frac{1}{\sqrt{1 + (\partial u/\partial x)^2 + (\partial u/\partial y)^2}} \approx 1$$

that $S \approx \sigma$ at any arbitrary instant of time t; that is, we may neglect the change in the area (on displacement) of an arbitrarily chosen section of the membrane. We may then assume that, when the vibrations of the membrane are small, the section (S) will be subject to the action of the original tension T.

Let us now compute the projection onto the u-axis of the resultant of the forces of tension applied to the section (S). We denote by ds' an element of the curve l'. The vector representing the tension on the element ds' lies

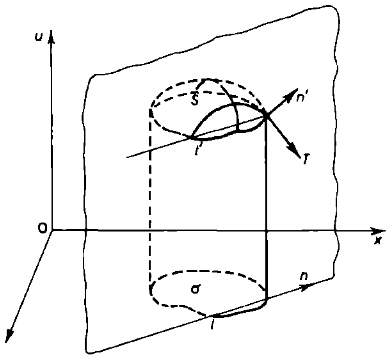

Fig. 22.

(because there is no resistance to bending) in the plane tangential to the surface of the membrane and is normal to the element ds' itself. The cosine of the angle between this vector and the u-axis is obviously equal to $\partial u/\partial n'$, where n' is the normal (directed outward) to the curve l'. It then follows that the projection onto the u-axis of the tension T calculated for an element ds' of the curve l' will be equal to

$$T \frac{\partial u}{\partial n'} ds' .$$

If we integrate this product over the entire curve l', we get the following expression for uniformly acting tension along this curve:

$$T \int_{l'} \frac{\partial u}{\partial n'} ds' .$$

Since ds is approximately equal to ds', when the vibrations of the membrane are of small amplitude, we may replace the path of integration l' with l. Then, using Green's formula, we obtain

$$T \int_{l} \frac{\partial u}{\partial n} ds = T \iint_{\sigma} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dx\, dy . \tag{1}$$

Suppose, also, that an external force is acting on the membrane in a direction parallel to the u-axis and that its value per unit of area is $p(x,y,t)$. Then, the external force acting on the section (σ) of the membrane will be equal to

$$\iint_{\sigma} p(x, y, t)\, dx\, dy . \tag{2}$$

On the basis of the principle of kinetic equilibrium, these two forces will, at any instant, be counterbalanced by inertial forces acting on the section (S) of the membrane. The sum of the inertial forces will be equal to

$$- \iint_\sigma \rho(x, y) \frac{\partial^2 u}{\partial t^2} \, dx \, dy ,$$

where $\rho(x, y)$ denotes the surface density of the membrane. Thus, we obtain the equation

$$\iint_\sigma \left\{\rho(x, y) \frac{\partial^2 u}{\partial t^2} - T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - p(x, y, t)\right\} dx \, dy = 0 ,$$

from which, since the section σ was arbitrary, it follows that

$$\rho(x, y) \frac{\partial^2 u}{\partial t^2} = T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + p(x, y, t) . \tag{3}$$

This is the differential equation for the transverse vibrations of a membrane.

In the case of a homogeneous membrane, ρ = constant and the equation for vibrations of small amplitude can be written in the form

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g(x, y, t) , \tag{4}$$

where

$$a = \sqrt{T/\rho} , \qquad g(x, y, t) = \frac{p(x, y, t)}{\rho} . \tag{5}$$

If there is no external force, that is, if $p(x, y, t) = 0$, we obtain, from eq. (4), the equation for *free vibrations* of a homogeneous membrane:

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) . \tag{6}$$

Eq. (6) is called the wave equation for a plane.

As we know, eq. (6) alone is not enough for a complete determination of the motion of the membrane. We still need to know the position and the velocity of all points of the membrane at some initial instant $t = 0$:

$$u|_{t=0} = f(x, y) , \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = F(x, y) , \tag{7}$$

and we also need to have the boundary conditions. For example, when the membrane is fastened at the curve L,

$$u|_L = 0 . \tag{8}$$

2. *The hydrodynamic equations and the propagation of sound waves*

In hydrodynamics, a liquid is regarded as a continuous medium. This means that any small element of volume of the liquid is treated as still being large enough to contain a very large number of molecules.

A mathematical description of the motion of a liquid is obtained by using the distribution functions for the velocity $v = v(x, y, z, t)$, the pressure

$p(x,y,z,t)$, and the density $\rho(x,y,z,t)$ of the liquid. We note that $v(x,y,z,t)$ is the velocity of the liquid at a given point (x,y,z) of space at the time t; that is, it refers to definite points in space rather than to definite particles of the liquid that are being displaced throughout the space with the passage of time. The same applies to the values of p and ρ.

Let us begin the derivation of the basic hydrodynamic equations by deducing the equation expressing the law of conservation of matter in hydrodynamics.

Let us examine some volume of the liquid V, bounded by a surface S. If there are no sources or sinks within the volume V, the change in the mass of the liquid enclosed within V in a unit of time will be equal to the flow of the liquid through the surface S:

$$\frac{\partial}{\partial t}\iiint_V \rho\, dV = -\iint_S \rho\, v_n\, dS ,$$

where $\rho(x,y,z,t)$ is the density of the liquid at the time t at the point (x,y,z) and v_n is the projection of $v(r,t)$ onto the outwardly directed normal n to the surface S.

According to Gauss' divergence theorem,

$$\iint_S \rho\, v_n\, dS = \iiint_V \operatorname{div}(\rho v)\, dV ,$$

where

$$\operatorname{div}(\rho v) = \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} .$$

Consequently,

$$\iiint_V \frac{\partial \rho}{\partial t}\, dV = -\iiint_V \operatorname{div}(\rho v)\, dV$$

or

$$\iiint_V \left\{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v)\right\} dV = 0 .$$

Since this last equation is valid for an arbitrary volume within the liquid, it follows that

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0 ; \tag{9}$$

this equation is called the *continuity equation*.

Let us now derive the equations of motion of an ideal liquid.

By an ideal liquid, we mean a deformable continuous liquid for which the internal forces (whether the medium is in equilibrium or in motion) can be reduced to a normal pressure. Thus, if we choose an arbitrary element of volume of the liquid, bounded by a surface S, the effect of the remaining portion of the liquid on it can be reduced to a force that is directed inwardly along the normal at every point on the surface S. We denote the value of this force per unit area (pressure) by $p(x,y,z,t)$.

Thus, the resultant of the pressure forces that are applied to the surface S will be equal to

$$-\iint_S p\boldsymbol{n}\,\mathrm{d}S\ ,$$

where $\boldsymbol{n}$ is the outwardly directed unit normal vector to the surface.

From Gauss's theorem, we have

$$-\iint_S p\boldsymbol{n}\,\mathrm{d}S = -\iiint_V \operatorname{grad} p\ \mathrm{d}V\ ,$$

where

$$\operatorname{grad} p = i\,\frac{\partial p}{\partial x} + j\,\frac{\partial p}{\partial y} + k\,\frac{\partial p}{\partial z}\ .$$

Suppose also that the external force per unit of mass acting on the liquid is $\boldsymbol{F}(F_x, F_y, F_z)$, so that the resultant of these forces applied to the volume V will be equal to

$$\iiint_V \rho \boldsymbol{F}\,\mathrm{d}V\ .$$

Finally, an inertial force equal to

$$-\iiint_V \rho\,\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\,\mathrm{d}V$$

acts on the volume V. Using d'Alembert's principle, we obtain

$$\iiint_V \left\{\rho\boldsymbol{F} - \rho\,\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} - \operatorname{grad} p\right\}\mathrm{d}V = 0\ .$$

Therefore, since the element of volume V was arbitrary, it follows that

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{F} - \frac{1}{\rho}\operatorname{grad} p\ , \tag{10}$$

or, in scalar form,

$$\begin{aligned}
\frac{\partial v_x}{\partial t} + \frac{\partial v_x}{\partial x}\,v_x + \frac{\partial v_x}{\partial y}\,v_y + \frac{\partial v_x}{\partial z}\,v_z &= F_x - \frac{1}{\rho}\frac{\partial p}{\partial x}\ ,\\
\frac{\partial v_y}{\partial t} + \frac{\partial v_y}{\partial x}\,v_x + \frac{\partial v_y}{\partial y}\,v_y + \frac{\partial v_y}{\partial z}\,v_z &= F_y - \frac{1}{\rho}\frac{\partial p}{\partial y}\ ,\\
\frac{\partial v_z}{\partial t} + \frac{\partial v_z}{\partial x}\,v_x + \frac{\partial v_z}{\partial y}\,v_y + \frac{\partial v_z}{\partial z}\,v_z &= F_z - \frac{1}{\rho}\frac{\partial p}{\partial z}\ .
\end{aligned} \tag{10a}$$

These are the equations of motion of an ideal liquid in Euler's form.

Thus, for the five unknown scalar functions, v_x, v_y, v_z, ρ, and p, that characterize the motion of an ideal liquid or gas, we have, in all, the four equations (9) and (10).

To obtain one more equation, we shall assume that the motion of a

compressible liquid or gas takes place *adiabatically*. In this case, it can be shown, under certain supplementary assumptions, that the density ρ depends only on the pressure p and that this relationship is expressed by the formula

$$p = p_0(\rho/\rho_0)^\gamma \qquad (\gamma = C_p/C_v) \,, \tag{11}$$

where ρ_0 and p_0 are the initial density and the initial pressure, and C_p and C_v are the specific heats at constant pressure and volume, respectively.

Thus, we have the five equations (9), (10), and (11), which contain exactly five unknown functions: v_x, v_y, v_z, ρ, and p.

Let us apply the hydrodynamic equations to the propagation of sound in a gas.

We shall consider only small vibrations of the gas, so that in the Euler equations (10a), we may neglect the terms $(\partial v_x/\partial x)v_x, \ldots$. Assuming that there are no external forces, we obtain

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}\,, \qquad \frac{\partial v_y}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y}\,, \qquad \frac{\partial v_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z}\,, \tag{12}$$

or in vector form,

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\,\mathrm{grad}\, p \,. \tag{12a}$$

The quantity $s(x, y, z, t)$, which is equal to the proportional change in the density

$$s(x, y, z, t) = \frac{\rho - \rho_0}{\rho_0} \,, \tag{13}$$

is called the *condensation* of the gas. By rewriting eq. (13), we have

$$\rho = \rho_0(1 + s) \,. \tag{14}$$

Then, eq. (11) can be rewritten in the form

$$p = p_0(1 + s)^\gamma \,. \tag{15}$$

For small vibrations of the gas, its condensation s is sufficiently small that higher powers of s may be neglected and we obtain

$$p = p_0(1 + \gamma s) \,. \tag{16}$$

When we substitute eq. (14) into the continuity equation (9) and neglect the second-order terms, eq. (9) takes the form

$$\frac{\partial s}{\partial t} + \mathrm{div}\, v = 0 \,, \tag{17}$$

since

$$\mathrm{div}\,(\rho v) = \rho\,\mathrm{div}\, v + v\,\mathrm{grad}\,\rho = \rho_0\,\mathrm{div}\, v + \rho_0 s\,\mathrm{div}\, v + v\,\mathrm{grad}\,\rho \,,$$

and the last two terms can be neglected.

With this approximation, the Euler equation (12a) is reduced to the equation

$$\frac{\partial v}{\partial t} = - a^2 \operatorname{grad} s \ , \tag{18}$$

where

$$a = \sqrt{\gamma p_0/\rho_0} \ . \tag{19}$$

Taking the divergence of both sides of eq. (18) and reversing the order of differentiation, we obtain

$$\frac{\partial}{\partial t} \operatorname{div} v = - a^2 \operatorname{div} \operatorname{grad} s = - a^2 \nabla^2 s \ , \tag{20}$$

where

$$\nabla^2 s = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2}$$

is called the *Laplacian operator* (here operating on the function s). By using eq. (17), we obtain

$$\frac{\partial^2 s}{\partial t^2} = a^2 \left(\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} \right) . \tag{21}$$

We can also obtain a wave equation of the form (21) for the pressure p and the velocity v.

Let us now assume that, at the initial time, there is a potential for the velocities, $U_0(x, y, z)$; that is,

$$v|_{t=0} = - \operatorname{grad} U_0(x, y, z) \ . \tag{22}$$

From eq. (18), we have

$$v(x, y, z, t) = v|_{t=0} - a^2 \operatorname{grad} \left(\int_0^t s \ dt \right)$$

or, on the basis of eq. (22),

$$v = - \operatorname{grad} \left\{ U_0(x, y, z) + a^2 \int_0^t s \ dt \right\} = - \operatorname{grad} U(x, y, z, t) \ . \tag{23}$$

This means that there is a velocity potential $U(x, y, z, t)$ at an arbitrary instant of time t:

$$U(x, y, z, t) = U_0(x, y, z) + a^2 \int_0^t s \ dt \ . \tag{24}$$

Let us show that the velocity potential $U(x, y, z, t)$ satisfies eq. (21). If we differentiate eq. (24) twice with respect to t, we obtain

$$\frac{\partial^2 U}{\partial t^2} = a^2 \frac{\partial s}{\partial t} \ . \tag{25}$$

On the other hand, if we substitute eq. (23) into eq. (17), we shall obtain

$$\frac{\partial s}{\partial t} = \operatorname{div} \operatorname{grad} U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \ . \tag{26}$$

From eqs. (25) and (26), we obtain

$$\frac{\partial^2 U}{\partial t^2} = a^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) . \tag{27}$$

Thus, if the value of the velocity potential $U(x, y, z, t)$ is known, we can determine the entire motion of the liquid or gas, since

$$v = -\operatorname{grad} U , \qquad s = \frac{1}{a^2}\frac{\partial U}{\partial t} . \tag{28}$$

In the case of oscillations of a liquid or gas in a bounded region, definite boundary conditions must be given. If the boundary is a solid impenetrable wall, the normal component of the velocity will be equal to zero, which leads to the condition

$$\left.\frac{\partial U}{\partial n}\right|_{\Sigma} = 0 \qquad \text{or} \qquad \left.\frac{\partial s}{\partial n}\right|_{\Sigma} = 0 , \tag{29}$$

where Σ is the boundary of the region.

3. *Poisson's formula*

In the present section, we shall examine the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{30}$$

and seek a solution to it that satisfies the initial conditions

$$u|_{t=0} = f(x, y, z) , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x, y, z) . \tag{31}$$

This problem was first solved by Poisson in 1818.

Let us first show that the double integral

$$u(x, y, z, t) = \frac{1}{4\pi a}\iint_{\Sigma} \frac{\varphi(\xi, \eta, \zeta)}{r}\, d\sigma_r , \tag{32}$$

taken over the surface of the sphere Σ of radius $r = at$ and with center at $M(x, y, z)$, is a solution to the wave equation (30) with the particular initial conditions

$$u|_{t=0} = 0 , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \varphi(x, y, z) \tag{33}$$

The first of the initial conditions (33) is satisfied because

$$\left|\iint_{\Sigma} \frac{\varphi(\xi, \eta, \zeta)}{r}\, d\sigma_r\right| \leqslant \max|\varphi| \,\frac{4\pi a^2 t^2}{at} = 4\pi a t \max|\varphi|$$

and, consequently,

$$u(x, y, z, t) \to 0 \quad \text{as} \quad t \to 0$$

To verify the second condition, we note that the coordinates of points of the sphere Σ can be expressed by the formulae

$$\xi = x + \alpha a t\,, \qquad \eta = y + \beta a t\,, \qquad \zeta = z + \gamma a t\,, \tag{34}$$

where (α, β, γ) are the direction cosines of the radii of the sphere Σ. Then, the integral (32) is reduced to the form

$$u(x, y, z, t) = \frac{t}{4\pi} \iint_{S_1} \varphi(x + \alpha a t, y + \beta a t, z + \gamma a t)\, d\sigma_1\,, \tag{35}$$

where the integration is performed over a unit sphere S_1 that is fixed for all x, y, z, and t:

$$\alpha^2 + \beta^2 + \gamma^2 = 1\,, \qquad d\sigma_r = r^2\, d\sigma_1 = a^2 t^2\, d\sigma_1\,.$$

From formula (35), we obtain

$$\frac{\partial u}{\partial t} = \frac{1}{4\pi} \iint_{S_1} \varphi(x + \alpha a t, y + \beta a t, z + \gamma a t)\, d\sigma_1 + \frac{at}{4\pi} \iint_{S_1} \left(\alpha \frac{\partial \varphi}{\partial \xi} + \beta \frac{\partial \varphi}{\partial \eta} + \gamma \frac{\partial \varphi}{\partial \zeta}\right) d\sigma_1\,. \tag{36}$$

From this, it is easy to see that the first term on the right side tends to $\varphi(x, y, z)$ as t tends to zero, and that the second tends to zero as t tends to zero, since the integral in it remains bounded.

Let us now show that the function $u(x, y, z, t)$, defined by formula (32), satisfies the wave equation (30). From eq. (35), we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{t}{4\pi} \iint_{S_1} \left(\frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \varphi}{\partial \eta^2} + \frac{\partial^2 \varphi}{\partial \zeta^2}\right) d\sigma_1 = \frac{1}{4\pi a^2 t} \iint_{\Sigma} \left(\frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \varphi}{\partial \eta^2} + \frac{\partial^2 \varphi}{\partial \zeta^2}\right) d\sigma_r\,. \tag{37}$$

To compute $\partial^2 u/\partial t^2$, let us rewrite eq. (36) in the form

$$\frac{\partial u}{\partial t} = \frac{u}{t} + \frac{1}{4\pi a t} \iint_{\Sigma} \left(\alpha \frac{\partial \varphi}{\partial \xi} + \beta \frac{\partial \varphi}{\partial \eta} + \gamma \frac{\partial \varphi}{\partial \zeta}\right) d\sigma_r$$

or, on the basis of Gauss's theorem

$$\frac{\partial u}{\partial t} = \frac{u}{t} + \frac{1}{4\pi a t} \iiint_{D} \left(\frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \varphi}{\partial \eta^2} + \frac{\partial^2 \varphi}{\partial \zeta^2}\right) d\xi\, d\eta\, d\zeta = \frac{u}{t} + \frac{J(t)}{4\pi a t}\,, \tag{38}$$

where

$$J(t) = \iiint_{D} \left(\frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \varphi}{\partial \eta^2} + \frac{\partial^2 \varphi}{\partial \zeta^2}\right) d\xi\, d\eta\, d\zeta\,,$$

and D is a sphere of radius $r = at$ with center at the point (x, y, z).

Differentiating eq. (38) once more with respect to t, we obtain

$$\frac{\partial^2 u}{\partial t^2} = -\frac{u}{t^2} + \frac{1}{t}\left\{\frac{u}{t} + \frac{J(t)}{4\pi at}\right\} - \frac{J(t)}{4\pi at^2} + \frac{1}{4\pi at}\frac{\partial J(t)}{\partial t} = \frac{1}{4\ at}\frac{\partial J(t)}{\partial t}\ . \tag{39}$$

It is easy to see that

$$\frac{\partial J(t)}{\partial t} = a \iint\limits_{\Sigma} \left(\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta^2} + \frac{\partial^2\varphi}{\partial\zeta^2}\right) d\sigma_r\ . \tag{40}$$

In fact, changing to spherical coordinates (ρ, θ, ψ), with origin at the center of the sphere D, we obtain

$$J(t) = \int_0^{at}\int_0^{2\pi}\int_0^{\pi} \left(\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta^2} + \frac{\partial^2\varphi}{\partial\zeta^2}\right) \rho^2 \sin\theta\ d\theta\ d\psi\ d\rho\ .$$

Differentiating with respect to t, we obtain

$$\frac{\partial J(t)}{\partial t} = a\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta^2} + \frac{\partial^2\varphi}{\partial\zeta^2}\right)_{\rho=at} a^2t^2 \sin\theta\ d\theta\ d\psi$$

$$= a\iint\limits_{\Sigma}\left(\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta^2} + \frac{\partial^2\varphi}{\partial\zeta^2}\right) d\sigma_r\ .$$

Comparing eqs. (37), (39), and (40), we see that the function $u(x, y, z, t)$, defined by eq. (32), actually does satisfy the wave equation (30), regardless of the choice of the continuous function $\varphi(x, y, z)$ with continuous derivatives up to the second order inclusive.

Since eq. (30) is a linear homogeneous equation with constant coefficients, it is not difficult to show that the function

$$v(x, y, z, t) = \frac{\partial u}{\partial t}$$

will also be a solution to the wave equation (30) satisfying the initial conditions

$$v|_{t=0} = \varphi(x, y, z)\ , \quad \frac{\partial v}{\partial t}\Big|_{t=0} = \frac{\partial^2 u}{\partial t^2}\Big|_{t=0} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\Big|_{t=0} = 0\ . \tag{41}$$

Thus, the solution to the wave equation (30) satisfying the initial conditions (31) is given by the formula

$$u(x, y, z, t) = \frac{1}{4\pi a}\left\{\frac{\partial}{\partial t}\left[\iint\limits_{\Sigma}\frac{f(\xi, \eta, \zeta)}{r}\, d\sigma_r\right] + \iint\limits_{\Sigma}\frac{F(\xi, \eta, \zeta)}{r}\, d\sigma_r\right\}\ . \tag{42}$$

This formula is usually called Poisson's formula. Obviously, it can be rewritten in the form

$$u(x, y, z, t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\left[t\iint\limits_{S_1} f(\xi, \eta, \zeta)\, d\sigma_1\right] + \frac{t}{4\pi}\iint\limits_{S_1} F(\xi, \eta, \zeta)\, d\sigma_1\ , \tag{43}$$

where

$$\xi = x + \alpha at \ , \qquad \eta = y + \beta at \ , \qquad \zeta = z + \gamma at \ . \tag{34}$$

The preceding considerations indicate that the function $u(x, y, z, t)$ defined by eq. (42) does in fact satisfy eq. (30) and the initial conditions (31), if $f(x, y, z)$ and its derivatives up to the third order (inclusive) are continuous, and $F(x, y, z)$ and its derivatives up to the second order (inclusive) are continuous.

From Poisson's formula (42), it is easy to show that the solution to the Cauchy problem (30), (31) is a continuous function of the initial conditions. Formula (42) contains integrals of the initial functions, and the derivatives of these integrals with respect to time. Therefore, if we replace the initial functions $f(x, y, z)$ and $F(x, y, z)$ in such a way that they and their first derivatives change sufficiently slightly, the function $u(x, y, z, t)$ will also change only slightly, giving a solution to the Cauchy problem. Of course, it is assumed here that only finite values of t are being considered.

4. *The propagation of sound waves in space*

Let us apply Poisson's formula to the investigation of certain fundamental questions in the study of sound.

Let us suppose that the space occupied by a gas is sufficiently large that we may consider it as extending infinitely far in all directions. Suppose that some portion (R) of this space, bounded by a closed surface (S), is put into a disturbed state at an initial time $t = 0$. Then, the equations

$$U|_{t=0} = f(x, y, z) \ , \qquad \frac{\partial U}{\partial t}\Big|_{t=0} = F(x, y, z) \ , \tag{31}$$

which express the initial conditions for the motion of the particle of the gas that is at the point $\mathrm{M}(x, y, z)$, have the following physical meaning: the function $U(x, y, z, t)$, as was explained above, is the velocity potential, and the derivative $\partial U/\partial t$ is related to the condensation s by the equation

$$s = \frac{1}{a^2}\frac{\partial U}{\partial t} \ . \tag{28}$$

Thus, it follows that the function $f(x, y, z, t)$ is equal to zero if the point M lies within the region (R), since, there, the particles of the gas are at rest at the initial time. Since the condensation at these points is also equal to zero, it follows that $F(x, y, z, t) = 0$ outside the region (R). With this in mind, we now turn to the potential $U(x, y, z, t)$; as we have shown, it satisfies the wave equation (30).

Consequently, to study the laws of oscillation of a gas, we need to find the solution to eq. (30) that satisfies the initial conditions (31). But this problem was already solved in the preceding section, where it was shown that the function sought is given by Poisson's formula (42).

When we examine eq. (42), we can see that the integrals appearing in it extend only over that portion of the sphere Σ that is contained within the region (R) because, as was remarked above, the functions f and F are equal to zero outside the region (R). This fact makes possible the study of the

vibrations of a particle of gas at a point M situated outside the region of the initial oscillations.

Let us describe a sequence of spheres with positive increasing radius at and with center at the point M. At the initial time, these spheres will not intersect the region (R). Later, at the time $t_1 = d/a$, where d is the shortest distance from M to (R), they will be tangent to this region, and they will intersect it until the time $t_2 = D/a$, where D is the greatest distance from M to (R). After this, they will again have no points in common with the region (R) (fig. 23). We know that the integrals appearing on the right side of eq. (42) are different from zero only when the sphere Σ intersects the region (R). From this, we draw the following conclusion: a particle of gas that is situated at the point M is at rest during the time between $t = 0$ and $t_1 = d/a$. Afterwards, it begins to oscillate and these oscillations will continue until the time $t_2 = D/a$, after which the particle will again be at rest.

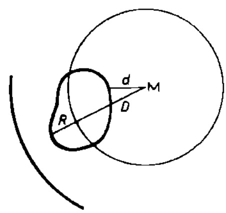

Fig. 23.

Now, let us consider the region (R) of the initial oscillations. As we know, it is bounded by the surface (S) which, at the initial time $t = 0$, separates the region in the disturbed state from the space that is at rest. With the passage of time, the oscillations in the region (R) are transmitted to the surrounding points of the space. At any given subsequent instant, by using Huygens' principle, we can construct a surface separating the points that the disturbance has not yet reached from those that it has reached. Huygens' principle states that if at every point of the surface (S), we construct a sphere of radius at and then form the envelope of these spheres, then this envelope will be the surface of a wave that is being propagated through space with velocity a.

A wave moving from the region (R) has *two wave fronts – a leading and a trailing front.* At the leading front, the points of the space previously at rest are set into oscillation; at the trailing front, the opposite is observed – the previously oscillating points pass into a state of rest. A wave, with both its fronts, passes through every point lying outside the region of initial oscillations. When this happens, its leading front reaches the given point at the time $t_1 = d/a$, and its trailing front leaves the given point at the time $t_2 = D/a$.

To obtain the surfaces of both wave fronts that pass through the point M, we must lay off a segment equal to d on all the outwardly directed normals to the surface (R), and a segment equal to D on all the inwardly directed normals. Then, the geometric position of the ends of these segments

will form two surfaces. These surfaces, or parts of them, represent the leading and trailing fronts.

5. *Cylindrical waves*

Let us suppose that the initial conditions of motion of the gas are such that both functions $f(x, y, z)$ and $F(x, y, z)$ are independent of z. This will be the case if the region of initial oscillations has the shape of an infinite cylinder with generators parallel to the z-axis. It is easy to see that, in this case, the velocity potential $U(x, y, z, t)$ will also be independent of z. In fact, the values of this potential can be computed from Poisson's formula (42), the right side of which will obviously remain constant if we displace the point $M(x, y, z)$ parallel to the z-axis. Thus, we conclude that the conditions will be the same for all particles of the gas that are situated on a straight line parallel to the z-axis, and therefore, we need only study the oscillation of a particle of gas that is situated in the xy-plane.

If we denote by $U(x, y, t)$ the value of the velocity potential for a point $M(x, y)$ lying in this plane, we obtain the equation

$$\frac{\partial^2 U}{\partial t^2} = a^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) \tag{44}$$

and the initial conditions

$$U|_{t=0} = f(x, y) , \qquad \frac{\partial U}{\partial t}\Big|_{t=0} = F(x, y) . \tag{45}$$

The value of the function $U(x, y, t)$ can be determined by formula (42), which, in the present case, acquires a somewhat different form. To show this, let us project onto the xy-plane a surface element $d\sigma_r$ of a sphere Σ with center at $M(x, y)$ and with radius $r = at$ (fig. 24). The projection of this element is equal to the product $d\xi\, d\eta$ where ξ and η denote the coordinates of a variable point $A(\xi, \eta)$ lying in the plane of the great circle BC.

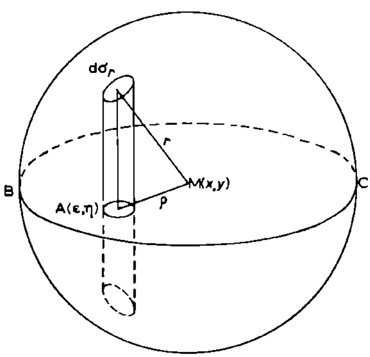

Fig. 24.

From a theorem on projections, we obtain

$$d\sigma_r = \frac{d\xi\, d\eta}{\sin(\rho, r)} = \frac{r\, d\xi\, d\eta}{\sqrt{r^2 - (x-\xi)^2 - (y-\eta)^2}}.$$

Noting that the element $d\xi\, d\eta$ is the projection of two symmetric elements of the sphere Σ, we can easily show that formula (42) takes the following form:

$$U(x,y,t) = \frac{1}{2\pi a}\frac{\partial}{\partial t}\iint_{\Gamma}\frac{f(\xi,\eta)\, d\xi\, d\eta}{\sqrt{a^2t^2 - (x-\xi)^2 - (y-\eta)^2}} + \frac{1}{2\pi a}\iint_{\Gamma}\frac{F(\xi,\eta)\, d\xi\, d\eta}{\sqrt{a^2t^2 - (x-\xi)^2 - (y-\eta)^2}}, \qquad (46)$$

where the region of integration (Γ) is a circle whose center is the point $M(x,y)$ and whose radius is equal to at.

Let us now investigate, in greater detail, the oscillation of the particle of the gas that is located at the point $M(x,y)$. We denote by (B) that portion of the plane that is bounded by the curve (C) formed by the intersection of the infinite cylinder with the plane $z = 0$. Let us assume that both the functions $f(x,y)$ and $F(x,y)$ are equal to zero for every point M lying *outside* the region (B). Then, by the reasoning of section 4, we see that the particle of the gas at the point M will be at rest until the time $t_1 = d/a$, where d is the shortest distance from M to the curve (C). After this time, not only the particle situated at the point M, but also all other particles on the straight line passing through the point M parallel to the z-axis, will be set in oscillation.

However, as compared with the general case analyzed in section 4, there is one distinctive feature about the present problem. This is the fact that particles of the gas that have been set in motion will never again return to a state of rest, as in the general case, since all the circles with radius $at > d$ and with center M will always enclose points of the region (B) for which, as we know, the functions f and F are different from zero. It follows from this that the velocity potential $U(x,y,t)$, after some given instant of time, will not be reduced either to zero or to a constant value, and the particles situated at the point M will always oscillate. These oscillations are subject to damping, since, from formula (46), it follows that U, $\partial U/\partial x$, and $\partial U/\partial y$ approach zero as t approaches ∞, and the derivatives $\partial U/\partial x$ and $\partial U/\partial y$ are the projections onto the coordinate axes of the velocity of the particle of the gas.

It is easy to see that, in the above case, we are dealing with the propagation of a *cylindrical* wave whose surface is parallel to that of the initial cylinder. This cylindrical wave has only a leading front; there is no trailing front.

6. *Plane waves*

Let us now analyze the case in which the functions $f(x,y,z)$ and $F(x,y,z)$ that appear in the initial conditions depend only on the coordinate x. This is

the case when the region of initial oscillations is a portion of infinite space included between two planes perpendicular to the x-axis. If we draw a plane parallel to the above two, it is obvious that all particles located in it will oscillate in the same way. Thus, it is clear that we need only study oscillations of a particle of the gas that is situated at the point where this plane intersects the x-axis.

In this case, the velocity potential $U(x, t)$ satisfies the wave equation

$$\frac{\partial^2 U}{\partial t^2} = a^2 \frac{\partial^2 U}{\partial x^2}$$

and the initial conditions

$$U|_{t=0} = f(x) , \qquad \frac{\partial U}{\partial t}\Big|_{t=0} = F(x) .$$

Formula (42) then becomes the well-known d'Alembert formula:

$$U(x, t) = \frac{f(x - at) + f(x + at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} F(\xi)\, d\xi . \tag{47}$$

To show this, let us introduce a spherical coordinate system with polar axis directed along the x-axis. A surface element $d\sigma_r$ is expressed by

$$d\sigma_r = r^2 \sin\theta \, d\theta \, d\varphi = - r \, d\varphi \, d\xi ,$$

since

$$\xi = x + r\cos\theta , \qquad d\xi = - r \sin\theta \, d\theta .$$

Then, Poisson's formula (42) is reduced to the form

$$U(x, t) = \frac{1}{4\pi a}\frac{\partial}{\partial t}\int_0^{2\pi}\int_{x-at}^{x+at} \frac{f(\xi)\, at\, d\varphi\, d\xi}{at} + \frac{1}{4\pi a}\int_0^{2\pi}\int_{x-at}^{x+at} \frac{F(\xi)\, at\, d\varphi\, d\xi}{at}$$

$$= \frac{1}{2a}\frac{\partial}{\partial t}\int_{x-at}^{x+at} f(\xi)\, d\xi + \frac{1}{2a}\int_{x-at}^{x+at} F(\xi)\, d\xi .$$

If we differentiate with respect to t as indicated, we obtain eq. (47), which was to be derived.

Let us now assume that the initial conditions of motion of the gas are such that the functions f and F are equal to zero outside some interval (x_1, x_2). Let us take some point M with coordinate $x > x_2$. Eq. (47) shows that, in the interval of time between $t = 0$ and $t = t_1 = (x - x_2)/a$, the particle of gas situated at the point M is at rest because, obviously, in this case $U = 0$.

At the time $t = t_1$, this particle will begin to oscillate, and the oscillations will continue until the time $t_2 = (x - x_1)/a$. In this time interval, the values of the potential function can be computed from the formula

$$U = \tfrac{1}{2} f(x - at) + \frac{1}{2a}\int_{x-at}^{x_2} F(\xi)\, d\xi .$$

From the time $t = t_2$, the particle of the gas will again be at rest because, in this case, it follows from the equation

$$U = \frac{1}{2a}\int_{x_1}^{x_2} F(\xi)\, d\xi = \text{constant} \tag{48}$$

that $\partial U/\partial x = 0$.

We obtain an analogous picture when we examine a point M lying to the left of x_1.

All these considerations indicate that *plane waves* are propagated from the region of the initial oscillations with velocity a in the direction of the x-axis.

As compared with the general case considered in section 4, we note that, after the plane wave passes through the point M, the potential $U(x, t)$ is no longer equal to zero but, as eq. (48) shows, takes a constant value. Thus, a plane wave leaves a "trace" of its origin.

7. *Spherical waves*

Let us assume that the region of initial oscillations is a sphere of radius R, from which oscillations are propagated uniformly in all directions. Let us place the coordinate origin at the center of this sphere, and let us denote by r the distance from the origin to a variable point M. Then, the initial conditions of the oscillations will be of the form

$$U|_{t=0} = f(r)\,, \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = F(r)\,, \tag{49}$$

where $f(r)$ and $F(r)$ are known functions for positive values of r.

Because of the complete symmetry of the oscillations, the velocity potential U and the condensation $s = (1/a^2)(\partial U/\partial t)$ for all subsequent instants of time will also be functions of only the distance r and the time t. We know that the potential U satisfies the wave equation (30). If we now introduce spherical coordinates (r, θ, φ), and recall that the potential U does not depend on the angular coordinates θ and φ, we obtain

$$\frac{\partial^2 U}{\partial t^2} = a^2\left(\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r}\right)$$

or, after elementary transformations,

$$\frac{\partial^2 (rU)}{\partial t^2} = a^2\,\frac{\partial^2 (rU)}{\partial r^2} \tag{50}$$

The general solution to this equation will be of the form

$$U(r,t) = \frac{\varphi(at - r)}{r} + \frac{\psi(at + r)}{r} \tag{51}$$

The first term of this expression is a *spherical wave* that is propagated radially with velocity a from the region of initial disturbance; the second

term is a spherical wave that is propagated with the same velocity in the opposite direction. The functions φ and ψ that appear in eq. (51) can be determined from the finiteness condition for the potential at the center of oscillation, and from the initial conditions (49).

Let us now consider, as an example, the following given initial conditions: the initial velocities are everywhere equal to zero. The initial condensation is constant within the sphere (R) and is equal to zero outside this sphere. These conditions can be expressed as follows:

$$f(r) = 0 \quad \text{for arbitrary } r, \qquad F(r) = \begin{cases} a^2 s_0 & \text{for } r < R\,, \\ 0 & \text{for } r > R\,, \end{cases} \tag{52}$$

where s_0 is the initial condensation.

Let us determine the condensation s for an arbitrary instant of time, at a point M lying outside the region of initial disturbance.

Since the potential at the point $r = 0$ is finite, we may rewrite eq. (51) in the form

$$U(r, t) = \frac{\varphi(at - r) - \varphi(at + r)}{r}\,. \tag{53}$$

Hence,

$$s = \frac{1}{a^2}\frac{\partial U}{\partial t} = \frac{\varphi'(at - r) - \varphi'(at + r)}{ar}\,. \tag{54}$$

To determine the form of the function φ', we use eqs. (53) and (54) and the initial conditions (52). From these, it is clear that

$$\varphi(-r) - \varphi(r) = 0\,, \tag{55}$$

$$\varphi'(-r) - \varphi'(r) = \frac{r}{a} F(r)\,, \tag{56}$$

where the function $F(r)$ us determined by the second of the conditions (52).

If we differentiate eq. (55) and solve the system of equations thus obtained for the derivatives $\varphi'(r)$ and $\varphi'(-r)$, we have the following necessary formulae:

$$\varphi'(r) = -\varphi'(-r) = -\frac{r}{2a} F(r)\,. \tag{57}$$

Let us now turn to eq. (54) and let us examine the function $\varphi'(at + r)$ that appears in it. Since we are interested in the picture of the phenomenon at an instant of time t greater than zero and wish to compute the condensation at the point M (which is at a distance $r > R$ from the origin), the argument of the derivative $\varphi'(at + r)$ will satisfy the inequality

$$at + r > R\,,$$

from which, on the basis of (52) and (57), it follows that

$$\varphi'(at + r) = 0\,.$$

Now it remains to determine the derivative $\varphi'(at - r)$ that appeared in the expression

$$s = \frac{\varphi'(at - r)}{ar} . \tag{58}$$

To do this, we partition the time of oscillation of the gas into three intervals (beginning with the initial time):

$$\left(0, \frac{r-R}{a}\right), \quad \left(\frac{r-R}{a}, \frac{r+R}{a}\right), \quad \left(\frac{r+R}{a}, \infty\right) .$$

When t varies in the first interval, we have the inequality

$$r - at > R ,$$

from which, on the basis of (52) and (57), it follows that

$$\varphi'[-(r - at)] = 0$$

and, consequently,

$$s = 0 ; \tag{59}$$

As t varies in the second interval, we have the inequality

$$at - r < R ,$$

from which, on the basis of the same eqs. (52) and (57), it follows that

$$\varphi'(at - r) = \tfrac{1}{2} a s_0 (r - at)$$

and, hence,

$$s = \frac{s_0(r - at)}{2r} . \tag{60}$$

Finally, as t varies in the third interval, we have the inequality

$$at - r > R ,$$

from which it is clear that, in this case,

$$s = 0 . \tag{61}$$

Thus, for an arbitrary instant of time, the condensation s may be calculated from eqs. (59) - (61). A graph of the dependence of the condensation on time is shown in fig. 25.

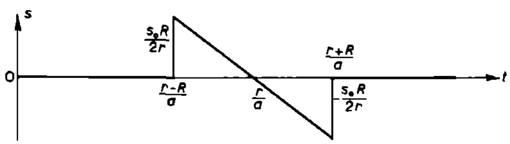

Fig. 25.

We now solve the same problem by beginning with Poisson's formula. If, as before, we denote by d and D the smallest and the greatest distances between the point M and the sphere (R), formula (42) will immediately give

$$U(r, t) = 0$$

for instants of time that satisfy the inequalities

$$t < \frac{d}{a} = \frac{r - R}{a} \quad \text{or} \quad t > \frac{D}{a} = \frac{r + R}{a},$$

since in these intervals, on the basis of the initial conditions (52), the functions f and F are equal to zero outside the region (R). It follows from this, that, for the instants of time that we are considering,

$$s = 0,$$

and we thus obtain eqs. (59) and (61)

Let us now calculate the value of the potential U and the condensation s for an instant of time that satisfies the inequalities

$$\frac{d}{a} < t < \frac{D}{a}$$

By using eq. (42), we obtain

$$U(r, t) = \frac{1}{4\pi a} \iint_{\Sigma} \frac{a^2 s_0}{at}\, d\sigma = \frac{s_0}{4\pi t} \iint_{\Sigma} d\sigma, \tag{62}$$

where Σ is that portion of the surface of the sphere of radius at that is included within the sphere (R) (fig. 26). But the integral $\iint_{\Sigma} ds$ is the area of this portion of the sphere, and, consequently,

$$\iint_{\Sigma} d\sigma = 2\pi(1 - \cos\alpha)\,(at)^2$$

On the other hand, it is clear from the drawing that

$$\cos\alpha = \frac{r^2 + (at)^2 - R^2}{2art}$$

and, thus,

$$\iint_{\Sigma} d\sigma = \frac{R^2 - (r - at)^2}{r}\, at. \tag{63}$$

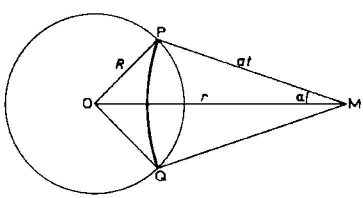

Fig. 26.

Substituting eq. (63) into eq. (62), we obtain the following result: for an arbitrary instant of time satisfying the condition

$$\frac{r - R}{a} < t < \frac{r + R}{a},$$

the values of the velocity potential and condensation of the gas can be calculated from the formulae

$$U = \frac{as_0}{4r}[R^2 - (r - at)^2] \tag{64}$$

and

$$s = \frac{1}{a^2}\frac{\partial U}{\partial t} = \frac{s_0(r - at)}{2r} . \tag{65}$$

This coincides with the results above.

Let us examine the oscillations of particles of the gas situated at a distance r from the center of oscillations. Recalling the considerations mentioned at the end of section 4, we may draw the following picture.

At the initial time, all points at a distance r from the center O are at rest. Then, at the time $t_1 = (r-R)/a$, the leading front of the spherical wave reaches these points and they begin to oscillate. The oscillations will continue for a period of time equal to $2R/a$, that is, until the trailing wave front reaches these points. After this, the points will again be at rest. Thus, a spherical wave has the shape of a spherical layer of thickness $2R$ (fig. 27).

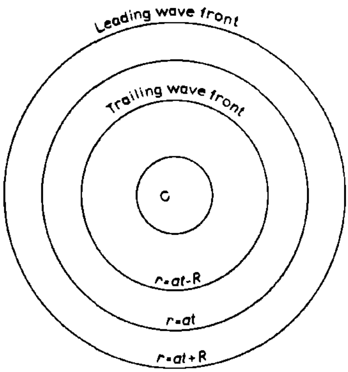

Fig. 27.

We note also that the trailing wave front does not form immediately, as does the leading front. To show this, let us construct, for a sufficiently small value of t, both envelopes of the spheres of radius at, at all points of the surface of the sphere (R). With increasing t, the outer envelope will expand, becoming the leading wave front. The inner envelope will initially be compressed, and will become a point at the instant R/a; only then will it expand in the form of a sphere and produce the trailing wave front.

In conclusion, we note the following: if a spherical layer, constituting a spherical wave, is divided by the sphere $r = at$ into two equal portions, then, as can be seen from eqs. (65), in the outer portion of the layer, where $r > at$, the condensation of the gas will be positive, and in the inner portion, it will be negative.

8. *The inhomogeneous wave equation*

Let us examine the inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + g(x, y, z, t) \tag{66}$$

and seek a solution to it that satisfies the initial conditions

$$u\big|_{t=0} = 0 , \qquad \frac{\partial u}{\partial t}\bigg|_{t=0} = 0 \tag{67}$$

To solve this problem, let us examine the solution to the homogeneous equation

$$\frac{\partial^2 v}{\partial t^2} = a^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) , \tag{68}$$

satisfying the initial conditions

$$v\big|_{t=\tau} = 0 , \qquad \frac{\partial v}{\partial t}\bigg|_{t=\tau} = g(x, y, z, \tau) , \tag{69}$$

where, at the initial time, it is assumed not that $t = 0$, but that $t = \tau$ (where τ is some parameter). The solution to the problem (68) - (69) can be expressed by Poisson's formula, but, in that formula, we must replace t by $t - \tau$ (since the initial time is $t = \tau$ and not $t = 0$).

Thus, we have

$$v(x, y, z, t; \tau) = \frac{t - \tau}{4\pi} \iint_{S_1} g[x + \alpha a(t - \tau), y + \beta a(t - \tau), z + \gamma a(t - \tau), \tau]\, d\sigma_1 . \tag{70}$$

Let us show that the function $u(x, y, z, t)$, defined by

$$u(x, y, z, t) = \int_0^t v(x, y, z, t; \tau)\, d\tau , \tag{71}$$

is a solution to the inhomogeneous wave equation (66) with initial conditions (67). From eq. (71), we have

$$\nabla^2 u = \int_0^t \nabla^2 v(x, y, z, t; \tau)\, d\tau . \tag{72}$$

Differentiating eq. (71) with respect to t, we obtain

$$\frac{\partial u}{\partial t} = \int_0^t \frac{\partial v(x, y, z, t; \tau)}{\partial t}\, d\tau + v(x, y, z, t; \tau)\bigg|_{\tau=t} \tag{73}$$

Here, the integrand is equal to zero because of the first of conditions (69).

If we again differentiate with respect to t, we obtain

$$\frac{\partial^2 u}{\partial t^2} = \int_0^t \frac{\partial^2 v(x, y, z, t; \tau)}{\partial t^2}\, d\tau + \frac{\partial v(x, y, z, t; \tau)}{\partial t}\bigg|_{\tau=t} ;$$

here, the integrand is equal to $g(x, y, z, t)$, because of the second of conditions (69); that is,

$$\frac{\partial^2 u}{\partial t^2} = \int_0^t \frac{\partial^2 v(x, y, z, t; \tau)}{\partial t^2} \, d\tau + g(x, y, z, t) \ . \tag{74}$$

From eqs. (72), (74), and (68), it is easy to see that the function $u(x, y, z, t)$ satisfies the inhomogeneous equation (66). The initial conditions (67) follow immediately from (71) and (73).

If, in eq. (71), we replace the function $z(x, y, z, t; \tau)$ with the expression given for it by eq. (70), we obtain

$$u(x, y, z, t) = \frac{1}{4\pi} \int_0^t (t-\tau) \left\{ \iint_{S_1} g[x+\alpha a(t-\tau), y+\beta a(t-\tau), z+\gamma a(t-\tau), \tau] \, d\sigma_1 \right\} d\tau \, .$$

Instead of τ, let us introduce a new variable of integration $r = a(t-\tau)$. We then have

$$u(x, y, z, t) = \frac{1}{4\pi a^2} \int_0^{at} \int_0^{2\pi} \int_0^{\pi} g\left(x+\alpha r, y+\beta r, z+\gamma r, t-\frac{r}{a}\right) r \sin\theta \, d\theta \, d\varphi \, dr$$

and, if we both multiply and divide by r,

$$u(x, y, z, t) = \frac{1}{4\pi a^2} \int_0^{at} \int_0^{2\pi} \int_0^{\pi} \frac{g(x+\alpha r, y+\beta r, z+\gamma r, t-(r/a))}{r} r^2 \sin\theta \, d\theta \, d\varphi \, dr \, .$$

If, instead of the spherical, we now introduce the rectangular coordinates

$$\xi = x+\alpha r \, , \qquad \eta = y+\beta r \, , \qquad \zeta = z+\gamma r$$

and remember that $\alpha^2+\beta^2+\gamma^2 = 1$, we obtain

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2} \, ,$$

and the expression for $u(x, y, z, t)$ can finally be written in the form

$$u(x, y, z, t) = \frac{1}{4\pi a^2} \iiint_{D_{at}} \frac{g(\xi, \eta, \zeta, t-(r/a))}{r} \, d\xi \, d\eta \, d\zeta \, , \tag{75}$$

where D_{at} is a sphere of radius at with center at the point (x, y, z).

The expression shown in eq. (75) is called the *retarded potential*.

We note that, in the integration in eq. (75), g is taken not at the instant of time t being considered, but at the time $t = r/a$. The difference between the two times is that necessary for the process being propagated with velocity a to pass from the point (ξ, η, ζ) to the point (x, y, z).

Let us examine a particular case in which $g(x, y, z, t)$ is a periodic function of time

$$g(x, y, z, t) = g(x, y, z) \, e^{i\omega t} \, ,$$

where ω is the given oscillational frequency.

From eq. (75), we obtain

$$u(x,y,z,t) = \frac{e^{i\omega t}}{4\pi a^2} \iiint\limits_{D_{at}} g(\xi,\eta,\zeta) \frac{e^{-ikr}}{r} \, d\xi \, d\eta \, d\zeta \qquad (k = \omega/a) . \tag{76}$$

Suppose now that the function $g(x,y,z)$ is equal to zero outside some finite region (R). If the point $M(x,y,z)$ lies outside the region (R), the integral on the right side of eq. (76) will be equal to zero for instants of time $t < d/a$, where d is the shortest distance from M to (R), and, consequently, eq. (76) will give $u(x,y,z,t) = 0$ (that is, a state of rest at the point M). Beginning at the time $t = d/a$, oscillations will take place at the point M. Later, after the time $t > D/a$, where D is the greatest distance from M to (R), the region D_{at} will contain the entire region (R) and the integral on the right side of eq. (76) will be constant and will be reduced to an integral taken over the entire region (R). Thus, at the point M, beginning at the time $t = D/a$, periodic oscillations will be set up with amplitude

$$v(x,y,z) = \frac{1}{4\pi a^2} \iiint\limits_{(R)} g(\xi,\eta,\zeta) \frac{e^{-ikr}}{r} \, d\xi \, d\eta \, d\zeta , \tag{77}$$

so that

$$u(x,y,z,t) = v(x,y,z) \, e^{i\omega t} .$$

Substituting expression (76) for u (with $t > D/a$) in the homogeneous wave equation (66), we see that the function $v(x,y,z)$ satisfies the equation

$$\nabla^2 v + k^2 v = -\frac{1}{a^2} g(x,y,z) .$$

In the same way, we may obtain the solution to the inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + g(x,y,t) \tag{78}$$

with initial conditions

$$u \big|_{t=0} = 0 , \qquad \frac{\partial u}{\partial t} \bigg|_{t=0} = 0 . \tag{79}$$

This solution is obtained in the form

$$u(x,y,t) = \frac{1}{2\pi a} \int_0^t \left[\iint\limits_{\rho \le a(t-\tau)} \frac{g(\xi,\eta,\zeta) \, d\xi \, d\eta}{\sqrt{a^2(t-\tau)^2 - \rho^2}} \right] d\tau , \tag{80}$$

where

$$\rho^2 = (x-\xi)^2 + (y-\eta)^2 .$$

In the case of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + g(x,t) , \tag{81}$$

the solution satisfying the above initial conditions is obviously

$$u(x, t) = \frac{1}{2a}\int_0^t \left[\int_{x-a(t-\tau)}^{x+a(t-\tau)} g(\xi, \tau)\, d\xi\right] d\tau \tag{82}$$

9. *A uniqueness theorem*

Let us show the uniqueness of the solution to the wave equation with given initial conditions. For convenience, we shall assume that $a = 1$ (this is equivalent to replacing t by t/a in the wave equation). For greater clarity, let us take the case of three independent variables, that is, the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{83}$$

with the initial conditions

$$u|_{t=0} = f(x, y)\ , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x, y)\ . \tag{84}$$

We shall show the uniqueness of the solution to the Cauchy problem (83) - (84), assuming that the solution $u(x, y, t)$ has continuous derivatives up to the second order, inclusive.

Suppose that $u_1(x, y, t)$ and $u_2(x, y, t)$ are two solutions to eq. (83), both satisfying the initial conditions (84). Then, the difference

$$u(x, y, t) = u_1(x, y, t) - u_2(x, y, t)$$

will also satisfy the wave equation (83) and the initial conditions

$$u|_{t=0} = 0\ , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0\ . \tag{85}$$

The uniqueness theorem will be proved if we can show that u is identically equal to zero for arbitrary (x, y) and for arbitrary $t > 0$.

Let us consider the three-dimensional space (x, y, t), and choose an arbitrary point $N(x_0, y_0, t_0)$ in it, where $t_0 > 0$. Let us draw a cone

$$(x - x_0)^2 + (y - y_0)^2 - (t - t_0)^2 = 0$$

with the point N as vertex, and extend the cone until it intersects the plane $t = 0$. Let us draw another plane $t = t_1$, where $0 < t_1 < t_0$, and let D be the region bounded by the lateral surface Γ of the cone and by the portions of the planes $t = 0$ and $t = t_1$ that are within the cone (D is a truncated circular cone). Let us denote by σ_0 and σ_1 the lower and upper bases, respectively, of the truncated cone.

It is easy to verify the following identity:

$$2\frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial}{\partial t}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right] - 2\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right) - 2\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial y}\right)$$

Let us integrate this identity over the region D. The integral on the left side is equal to zero, since u is a solution to eq. (83). Let us transform the integral on the right side into an integral over the surface of the region D by using Ostrogradski's formula (Green's theorem). Then, we obtain

$$\iint_{\Gamma}\left\{\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right]\cos(nt)-2\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\cos(nx)\right.$$
$$\left.-2\frac{\partial u}{\partial t}\frac{\partial u}{\partial y}\cos(ny)\right\}ds+\iint_{\sigma_0}\{\ldots\}\,ds+\iint_{\sigma_1}\{\ldots\}\,ds=0\,. \qquad (86)$$

On the lower base σ_0 of the truncated cone D the function u and all its first-order partial derivatives will, because of the initial conditions (85), be equal to zero. Consequently, the second integral in eqs. (86) will be equal to zero. On the upper base σ_1, we have

$$\cos(nx)=\cos(ny)=0\,,\qquad \cos(nt)=1\,.$$

On the lateral surface Γ of the cone, the direction cosines of the normal satisfy the equation

$$\cos^2(nt)-\cos^2(nx)-\cos^2(ny)=0\,.$$

Thus, eq. (86) can be rewritten in the form

$$\iint_{\Gamma}\frac{1}{\cos(nt)}\left\{\left[\frac{\partial u}{\partial x}\cos(nt)-\frac{\partial u}{\partial t}\cos(nx)\right]^2+\left[\frac{\partial u}{\partial y}\cos(nt)-\frac{\partial u}{\partial t}\cos(ny)\right]^2\right\}ds$$
$$+\iint_{\sigma_1}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right]ds=0\,.$$

On the lateral surface Γ, we have $\cos(nt)=\frac{1}{2}\sqrt{2}$ and, consequently, the first integral is non-negative, and therefore,

$$\iint_{\sigma_1}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right]ds=0\,.$$

From this, it follows that at all points within the entire cone with vertex $N(x_0, y_0, t_0)$, the first-order partial derivatives of the function u will be equal to zero and, consequently, the function u itself will be constant. On the lower base of the cone, u will be zero because of (85), and consequently, $u=0$ at the point $N(x_0, y_0, t_0)$.

Problems

1. Suppose that a uniformly distributed force acts on a gas. Denote the force per unit mass by F. Show that the condensation $s(x, y, z, t)$ satisfies the differential equation

$$\frac{\partial^2 s}{\partial t^2}=a^2\left(\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}+\frac{\partial^2 s}{\partial z^2}\right)-\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right).$$

2. Find the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),$$

that satisfies the initial conditions

$$u|_{t=0} = \begin{cases} u_0 & \text{within a sphere of radius } R, \\ 0 & \text{outside this sphere,} \end{cases} \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 .$$

Answer:

$$u(r, t) = \begin{cases} u_0 & \text{for} \quad 0 \leqslant t < \dfrac{R - r}{a} \\ u_0 \dfrac{r - at}{2r} & \text{for} \quad \dfrac{R - r}{a} < t < \dfrac{R + r}{a} \\ 0 & \text{for} \quad \dfrac{R + r}{a} < t < \infty , \end{cases} \qquad (0 < r < R)$$

$$u(r, t) = \begin{cases} 0 & \text{for} \quad 0 \leqslant t < \dfrac{r - R}{a} \\ u_0 \dfrac{r - at}{2r} & \text{for} \quad \dfrac{r - R}{a} < t < \dfrac{r + R}{a} \\ 0 & \text{for} \quad \dfrac{r + R}{a} < t < \infty . \end{cases} \qquad (R < r < \infty)$$

3. Find the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

in the half-space $z > 0$ that satisfies the initial conditions

$$u|_{t=0} = f(x, y, z) , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x, y, z) \qquad (z \geqslant 0) ,$$

and the boundary condition

$$u|_{z=0} = 0 \quad \text{or} \quad \left.\frac{\partial u}{\partial z}\right|_{z=0} = 0 .$$

Method of solution: Extend the initial conditions throughout all space by extending the functions f and F as odd functions (for $u|_{z=0} = 0$):

$$f(x, y, z) = -f(x, y, -z) , \qquad F(x, y, z) = -F(x, y, -z)$$

or as even functions (for $\partial u/\partial z|_{z=0} = 0$):

$$f(x, y, z) = f(x, y, -z) , \qquad F(x, y, z) = F(x, y, -z) ,$$

and then use Poisson's formula.

Chapter VII

FUNCTIONALLY INVARIANT SOLUTIONS

1. *Functionally invariant solutions to equations of the hyperbolic type with two independent variables*

1. Let us examine the hyperbolic equations *

$$L(u) = A\,\frac{\partial^2 u}{\partial x^2} + 2B\,\frac{\partial^2 u}{\partial x\,\partial y} + C\,\frac{\partial^2 u}{\partial y^2} + D\,\frac{\partial u}{\partial x} + E\,\frac{\partial u}{\partial y} = 0\,, \tag{1}$$

where the coefficients A, B, C, D, and E are functions of x and y with continuous first- and second-order derivatives.

We shall call the function $u(x, y)$ that appears in the solution to eq. (1) a *functionally invariant* solution in some domain D of the real variables x and y if an arbitrary twice-differentiable function $F(u)$ is also a solution to eq. (1).

From the definition of a functionally invariant solution, we have

$$A\,\frac{\partial^2 F(u)}{\partial x^2} + 2B\,\frac{\partial^2 F(u)}{\partial x\,\partial y} + C\,\frac{\partial^2 F(u)}{\partial y^2} + D\,\frac{\partial F(u)}{\partial x} + E\,\frac{\partial F(u)}{\partial y} = 0$$

or, after certain transformations,

$$F''(u)\left[A\left(\frac{\partial u}{\partial x}\right)^2 + 2B\,\frac{\partial u}{\partial x}\,\frac{\partial u}{\partial y} + C\left(\frac{\partial u}{\partial y}\right)^2\right] + F'(u)\left[A\,\frac{\partial^2 u}{\partial x^2} + 2B\,\frac{\partial^2 u}{\partial x\,\partial y} + C\,\frac{\partial^2 u}{\partial y^2} + D\,\frac{\partial u}{\partial x} + E\,\frac{\partial u}{\partial y}\right] = 0\,.$$

Obviously, for this equation to be satisfied, it is necessary and sufficient that the function $u(x, y)$ satisfy simultaneously the two equations

$$A\,\frac{\partial^2 u}{\partial x^2} + 2B\,\frac{\partial^2 u}{\partial x\,\partial y} + C\,\frac{\partial^2 u}{\partial y^2} + D\,\frac{\partial u}{\partial x} + E\,\frac{\partial u}{\partial y} = 0\,, \tag{1}$$

$$A\left(\frac{\partial u}{\partial x}\right)^2 + 2B\,\frac{\partial u}{\partial x}\,\frac{\partial u}{\partial y} + C\left(\frac{\partial u}{\partial y}\right)^2 = 0\,. \tag{2}$$

Thus, a functionally invariant solution $u(x, y)$ satisfies both eq. (1) and eq. (2). Eq. (2) can be broken down into two equations with real coefficients

* N.P.Erugin, *Funktsional'no-invariantnye resheniya uravnenii vtorogo poryadka s dvumya nezavisimymi peremennymi* (functionally invariant solutions to second-order equations with two independent variables), Uchenye zapiski, Leningrad State University, Mathematical Sciences Series, No. 16 (1949).

$$\frac{\partial u}{\partial x} + \alpha_1 \frac{\partial u}{\partial y} = 0 , \qquad \frac{\partial u}{\partial x} + \alpha_2 \frac{\partial u}{\partial y} = 0 , \tag{3}$$

where $\alpha_1(x, y)$ and $\alpha_2(x, y)$ are roots of the equation

$$A\alpha^2 - 2B\alpha + C = 0 . \tag{4}$$

We may assume without loss of generality that $A \neq 0$.

Suppose that $\xi(x, y)$ and $\eta(x, y)$ are solutions to eq. (2), that is, that

$$\frac{\partial \xi}{\partial x} + \alpha_1 \frac{\partial \xi}{\partial y} = 0 , \tag{5}$$

$$\frac{\partial \eta}{\partial x} + \alpha_2 \frac{\partial \eta}{\partial y} = 0 . \tag{6}$$

Introducing, in place of x and y, the new independent variables ξ and η, where $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$, we reduce eq. (1) to the form

$$2\bar{B} \frac{\partial^2 u}{\partial \xi \, \partial \eta} + \bar{D} \frac{\partial u}{\partial \xi} + \bar{E} \frac{\partial u}{\partial \eta} = 0 , \tag{7}$$

where

$$\begin{gathered} \bar{B}(\xi, \eta) = A \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x}\right) + C \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} , \\ \bar{D} = L(\xi) , \qquad \bar{E} = L(\eta) \end{gathered} \tag{8}$$

The function $\xi(x,y)$ will be a functionally invariant solution to eq. (1) if

$$L(\xi) = 0 . \tag{9}$$

Let us find those conditions which the coefficients in eq. (1) must satisfy for eqs. (5) and (9) to be satisfied simultaneously. Let us differentiate eq. (5) separately with respect to x and y. Combining the resulting equations with eq. (1), in which we set $u = \xi(x, y)$ (denoting the results of this substitution by $L(\xi)$), we obtain the following system of equations:

$$\begin{gathered} \frac{\partial^2 \xi}{\partial x^2} + \alpha_1 \frac{\partial^2 \xi}{\partial x \, \partial y} + \frac{\partial \alpha_1}{\partial x} \frac{\partial \xi}{\partial y} = 0 , \qquad \frac{\partial^2 \xi}{\partial x \, \partial y} + \alpha_1 \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial \alpha_1}{\partial y} \frac{\partial \xi}{\partial y} = 0 , \\ A \frac{\partial^2 \xi}{\partial x^2} + 2B \frac{\partial^2 \xi}{\partial x \, \partial y} + C \frac{\partial^2 \xi}{\partial y^2} + D \frac{\partial \xi}{\partial x} + E \frac{\partial \xi}{\partial y} - L(\xi) = 0 . \end{gathered} \tag{10}$$

The determinant of the coefficients of the second-order derivatives of $\xi(x, y)$ in the system (10) is equal to

$$\Delta = A\alpha_1{}^2 - 2B\alpha_1 + C = 0$$

by virtue of eq. (4). It is easy to show that the rank of the matrix of the system (10) is equal to 2. For us to be able to find the second derivatives in this system, it is necessary and sufficient that

$$\begin{vmatrix} 1 & \alpha_1 & -\dfrac{\partial\alpha_1}{\partial x}\dfrac{\partial\xi}{\partial y} \\ 0 & 1 & -\dfrac{\partial\alpha_1}{\partial y}\dfrac{\partial\xi}{\partial y} \\ A & 2B & L(\xi) - D\dfrac{\partial\xi}{\partial x} - E\dfrac{\partial\xi}{\partial y} \end{vmatrix} = 0 .$$

Expanding the determinant, we obtain

$$L(\xi) = (A\alpha_1 - 2B)\frac{\partial\alpha_1}{\partial y}\frac{\partial\xi}{\partial y} - A\frac{\partial\alpha_1}{\partial x}\frac{\partial\xi}{\partial y} + D\frac{\partial\xi}{\partial x} + E\frac{\partial\xi}{\partial y}$$

or, on the basis of eq. (5), we have

$$L(\xi) = \left[(A\alpha_1 - 2B)\frac{\partial\alpha_1}{\partial y} - A\frac{\partial\alpha_1}{\partial x} - D\alpha_1 + E\right]\frac{\partial\xi}{\partial y} . \tag{11}$$

In the same manner, we obtain

$$L(\eta) = \left[(A\alpha_2 - 2B)\frac{\partial\alpha_2}{\partial y} - A\frac{\partial\alpha_2}{\partial x} - D\alpha_2 + E\right]\frac{\partial\eta}{\partial y} . \tag{12}$$

It is easy to see from eqs. (11) and (12) that if α_1 and α_2 are solutions to the equation

$$(A\alpha - 2B)\frac{\partial\alpha}{\partial y} - A\frac{\partial\alpha}{\partial x} - D\alpha + E = 0 , \tag{13}$$

then eq. (1) has two functionally invariant solutions, determined by eqs. (5) and (6). In this case, eq. (7) takes the form

$$\frac{\partial^2 u}{\partial\xi\,\partial\eta} = 0$$

and the general solution to eq. (1) is of the form

$$u = \varphi(\xi) + \psi(\eta) ,$$

where φ and ψ are arbitrary functions.

If only α_1 (α_2) is a solution to eq. (13), then eq. (1) will have one functionally invariant solution. In this case, eq. (7) is reduced to the form

$$2\bar{B}\frac{\partial^2 u}{\partial\xi\,\partial\eta} + \bar{E}\frac{\partial u}{\partial\eta} = 0 . \tag{14}$$

Integrating eq. (14), we obtain

$$u = \int \psi(\eta)\exp\left[-\int\frac{\bar{E}}{2\bar{B}}\,d\xi\right]d\eta + \varphi(\xi) ,$$

where $\varphi(\xi)$ and $\psi(\eta)$ are arbitrary functions. Returning to the original variables x and y, we obtain the general solution to eq. (1).

However, if α_1 and α_2 are not solutions to eq. (13), eq. (1) does not have any functionally invariant solutions.

2. Let us now examine the hyperbolic equation

$$A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \, \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Gu = 0 \,. \tag{15}$$

We again take the solutions $\xi(x, y)$ and $\eta(x, y)$ of eqs. (5) and (6) as our new independent variables. Then, eq. (15) is transformed into canonical form:

$$\frac{\partial^2 u}{\partial \xi \, \partial \eta} + a \frac{\partial u}{\partial \xi} + b \frac{\partial u}{\partial \eta} + cu = 0 \,, \tag{16}$$

where

$$a(\xi, \eta) = \frac{\bar{D}}{2\bar{B}} \,, \qquad b(\xi, \eta) = \frac{\bar{E}}{2\bar{B}} \,, \qquad c(\xi, \eta) = \frac{G}{2\bar{B}} \,.$$

The coefficients $\bar{B}$, $\bar{D}$, and $\bar{E}$ are defined by eqs. (8).

We wish to find a solution to eq. (16) in the form

$$u = v \cdot w \,, \tag{17}$$

where v and w are, as yet, arbitrary functions. Substituting eq. (17) into eq. (16), we obtain

$$w \frac{\partial^2 v}{\partial \xi \, \partial \eta} + \left(\frac{\partial w}{\partial \eta} + aw\right) \frac{\partial v}{\partial \xi} + \left(\frac{\partial w}{\partial \xi} + bw\right) \frac{\partial v}{\partial \eta} + \left(\frac{\partial^2 w}{\partial \xi \, \partial \eta} + a \frac{\partial w}{\partial \xi} + b \frac{\partial w}{\partial \eta} + cw\right) v = 0 \,. \tag{18}$$

Let us choose the function w so that

$$\frac{\partial w}{\partial \eta} + aw = 0 \,, \tag{19}$$

that is,

$$w = \varphi(\xi) \exp\left[-\int a \, d\eta\right] . \tag{20}$$

Substituting eq. (20) into the equation

$$\frac{\partial w}{\partial \xi} + bw = 0 \,,$$

we obtain, after performing the differentiation and dividing by $\exp\left[-\int a \, d\eta\right]$,

$$\varphi'(\xi) - \varphi(\xi) \left[\int \frac{\partial a}{\partial \xi} \, d\eta - b\right] = 0 \,. \tag{21}$$

We can then find $\varphi(\xi)$, provided $\int (\partial a/\partial \xi) d\eta - b$ is independent of η, that is, if

$$\frac{\partial a}{\partial \xi} = \frac{\partial b}{\partial \eta} \,. \tag{22}$$

Thus, if condition (22) is satisfied, the function w defined by eq. (20) makes the coefficients of $\partial v/\partial \xi$ and $\partial v/\partial \eta$ vanish in eq. (18). In this case, eq. (18), after we divide by w, becomes

$$\frac{\partial^2 v}{\partial \xi \, \partial \eta} - \left(\frac{\partial a}{\partial \xi} + ab - c\right) v = 0 \,. \tag{23}$$

If, in addition to condition (22), we have

$$\frac{\partial a}{\partial \xi} + ab - c = 0 \ , \tag{24}$$

then

$$v = \psi_1(\xi) + \psi_2(\eta)$$

and on the basis of (17) and (20), the general solution to eq. (16) will be

$$u = \varphi(\xi) \exp\left[- \int a \, d\eta\right] \left[\psi_1(\xi) + \psi_2(\eta)\right] \ , \tag{25}$$

where $\varphi(\xi)$ is determined from eq. (21) and $\psi_1(\xi)$ and $\psi_2(\eta)$ are arbitrary functions.

Now suppose that condition (22) is not satisfied, but that condition (24) is satisfied. Then, eq. (18) takes the following form:

$$\frac{\partial^2 v}{\partial \xi \, \partial \eta} + w(\xi, \eta) \frac{\partial v}{\partial \eta} = 0 \ , \tag{26}$$

where

$$w(\xi, \eta) = b(\xi, \eta) - \int \frac{\partial a}{\partial \xi} \, d\eta + \frac{\varphi'(\xi)}{\varphi(\xi)} \tag{27}$$

and $\varphi(\xi)$ is an arbitrary function.

Integrating eq. (26), we obtain

$$v = \int \psi_2(\eta) \exp\left[- \int w(\xi, \eta) \, d\xi\right] d\eta + \psi_1(\xi)$$

and, taking account of (17) and (20), we obtain

$$u = \varphi(\xi) \exp\left[- \int a \, d\eta\right] \left\{\int \psi_2(\eta) \exp\left[- \int w(\xi, \eta) \, d\xi\right] d\eta + \psi_1(\xi)\right\} \ , \tag{28}$$

where ψ_1 and ψ_2 are arbitrary functions.

If condition (24) is not satisfied, but if

$$\frac{\partial b}{\partial \eta} + ab - c = 0 \ , \tag{29}$$

we obtain an analogous result.

Example 1. Consider the equation

$$\frac{\partial^2 u}{\partial x^2} + 2 \cos x \frac{\partial^2 u}{\partial x \, \partial y} - \sin^2 x \frac{\partial^2 u}{\partial y^2} - \sin x \frac{\partial u}{\partial y} = 0 \ . \tag{30}$$

In accordance with the general theory, we set up eq. (4):

$$\alpha^2 - 2 \cos x \alpha - \sin^2 x = 0 \ .$$

Its roots are

$$\alpha_1 = \cos x + 1 \ , \qquad \alpha_2 = \cos x - 1 \ .$$

It is easy to verify that α_1 and α_2 satisfy eq. (13) and that, consequently, eq. (30) has two functionally invariant solutions. To find the functionally invariant solutions, we set up eqs. (5) and (6):

$$\frac{\partial \xi}{\partial x} + (1 + \cos x)\frac{\partial \xi}{\partial y} = 0\ , \qquad \frac{\partial \eta}{\partial x} + (\cos x - 1)\frac{\partial \eta}{\partial y} = 0\ .$$

It is easy to see that these equations have the solutions

$$\xi = x + \sin x - y\ , \qquad \eta = x - \sin x + y\ .$$

These are the two solutions to eq. (30). Thus, the general solution to eq. (30) can be written in the form

$$u = \varphi(x + \sin x - y) + \psi(x - \sin x + y)\ ,$$

where φ and ψ are arbitrary functions.

We note that in finding the general solution to eq. (30) we are not required to reduce it to canonical form.

Example 2. Consider the equation

$$\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} + xyu = 0\ . \tag{31}$$

It is easy to verify that conditions (22) and (24) are satisfied. Since in our example

$$\int \frac{\partial a}{\partial x}\,dy - b = -x,$$

eq. (21) takes the form

$$\varphi'(x) + x\varphi(x) = 0\ ,$$

and hence

$$\varphi(x) = C\,e^{-\frac{1}{2}x^2}\ ,$$

and eq. (25) yields the general solution to eq. (31):

$$u(x,y) = e^{-\frac{1}{2}(x^2+y^2)}\,[\psi_1(x) + \psi_2(y)]\ , \tag{32}$$

where $\psi_1(x)$ and $\psi_2(x)$ are arbitrary functions.

2. *Functionally invariant solutions to the wave equation*

For the wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}, \tag{33}$$

Academicians V. I. Smirnov and S. L. Sobolev have obtained a class of functionally invariant solutions given by the formula

$$l(u)t + m(u)x + n(u)y - k(u) = 0\ , \tag{34}$$

where

$$l^2(u) = a^2[m^2(u) + n^2(u)] \tag{35}$$

The simplest functionally invariant solutions to the wave equation (33) are obtained if we assume l, m, and n are constant and set the function $k(u) = u$. Then, formula (34) takes the form

$$u = lt + mx + ny\ , \tag{36}$$

where

$$(m^2+n^2)\,a^2 = l^2 .$$

Consequently,

$$u = f(mx+ny+lt) \tag{37}$$

will also be a solution to eq. (33) for an arbitrary function f. If the numbers l, m, and n are all real, we have a so-called plane wave, which is the simplest solution to the wave equation. However, if any of the coefficients l, m, and n are complex, then we shall obtain an essentially new solution, which we shall call a complex plane wave.

The Smirnov-Sobolev formula (34), as Sobolev has shown, gives all the functionally invariant solutions to the wave equation in two-dimensional space.

In the case of the three-dimensional wave equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} ,$$

N. P. Erugin * has shown that all of the real and some of the complex functionally invariant solutions are given by the Smirnov-Sobolev formula:

$$m(u)x + n(u)y + p(u)z + l(u)t - k(u) = 0 ,$$

where

$$l^2(u) = a^2[m^2(u) + n^2(u) + p^2(u)] ,$$

and he has obtained two new classes of complex functionally invariant solutions

$$u = \varphi(x + iz\sqrt{1+C^2} + iact,\ y + Cz + a\sqrt{1+C^2}\,t) ,$$

where φ is an arbitrary analytic function and C is an arbitrary constant. We shall not introduce the second class of functionally independent solutions because of their complicated form.

Functionally invariant solutions to the wave equation have widespread applications in the theory of diffraction and in the problem of the reflection of elastic vibrations from a plane boundary [7]. We note that the Smirnov-Sobolev formula gives only a certain class of solutions to the wave equation but, as it turns out, this class contains solutions of great physical importance. By using this class of solutions, one can reduce many problems related to the reflection and diffraction of waves to a form that is convenient for calculation. Without going into details (such a digression would require a great deal of space), we shall discuss very briefly the application of functionally invariant solutions of the wave to the single and very simple problem of the reflection of elastic vibrations from a plane boundary.

* N. P. Erugin, *O funktsional'no-invariantnikh resheniyakh* (functionally invariant solutions), Uchenye zapiski, Leningrad State University, Mathematical Sciences Series, No. 15 (1948).

3. *The problem of reflection of plane elastic waves*

1. Let us consider the problem of the propagation of elastic vibrations in a half-space 7). We choose the coordinate axes so that the xz-plane is the boundary of the half-space and the y-axis is directed along the normal, into an elastic medium. For simplicity, we shall confine ourselves to a plane problem in the theory of elasticity. In this case, the components u and v of the displacement vector can be expressed by the formulae

$$u = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x}, \tag{38}$$

where the function φ is called the *vector potential* and the function ψ is called the *scalar potential*. These potentials must satisfy the wave equations

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = \frac{1}{a^2}\frac{\partial^2 \varphi}{\partial t^2}, \tag{39}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{1}{b^2}\frac{\partial^2 \psi}{\partial t^2}, \tag{40}$$

$$a^2 = \frac{\lambda + 2\mu}{\rho}, \qquad b^2 = \frac{\mu}{\rho}, \tag{41}$$

where ρ is the density of the medium and λ and μ are the Lamé elastic constants. Since λ and μ are positive, a^2 is greater than b^2.

Let us consider the case in which the boundary is free of stresses. Then, the boundary conditions are of the form

$$X_y|_{y=0} = 0, \qquad Y_y|_{y=0} = 0, \qquad Z_y|_{y=0} = 0,$$

where X_y, Y_y, and Z_y are the components of the stress vector acting on an area perpendicular to the y-axis.

The third condition is satisfied automatically, and the left sides of the first two can be expressed in terms of the potentials in accordance with the well-known formulae 7):

$$X_y|_{y=0} = \mu\left[2\frac{\partial^2 \varphi}{\partial x\,\partial y} + \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2}\right]\Bigg|_{y=0} = 0,$$

$$Y_y|_{y=0} = \left[\lambda\frac{\partial^2 \varphi}{\partial x^2} + (\lambda + 2\mu)\frac{\partial^2 \varphi}{\partial y^2} - 2\mu\frac{\partial^2 \psi}{\partial x\,\partial y}\right]\Bigg|_{y=0} = 0$$

or, by considering eqs. (41),

$$\left.2\frac{\partial^2 \varphi}{\partial x\,\partial y} + \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2}\right|_{y=0} = 0,$$

$$\left.a^2\frac{\partial^2 \varphi}{\partial y^2} + (a^2 - 2b^2)\frac{\partial^2 \varphi}{\partial x^2} - 2b^2\frac{\partial^2 \psi}{\partial x\,\partial y}\right|_{y=0} = 0. \tag{42}$$

Thus, our problem is reduced to integrating eqs. (39) and (40) with the boundary conditions (42).

2. *Plane waves:* In the plane problem of the propagation of elastic vibrations, that solution to eqs. (39) and (40) in which the vector potential is identically equal to zero and the scalar potential is given by a formula of the form (37) with real coefficients is called a *longitudinal plane wave.*

A solution to eqs. (39) and (40) in which the scalar potential is identically equal to zero, and the vector potential is given in the form (37) with real coefficients, is called a *transverse plane wave.*

Since, obviously, the coefficient $l \neq 0$, we can, without loss of generality, assume that it is equal to unity. If we set $m = -\theta$ for convenience, we can write the longitudinal plane wave in the form

$$\varphi = f\left(t - \theta x \pm \sqrt{\frac{1}{a^2} - \theta^2}\, y\right), \quad \psi = 0, \tag{43}$$

and the transverse plane wave in the form

$$\varphi = 0, \quad \psi = f\left(t - \theta x \pm \sqrt{\frac{1}{b^2} - \theta^2}\, y\right). \tag{44}$$

If a solution is of the form

$$\varphi = f\left(t - \theta x + \sqrt{\frac{1}{a^2} - \theta^2}\, y\right), \quad \psi = 0,$$

we shall call it a longitudinal wave travelling in the direction of the boundary of the half-space $y > 0$. We shall call a solution of the form

$$\varphi = f\left(t - \theta x - \sqrt{\frac{1}{a^2} - \theta^2}\, y\right), \quad \psi = 0$$

a longitudinal wave travelling in the direction away from the boundary.

We shall use the same terminology in the case of a transverse wave. It is easy to see that neither a wave travelling toward the boundary nor a wave travelling away from it will satisfy the homogeneous boundary conditions (42). However, the solution obtained by superimposing these waves may satisfy these conditions.

Let us consider the following problem. Given a wave that is travelling toward the boundary (called an ordinary incident wave), determine the two reflected waves travelling away from the boundary – the longitudinal and the transverse – in such a way that the sum of the incident wave and the two reflected waves will satisfy the boundary conditions (42).

3. *The reflection of a plane longitudinal wave from a free boundary:* Suppose that an incident longitudinal wave is of the form

$$\varphi_1 = f\left(t - \theta x + \sqrt{\frac{1}{a^2} - \theta^2}\, y\right), \quad |\theta| < \frac{1}{a}. \tag{45}$$

Let us seek reflected waves of the form

$$\varphi_2 = Af\left(t - \theta x - \sqrt{\frac{1}{a^2} - \theta^2}\, y\right), \quad \psi = Bf\left(t - \theta x - \sqrt{\frac{1}{b^2} - \theta^2}\, y\right). \tag{46}$$

Substituting these expressions for $\varphi = \varphi_1 + \varphi_2$ and ψ into the boundary conditions (42), we obtain

$$\left[-2\theta\sqrt{\frac{1}{a^2}-\theta^2}\,(1-A) + \left(\frac{1}{b^2}-2\theta^2\right)B\right]f''(t-\theta x) = 0\,,$$

$$\left[(1-2b^2\theta^2)(1+A) - 2b^2\theta\sqrt{\frac{1}{b^2}-\theta^2}\,B\right]f''(t-\theta x) = 0\,,$$

from which, by assuming that $f'' \neq 0$, we find the constants A and B:

$$A = \frac{-\left(2\theta^2-\frac{1}{b^2}\right)^2 + 4\theta^2\sqrt{\frac{1}{a^2}-\theta^2}\sqrt{\frac{1}{b^2}-\theta^2}}{\left(2\theta^2-\frac{1}{b^2}\right)^2 + 4\theta^2\sqrt{\frac{1}{a^2}-\theta^2}\sqrt{\frac{1}{b^2}-\theta^2}}\,,$$

$$B = \frac{-4\theta\left(2\theta^2-\frac{1}{b^2}\right)\sqrt{\frac{1}{a^2}-\theta^2}}{\left(2\theta^2-\frac{1}{b^2}\right)^2 + 4\theta^2\sqrt{\frac{1}{a^2}-\theta^2}\sqrt{\frac{1}{b^2}-\theta^2}}$$

We note several geometrical consequences of formulae (45) and (46). The angle θ_1 that the normal to the plane

$$t - \theta x + \sqrt{\frac{1}{a^2}-\theta^2}\,y = \text{constant}$$

makes with the negative y-axis is called the *angle of incidence*. The angles θ_2 and θ_3 that the normals to the planes

$$t - \theta x - \sqrt{\frac{1}{a^2}-\theta^2}\,y = \text{constant}\,, \qquad t - \theta x - \sqrt{\frac{1}{b^2}-\theta^2}\,y = \text{constant}$$

make, respectively, with the positive y-axis are called the *angles of reflection*. Then, for a reflected longitudinal wave, the angle of incidence is equal to the angle of reflection $\theta_1 = \theta_2$, and for a reflected transverse wave, the ratio of the sine of the angle of incidence to the sine of the angle of reflection is equal to the ratio of the velocity of the longitudinal wave to the velocity of the transverse wave:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{a\theta}{b\theta} = \frac{a}{b}\,.$$

4. *The reflection of transverse waves:* Suppose that an incident transverse wave is of the form

$$\psi_1 = f\left(t - \theta x + \sqrt{\frac{1}{b^2}-\theta^2}\,y\right), \tag{47}$$

where $|\theta| < 1/a$. Let us seek reflected waves of the form

$$\varphi = Cf\left(t - \theta x - \sqrt{\frac{1}{a^2}-\theta^2}\,y\right), \quad \psi_2 = Df\left(t - \theta x - \sqrt{\frac{1}{b^2}-\theta^2}\,y\right). \tag{48}$$

Substituting these values of φ and $\psi = \psi_1 + \psi_2$ into the boundary conditions (42), we obtain

$$\left[2\theta \sqrt{\frac{1}{a^2} - \theta^2}\, C + \left(\frac{1}{b^2} - 2\theta^2\right)(1 + D)\right] f''(t - \theta x) = 0 ,$$

$$\left[(1 - 2b^2\theta^2)\, C + 2b^2\theta \sqrt{\frac{1}{b^2} - \theta^2}(1 - D)\right] f''(t - \theta x) = 0 ,$$

from which, by assuming $f'' \neq 0$, we can find the constants C and D:

$$C = \frac{4\theta \sqrt{\frac{1}{b^2} - \theta^2}\left(2\theta^2 - \frac{1}{b^2}\right)}{\left(2\theta^2 - \frac{1}{b^2}\right)^2 + 4\theta^2 \sqrt{\frac{1}{a^2} - \theta^2}\sqrt{\frac{1}{b^2} - \theta^2}} ,$$

$$D = \frac{-\left(2\theta^2 - \frac{1}{b^2}\right)^2 + 4\theta^2 \sqrt{\frac{1}{a^2} - \theta^2}\sqrt{\frac{1}{b^2} - \theta^2}}{\left(2\theta^2 - \frac{1}{b^2}\right)^2 + 4\theta^2 \sqrt{\frac{1}{a^2} - \theta^2}\sqrt{\frac{1}{b^2} - \theta^2}} .$$

Chapter VIII

APPLICATION OF THE FOURIER METHOD TO THE STUDY OF FREE VIBRATIONS OF STRINGS AND RODS

1. *The Fourier method for the equation of free vibrations of a string*

The Fourier method or the method of separation of variables is one of the most widely-used methods of solving partial differential equations. We shall explain this method in a number of examples, beginning with the simplest problem of the vibrations of a string fixed at both ends. This problem, as we know, is reduced to solving the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

with the boundary conditions

$$u|_{x=0} = 0 , \qquad u|_{x=l} = 0 \tag{2}$$

and the initial conditions

$$u|_{t=0} = f(x) , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x) \qquad (0 \leqslant x \leqslant l) . \tag{3}$$

First, we wish to find particular solutions to eq. (1) (not identically equal to zero), in the form of a product

$$u(x,t) = X(x)\ T(t) , \tag{4}$$

satisfying the boundary conditions (2).

Substituting eq. (4) into eq. (1), we obtain

$$T''(t)\ X(x) = a^2\ T(t)\ X''(x)$$

or

$$\frac{T''(t)}{a^2\ T(t)} = \frac{X''(x)}{X(x)} . \tag{5}$$

In this last equation, the left side depends only on t and the right side only on x, so that equality is possible only if both sides are independent of both x and t, that is, if they are equal to the same constant. Let us denote this constant by $-\lambda$. Then, we obtain from eq. (5) two ordinary differential equations:

$$T''(t) + a^2 \lambda T(t) = 0 , \tag{6}$$

$$X''(x) + \lambda X(x) = 0 . \tag{7}$$

In order to obtain a non-trivial solution (that is, not identically equal to zero) of the form (4) satisfying the boundary conditions (2), we need to find the non-trivial solutions to eq. (7) that satisfy the boundary conditions

$$X(0) = 0 \ , \qquad X(l) = 0 \ . \tag{8}$$

Thus, we are confronted with the problem of finding those values of the parameter λ for which non-trivial solutions to eq. (7) satisfying the boundary conditions (8) exist.

This problem is often called the Sturm-Liouville problem.

Those values of the parameter λ for which the problem (7) - (8) has non-trivial solutions are called eigenvalues and these solutions themselves are called eigenfunctions.

Let us now find the eigenvalues and eigenfunctions for the problem (7) - (8). We need to consider separately the three cases of $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.

1. If $\lambda < 0$, the general solution to eq. (7) will be of the form

$$X(x) = C_1 \, e^{\sqrt{-\lambda}x} + C_2 \, e^{-\sqrt{-\lambda}x}$$

In satisfying the boundary conditions (8), we obtain

$$C_1 + C_2 = 0 \ , \qquad C_1 \, e^{\sqrt{-\lambda}l} + C_2 \, e^{-\sqrt{-\lambda}l} = 0 \tag{9}$$

Since the determinant of the system (9) is different from zero, C_1 and C_2 must both be equal to zero. Consequently, $X(x)$ is identically equal to zero.

2. For $\lambda = 0$, the general solution to eq. (7) is of the form

$$X(x) = C_1 + C_2 x \, .$$

The boundary conditions (8) give

$$C_1 + C_2 \cdot 0 = 0 \ , \qquad C_1 + C_2 l = 0$$

Hence, C_1 and C_2 are again both equal to zero, and, consequently, $X(x)$ is identically equal to zero.

3. For $\lambda > 0$, the general solution to eq. (7) will be of the form

$$X(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x$$

In satisfying the boundary conditions (8), we obtain

$$C_1 \cdot 1 + C_2 \cdot 0 = 0 \ , \qquad C_1 \cos \sqrt{\lambda} l + C_2 \sin \sqrt{\lambda} l = 0$$

It follows from the first of these equations that $C_1 = 0$ and from the second that $C_2 \sin \sqrt{\lambda} l = 0$. We must assume that $C_2 \neq 0$ because otherwise $X(x)$ is identically equal to zero. Therefore,

$$\sin \sqrt{\lambda} l = 0 \ , \qquad \text{i.e.,} \qquad \sqrt{\lambda} = k\pi/l \ ,$$

where k is an arbitrary integer.

Consequently, non-trivial solutions to the problem (7) - (8) are possible only for values

$$\lambda_k = (k\pi/l)^2 \qquad (k = 1, 2, 3, \dots)$$

To these eigenvalues correspond the eigenfunctions

$$X_k(x) = \sin\frac{k\pi x}{l}.$$

These are completely determined up to a constant factor, which we set equal to unity.

We note that positive and negative values of k that are equal in absolute value give eigenfunctions that differ only by a constant factor. Therefore, it will be sufficient to take only *positive* integral values for k.

For $\lambda = \lambda_k$, the general solution to eq. (6) is of the form

$$T_k(t) = a_k \cos\frac{k\pi at}{l} + b_k \sin\frac{k\pi at}{l},$$

where a_k and b_k are arbitrary constants.

Thus, the functions

$$u_k(x, t) = X_k(x)\, T_k(t) = \left(a_k \cos\frac{k\pi at}{l} + b_k \sin\frac{k\pi at}{l}\right) \sin\frac{k\pi x}{l}$$

satisfy eq. (1) and the boundary conditions (2) for arbitrary values of a_k and b_k.

Since eq. (1) is linear and homogeneous, every sum of a finite number of solutions to eq. (1) will also be a solution. The same is valid also for an infinite series

$$u(x, t) = \sum_{k=1}^{\infty} \left(a_k \cos\frac{k\pi at}{l} + b_k \sin\frac{k\pi at}{l}\right) \sin\frac{k\pi x}{l}, \tag{10}$$

provided it converges, and is twice termwise differentiable with respect to x and t. Since every term in the series (10) satisfies the boundary conditions (2), the sum of the series (that is, the function $u(x, t)$) will also satisfy these conditions. It remains to determine the constants a_k and b_k so as to satisfy the initial conditions (3).

Differentiating the series (10) with respect to t, we obtain

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{\infty} \frac{k\pi a}{l} \left(-\, a_k \sin\frac{k\pi at}{l} + b_k \cos\frac{k\pi at}{l}\right) \sin\frac{k\pi x}{l}. \tag{11}$$

Setting $t = 0$ in (10) and (11), we obtain, on the basis of the initial conditions (3),

$$f(x) = \sum_{k=1}^{\infty} a_k \sin\frac{k\pi x}{l}, \qquad F(x) = \sum_{k=1}^{\infty} \frac{k\pi a}{l}\, b_k \sin\frac{k\pi x}{l}. \tag{12}$$

Eqs. (12) are an expansion of the given functions $f(x)$ and $F(x)$ in a Fourier sine series over the interval $(0, l)$.

The coefficients in the expansions (12) can be calculated by the well-known formulae [1]:

$$a_k = \frac{2}{l}\int_0^l f(x) \sin\frac{k\pi x}{l}\, dx, \qquad b_k = \frac{2}{k\pi a}\int_0^l F(x) \sin\frac{k\pi x}{l}\, dx. \tag{13}$$

Thus, the solution to the problem (1) - (3) is given by the series (10), where a_k and b_k are determined by the formulae (13).

THEOREM. *Suppose that $f(x)$ is twice continuously differentiable on the interval $[0, l]$, that it has a piecewise-continuous third derivative, and that it satisfies the conditions*

$$f(0) = f(l) = 0\ , \qquad f''(0) = f''(l) = 0\ . \tag{14}$$

Suppose also that $F(x)$ is continuously differentiable, that it has a piecewise-continuous second derivative, and that it satisfies the conditions

$$F(0) = F(l) = 0\ . \tag{15}$$

Under these conditions, the function $u(x,t)$ defined by the series (10) *has continuous second derivatives and satisfies eq.* (1), *the boundary conditions* (2), *and the initial conditions* (3). *Also, the series* (10) *can be twice differentiated termwise with respect to x and t, and the series thus obtained will converge absolutely and uniformly for $0 \leq x \leq l$ and arbitrary t.*

Proof: If we integrate eqs. (13) by parts and take eqs. (14) and (15) into consideration, we obtain

$$a_k = -\left(\frac{l}{\pi}\right)^3 \frac{b_k^{(3)}}{k^3}\ , \qquad b_k = -\left(\frac{l}{\pi}\right)^3 \frac{a_k^{(3)}}{k^3}\ , \tag{16}$$

where

$$b_k^{(3)} = \frac{2}{l}\int_0^l f'''(x)\cos\frac{k\pi x}{l}\,dx\ , \qquad a_k^{(2)} = \frac{2}{l}\int_0^l \frac{F''(x)}{a}\sin\frac{k\pi x}{l}\,dx\ . \tag{17}$$

It is known from the theory of trigonometric series [34] that the series

$$\sum_{k=1}^{\infty}\frac{|a_k^{(2)}|}{k}\ , \qquad \sum_{k=1}^{\infty}\frac{|b_k^{(3)}|}{k} \tag{18}$$

converge. Substituting (16) into the series (10), we obtain

$$u(x,t) = -\left(\frac{l}{\pi}\right)^3 \sum_{k=1}^{\infty}\frac{1}{k^3}\left(b_k^{(3)}\cos\frac{k\pi at}{l} + a_k^{(2)}\sin\frac{k\pi at}{l}\right)\sin\frac{k\pi x}{l}\ . \tag{19}$$

This series is dominated by the series

$$\left(\frac{l}{\pi}\right)^3 \sum_{k=1}^{\infty}\frac{1}{k^3}\left(|b_k^{(3)}| + |a_k^{(2)}|\right)\ ,$$

which converges. Consequently, the series (10) converges absolutely and uniformly. By considering (18), we easily see that the series (10) can be twice differentiated termwise with respect to x and t. This proves the theorem.

If the initial functions $f(x)$ and $F(x)$ do not satisfy the conditions of this theorem, there may not be a twice continuously differentiable solution to the mixed problem (1) - (3). However, if $f(x)$ is a continuously differentiable function satisfying the conditions $f(0) = f(l) = 0$ and $F(x)$ is a continuous function such that $F(0) = F(l) = 0$, then the series (10) converges uniformly for $0 \leq x \leq l$ and arbitrary t and defines a continuous function $u(x, t)$.

Consider the function $u(x, t)$ which is the limit of the uniformly converging sequence $u_n(x, t)$ of solutions to eq. (1) which satisfy the boundary conditions (2) and the initial conditions

$$u_n|_{t=0} = f_n(x) , \qquad \left.\frac{\partial u_n}{\partial t}\right|_{t=0} = F_n(x) ,$$

where

$$\lim_{n\to\infty} \int_0^l [f(x) - f_n(x)]^2 \, dx = \lim_{n\to\infty} \int_0^l [F(x) - F_n(x)]^2 \, dx = 0 .$$

We shall call $u(x, t)$ the *generalized solution* to eq. (1) satisfying the conditions (2) and (3).

Under the conditions imposed above on the functions $f(x)$ and $F(x)$, the existence of a generalized solution follows from the fact that the partial sums of the series (10) form a sequence $u_n(x, t)$ that satisfies the required conditions and, hence, the series (10) is a generalized solution. It is easy to show that the generalized solution to the mixed problem (1) - (3) is unique.

Let us now turn to the solution (10) that we have found for the problem (1) - (3). If we introduce the notation

$$a_k = A_k \sin \varphi_k , \qquad b_k = A_k \cos \varphi_k ,$$

this solution can be written in the form

$$u(x, t) = \sum_{k=1}^{\infty} A_k \sin \frac{k\pi x}{l} \sin \left(\frac{k\pi at}{l} + \varphi_k\right) . \tag{20}$$

Every term of this series is a so-called *standing wave*, for which the points of the string perform a harmonic vibrational motion all with the same phase φ_k, with amplitude $A_k \sin (k\pi x/l)$, and with frequency $\omega_k = k\pi a/l$. When a string is vibrating in this manner, it produced a sound whose pitch depends on the frequency of the vibrations ω_k. The frequency of the fundamental (lowest) tone is expressed by the formula $\omega_1 = \pi(T_0/\rho)^{\frac{1}{2}}/l$. The remaining tones, corresponding to frequencies that are multiples of ω_1 are called the *harmonics* *. The first harmonic is the fundamental tone, the second harmonic is the tone with frequency $\omega_2 = 2\omega_1$, and so on.

The solution (20) is a composite of the separate harmonics. Their amplitude (and therefore their effect on the intensity of the sound made by the string) usually decreases rapidly with increase in the number of the harmonic, and their combined effect governs the *quality* (fr. *timbre*) of the sound. At the points

$$x = 0, \ \frac{l}{k}, \ \frac{2l}{k}, \ \ldots , \ \frac{k-1}{k} \, l, \ l$$

the amplitude of the vibrations of the k-th harmonic vanishes because at

* Tones corresponding to higher frequencies than the fundamental are called *overtones*. Overtones whose frequencies are multiples of the fundamental frequency are called *harmonics*.
There are very few vibrational systems with harmonic overtones, but these few are basic for the manufacture of all musical instruments. This is a consequence of the fact that a sound with harmonic overtones gives an especially pleasing effect from a musical standpoint.

these points $\sin(k\pi x/l) = 0$. These points are called the *nodes* of the k-th harmonic. On the other hand, at the points

$$x = \frac{l}{2k}, \frac{3l}{2k}, \ldots, \frac{(2k-1)l}{2k},$$

which are called the *crests* (antinodes), the amplitude of the k-th harmonic attains its maximum value, since $\sin(k\pi x/l)$ has its greatest absolute value at these points.

If we press a vibrating string exactly in the middle (that is, at the crest of its fundamental tone), then the amplitudes not only of that tone but also of all the others that have crests at that point (that is, the odd harmonics) will vanish. On the other hand, this will not effect the even harmonics, which have their nodes at the pressed point. Thus, only the even harmonics remain, and the lowest frequency will be $\omega_2 = 2\pi(T_0/\rho)^{\frac{1}{2}}/l$; the string will produce not its fundamental pitch but a pitch an octave higher (that is, a pitch with twice as many vibrations per second).

2. *The vibration of a plucked string*

Consider a string that is fixed at both ends. Suppose that we pull it up by plucking it at the point $x = c$ and then release it, allowing it to vibrate freely. In this case, the initial conditions will be (fig. 28)

$$u|_{t=0} = f(x) = \begin{cases} \dfrac{h}{c}x, & 0 \leqslant x \leqslant c, \\ \dfrac{h(l-x)}{l-c}, & c \leqslant x \leqslant l, \end{cases} \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0.$$

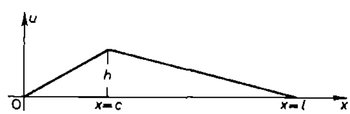

Fig. 28.

Applying formulae (13), we obtain

$$a_k = \frac{2hl^2}{\pi^2 c(l-c)k^2} \sin\frac{k\pi c}{l}, \qquad b_k = 0. \tag{21}$$

Consequently, the displacement of the plucked string will be expressed by the series

$$u(x,t) = \frac{2hl^2}{\pi^2 c(l-x)} \sum_{k=1}^{\infty} \sin\frac{k\pi c}{l} \sin\frac{k\pi x}{l} \cos\frac{k\pi at}{l}. \tag{22}$$

It is clear from eq. (21) that $a_k = 0$ if $\sin(k\pi c/l) = 0$; that is, in the solution (22), those harmonics whose nodes are at the point $x = c$ will be absent. Thus, if the point $x = c$ is at the middle of the string, for example, all the even harmonics will be absent in the solution (22).

3. *The vibrations of a struck string*

Let us now examine the case in which the initial displacements of the string fixed at both ends are equal to zero.

Suppose that, at the initial instant, the string receives a blow from a hammer at the point $x = c$. The head of the hammer is constructed so that the initial velocity given to the string is expressed by the following formula:

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \begin{cases} v_0 \cos \dfrac{\pi(x-c)}{h}, & \text{if} \quad |x-c| < \tfrac{1}{2}h, \\ 0, & \text{if} \quad |x-c| > \tfrac{1}{2}h. \end{cases}$$

By applying formulae (13), we find that

$$a_k = 0, \quad b_k = \frac{4hv_0}{\pi^2 ak} \frac{1}{1-(kh/l)^2} \sin \frac{k\pi c}{l} \cos \frac{k\pi n}{2l}. \tag{23}$$

Substituting these values into the series (10), we obtain the following expression for the desired displacement of the struck string:

$$u(x,t) = \frac{4hv_0}{\pi^2 a} \sum_{k=1}^{\infty} \frac{1}{k} \frac{\sin \dfrac{k\pi c}{l} \cos \dfrac{k\pi h}{2l}}{1-(kh/l)^2} \sin \frac{k\pi x}{l} \cos \frac{k\pi at}{l}. \tag{24}$$

4. *Longitudinal vibrations of a rod*

Let us examine the problem of the longitudinal vibrations of a homogeneous elastic rod of length l when one of its ends $x = 0$ is fixed and the other $x = l$ is free. It was shown in Chapter IV that this problem is reduced to the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad a^2 = \frac{E}{\rho}, \tag{25}$$

with the boundary conditions

$$u|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{x=l} = 0 \tag{26}$$

and the initial conditions

$$u|_{t=0} = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x) \quad (0 \leqslant x \leqslant l). \tag{27}$$

Following the Fourier procedure, we seek particular solutions to eq. (25) of the form

$$u(x,t) = X(x)\, T(t). \tag{28}$$

Substituting eq. (28) into eq. (25), we obtain

$$\frac{T''(t)}{a^2\, T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2,$$

from which we obtain the equations

$$X''(x) + \lambda^2 X(x) = 0 , \tag{29}$$

$$T''(t) + a^2\lambda^2 T(t) = 0 . \tag{30}$$

If the function (28) (not identically equal to zero) is to satisfy the boundary conditions (26), we obviously must require that the conditions

$$X(0) = 0 , \quad X'(l) = 0 \tag{31}$$

be satisfied.

Thus, our problem is to find the eigenvalues for eq. (29) with the boundary conditions (31). Integrating eq. (29), we obtain

$$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x .$$

From the boundary conditions (31), we obtain

$$C_1 = 0 , \quad C_2\lambda \cos \lambda l = 0 .$$

Assuming $C_2 \neq 0$ (for otherwise $X(x)$ would be identically equal to zero), we find that $\cos \lambda l = 0$, so that $\lambda l = \frac{1}{2}(2k+1)\pi$ (where k is an integer).

Thus, non-trivial solutions to the problem (29) - (31) are possible only for those values of λ for which

$$\lambda_k = \frac{(2k+1)\pi}{2l} .$$

To the eigenvalues λ_k^2 correspond the eigenfunctions

$$X_k(x) = \sin \frac{(2k+1)\pi x}{2l} \quad (k = 0, 1, 2, \ldots) ,$$

These are completely defined up to a constant factor, which we set equal to unity (negative integral values of k do not give new eigenfunctions).

For $\lambda = \lambda_k$, the general solution to eq. (30) is of the form

$$T_k(t) = a_k \cos \frac{(2k+1)\pi at}{2l} + b_k \sin \frac{(2k+1)\pi at}{2l} ,$$

where a_k and b_k are arbitrary constants.

On the basis of (28), we see that the functions

$$u_k(x, t) = T_k(t)\, X_k(x) = \left[a_k \cos \frac{(2k+1)\pi at}{2l} + b_k \sin \frac{(2k+1)\pi at}{2l}\right] \sin \frac{(2k+1)\pi x}{2l}$$

satisfy eq. (25) and the boundary conditions (26) for arbitrary values of a_k and b_k.

We define

$$u(x, t) = \sum_{k=0}^{\infty} \left[a_k \cos \frac{(2k+1)\pi at}{2l} + b_k \sin \frac{(2k+1)\pi at}{2l}\right] \sin \frac{(2k+1)\pi x}{2l} . \tag{32}$$

For the initial conditions (27) to be satisfied, it is necessary that

$$f(x) = \sum_{k=0}^{\infty} a_k \sin \frac{(2k+1)\pi x}{2l}, \tag{33}$$

$$F(x) = \sum_{k=0}^{\infty} b_k \frac{(2k+1)\pi a}{2l} \sin \frac{(2k+1)\pi x}{2l}. \tag{34}$$

Assuming that the series (33) and (34) converge uniformly, we may determine the coefficients a_k and b_k by multiplying both sides of eqs. (33) and (34) by $\sin (2n+1)\pi x/2l$ and integrating with respect to x between the limits $x = 0$ and $x = l$. Then, remembering that

$$\int_0^l \sin \frac{(2n+1)\pi x}{2l} \sin \frac{(2k+1)\pi x}{2l} \, dx = \begin{cases} 0 & \text{for} \quad k \neq n, \\ \frac{1}{2}l & \text{for} \quad k = n, \end{cases}$$

we obtain

$$a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{(2n+1)\pi x}{2l} \, dx,$$

$$b_n = \frac{4}{(2n+1)\pi a} \int_0^l F(x) \sin \frac{(2n+1)\pi x}{2l} \, dx. \tag{35}$$

Substituting the values that we have found for the coefficients in the series (32), we obviously obtain the solution to our problem, provided the series (32) (and the series obtained from it by twice differentiating termwise with respect to x and t) converge uniformly.

When we examine the solution (32), we see that the vibrational motion of the rod is the result of the composition of the simple harmonic vibrations

$$A_k \sin \frac{(2k+1)\pi x}{2l} \sin \left[\frac{(2k+1)\pi at}{2l} + \varphi_k\right],$$

where

$$A_k = \sqrt{a_k^2 + b_k^2}, \qquad \tan \varphi = a_k/b_k.$$

These vibrations have amplitudes $A_k \sin (2k+1)\pi x/2l$ and frequencies

$$\omega_k = \frac{(2k+1)\pi a}{2l} = \frac{(2k+1)\pi}{2l} \sqrt{\frac{E}{\rho}}.$$

The period of vibration of the fundamental tone, which is obtained for $k = 0$, is

$$T = \frac{2\pi}{\omega_0} = 4l \sqrt{\frac{\rho}{E}}.$$

Since the amplitude of the fundamental tone is equal to

$$A_0 \sin \frac{\pi x}{2l},$$

it is obvious that, at the fixed end of the rod ($x = 0$), we have a node and at the free end ($x = l$), we have a crest.

Using the Fourier method, it is easy to solve the problem of the longitudinal vibrations of the rod examined in section 2 of Chapter IV. We recall that the problem presented in Chapter IV was reduced to solving eq. (25) with boundary conditions (26) and initial conditions

$$u|_{t=0} = f(x) = rx \ , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \qquad (0 < x < l) \ ,$$

where r is a constant.

By applying formulae (35), we find that

$$a_k = \frac{(-1)^k \, 8lr}{(2k+1)^2 \, \pi^2} \ , \qquad b_k = 0 \ ,$$

from which it follows that the relative displacement of the section of the rod with abscissa x is expressed by the series

$$u(x,t) = \frac{8lr}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \cos \frac{(2k+1)\pi a t}{2l} \sin \frac{(2k+1)\pi x}{2l} \ .$$

5. *The general plan of the Fourier method*

In the present section, we shall give an exposition of the Fourier method for the solution of a mixed boundary-value problem without giving rigorous proofs of the results obtained.

Consider the hyperbolic equation

$$\frac{\partial}{\partial x}\left(p(x)\,\frac{\partial u}{\partial x}\right) - q(x)u = \rho(x)\,\frac{\partial^2 u}{\partial t^2} \ , \tag{36}$$

where $p(x)$, $p'(x)$, $q(x)$, and $\rho(x)$ are continuous functions for $0 \leqslant x \leqslant 1$, with $p(x) > 0$, $q(x) \geqslant 0$, and $\rho(0) > 0$.

Suppose that we are to find the solution to eq. (36) that satisfies the homogeneous boundary conditions

$$\alpha u(0,t) + \beta\,\frac{\partial u(0,t)}{\partial x} = 0 \ , \qquad \gamma u(l,t) + \delta\,\frac{\partial u(l,t)}{\partial x} = 0 \ , \tag{37}$$

where α, β, γ, and δ are constants (with $\alpha^2 + \beta^2 \neq 0$ and $\gamma^2 + \delta^2 \neq 0$); and the initial conditions

$$u|_{t=0} = f(x) \ , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x) \qquad (0 \leqslant x \leqslant l) \ . \tag{38}$$

Let us first seek the non-trivial solutions to eq. (36) in the form of the product

$$u(x,t) = X(x)\;T(t) \ , \tag{39}$$

satisfying only the boundary conditions (37).

Substituting eq. (39) into eq. (36), we obtain

$$T(t)\,\frac{\mathrm{d}}{\mathrm{d}x}\,[p(x)\;X'(x)] - q(x)\;X(x)\;T(t) = \rho(x)\;X(x)\;T''(t)$$

or

$$\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\, X'(x)\right] - q(x)\, X(x)}{\rho(x)\, X(x)} = \frac{T''(t)}{T(t)} . \tag{40}$$

The left side of eq. (40) depends only on x and the right side only on t; equality is possible only if the common value of the ratios shown in eq. (40) is a constant. Let us denote this constant by $-\lambda$. Then, we obtain from eq. (40) the two ordinary differential equations

$$T''(t) + \lambda T(t) = 0 , \tag{41}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\, X'(x)\right] + \left[\lambda\rho(x) - q(x)\right] X(x) = 0 . \tag{42}$$

In order to obtain non-trivial solutions to eq. (36) in the form (39) satisfying the boundary conditions (37), it is necessary that the function $X(x)$ satisfy the boundary conditions

$$\alpha X(0) + \beta X'(0) = 0 , \qquad \gamma X(l) + \delta X'(l) = 0 . \tag{43}$$

Thus, we are confronted with the eigenvalue problem of finding those values of λ for which non-trivial solutions to eq. (42) exist satisfying the boundary conditions (43).

This problem does not have a non-trivial solution (that is, one not identically equal to zero) for every value of λ. Those values of λ for which (42) - (43) has a non-trivial solution are called eigenvalues and the corresponding solutions are called eigenfunctions. Because of the homogeneity of eq. (42), and because of the boundary conditions (43), the eigenfunctions are defined up to a constant factor. It is easy to see that only one eigenfunction can correspond to any eigenvalue. To show this, suppose that to some value of λ there correspond two linearly independent solutions to eq. (42) that satisfy the boundary conditions (43). Then, the general solution to eq. (42) would also satisfy these conditions. But this is impossible because we may always find a solution to eq. (42) for initial conditions $X(0)$ and $X'(0)$ that do not satisfy the first of the boundary conditions (43).

It is possible to show that, for our problem, there exist an infinite number of real eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots$$

To each eigenvalue λ there corresponds an eigenfunction $X_k(x)$ that is completely defined up to a constant factor. Let us choose this factor such that

$$\int_0^l \rho(x)\, X_k^2(x)\, \mathrm{d}x = 1 . \tag{44}$$

The eigenfunctions that satisfy eq. (44) are said to be *normalized*.

Let us show that the eigenfunctions corresponding to different eigenvalues are *orthogonal with weight* $\rho(x)$, that is, that they satisfy the equations

$$\int_0^l \rho(x)\, X_k(x)\, X_n(x)\, dx = 0 \qquad (k \neq n)\ . \tag{45}$$

Suppose that λ_k and λ_n are two distinct eigenvalues and that $X_k(x)$ and $X_n(x)$ are the corresponding eigenfunctions, so that

$$\frac{d}{dx}[p(x)\, X_k'(x)] + [\lambda_k \rho(x) - q(x)]\, X_k(x) = 0\ ,$$

$$\frac{d}{dx}[p(x)\, X_n'(x)] + [\lambda_n \rho(x) - q(x)]\, X_n(x) = 0\ .$$

Multiplying the first of these equations by $X_n(x)$ and the second by $X_k(x)$ and subtracting the first from the second, we obtain

$$X_n(x)\frac{d}{dx}[p(x)\, X_k'(x)] - X_k(x)\frac{d}{dx}[p(x)\, X_n'(x)] + (\lambda_k - \lambda_n)\, \rho(x)\, X_k(x)\, X_n(x) = 0\ ,$$

which can be rewritten in the form

$$(\lambda_k - \lambda_n)\, \rho(x)\, X_k(x)\, X_n(x) + \frac{d}{dx}\{p(x)\, [X_n(x)\, X_k'(x) - X_k(x)\, X_n'(x)]\} = 0\ .$$

Integrating this equation with respect to x, from 0 to l, we obtain

$$(\lambda_n - \lambda_k)\int_0^l \rho(x)\, X_k(x)\, X_n(x)\, dx = p(x)\, [X_n(x)\, X_k'(x) - X_k(x)\, X_n'(x)]\Big|_{x=0}^{x=l}\ .$$

Taking the boundary conditions (43) into consideration, we easily see that the right side is equal to zero, that is, that

$$(\lambda_n - \lambda_k)\int_0^l \rho(x)\, X_k(x)\, X_n(x)\, dx = 0\ ,$$

from which (since $\lambda_n \neq \lambda_k$),

$$\int_0^l \rho(x)\, X_k(x)\, X_n(x)\, dx = 0\ ,$$

which proves the assertion.

Now suppose that the λ_k are eigenvalues and that the $X_k(x)$ are the eigenfunctions that form a normalized orthogonal (orthonormal) system. We have

$$\frac{d}{dx}[p(x)\, X_k'(x)] - q(x)\, X_k(x) = -\lambda_k \rho(x)\, X_k(x)\ .$$

Multiplying both sides by $X_k(x)$, integrating and taking account of (44), we obtain

$$\lambda_k = -\int_0^l \left\{\frac{d}{dx}[p(x)\, X_k'(x)] - q(x)\, X_k(x)\right\} X_k(x)\, dx\ ,$$

from which, by integrating the first term by parts, we obtain

$$\lambda_k = \int_0^l [p(x)\, X_k'^2(x) + q(x)\, X_k^2(x)]\, dx - [p(x)\, X_k(x)\, X_k'(x)]\Big|_{x=0}^{x=l} . \tag{46}$$

Suppose that $p(x) > 0$, $q(x) \geqslant 0$, $\rho(x) > 0$, and

$$[p(x)\, X_k(x)\, X_k'(x)]\Big|_{x=0}^{x=l} \leqslant 0 . \tag{46a}$$

Then, it follows immediately from eq. (46) that all the eigenvalues to the problem (42) - (43) are *non-negative*.

As it happens, the condition (46a) is satisfied in the boundary conditions most frequently encountered in applications:

$$X(0) = 0 , \qquad X(l) = 0 , \tag{43a}$$

$$X'(0) - h_1 X(0) = 0 , \quad X'(l) + h_2 X(l) = 0 , \quad h_1 \geqslant 0 , \quad h_2 \geqslant 0 . \tag{43b}$$

In conclusion, we note that the eigenfunctions $X_n(x)$ of the boundary problem (42) - (43a) or (42) - (43b) form a complete system *. (If $h_1 = h_2 = 0$ then $q(x) \geqslant q_0 > 0$.) Now let us turn to eq. (41). Its general solution for $\lambda = \lambda_k$, which we denote by $T_k(t)$, is of the form

$$T_k(t) = A_k \cos\sqrt{\lambda_k}t + B_k \sin\sqrt{\lambda_k}t ,$$

where A_k and B_k are arbitrary constants.

Each function

$$u_k(x, t) = X_k(x)\; T_k(t) = (A_k \cos\sqrt{\lambda_k}t + B_k \sin\sqrt{\lambda_k}t)\; X_k(x)$$

will be a solution to eq. (36) satisfying the boundary conditions (37).

In order to satisfy the initial conditions (38), we form the series

$$u(x, t) = \sum_{k=1}^{\infty} (A_k \cos\sqrt{\lambda_k}t + B_k \sin\sqrt{\lambda_k}t)\; X_k(x) . \tag{47}$$

If this series (and the series obtained from it by twice differentiating termwise with respect to x and t) converge uniformly, its sum will obviously be a solution to eq. (36) satisfying the boundary conditions (37). To satisfy the initial conditions (38), it is necessary that

$$u|_{t=0} = f(x) = \sum_{k=1}^{\infty} A_k X_k(x) , \tag{48}$$

$$\frac{\partial u}{\partial t}\bigg|_{t=0} = F(x) = \sum_{k=1}^{\infty} B_k \sqrt{\lambda_k}\; X_k(x) . \tag{49}$$

* A system of functions

$$\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \ldots$$

is said to be *complete* if there does not exist a square integrable function (not identically equal to zero) orthogonal with respect to all functions of the system.

Thus, we are confronted with the problem of expanding an arbitrary function in a series of eigenfunctions $X_k(x)$ of the boundary problem (42) - (43).

Suppose that an arbitrary function $\Phi(x)$ is represented in the form of the series

$$\Phi(x) = \sum_{k=1}^{\infty} a_k X_k(x) \tag{50}$$

of eigenfunctions $X_k(x)$ of the boundary problem (42) - (43).

Assuming that the series (50) converges uniformly, we may determine its coefficients a_k by multiplying both sides of eq. (50) by $\rho(x)X_k(x)$ and integrating with respect to x from 0 to l. Then, by taking (44) and (45) into account, we obtain

$$a_k = \int_0^l \rho(x)\, \Phi(x)\, X_k(x)\, dx\,. \tag{51}$$

We now put in correspondence with each function $\Phi(x)$ that is integrable on $[0, l]$ the Fourier series

$$\sum_{k=1}^{\infty} a_k X_k(x) \tag{52}$$

of that function, where the coefficients a_k are determined from the formulae (51).

We state without proof the following propositions:

THEOREM 1. *For every square integrable function* $\Phi(x)$ *on the interval* $[0, l]$, *the series* (52) *converges to that function in the limit*; *that is,*

$$\lim_{n\to\infty} \int_0^l \rho(x) \left[\Phi(x) - \sum_{k=1}^{n} a_k X_k(x)\right]^2 dx = 0\,. \tag{53}$$

THEOREM 2 (V. A. Steklov). *Every function* $\Phi(x)$ *satisfying the boundary conditions* (43) *that has a continuous first derivative and a piecewise-continuous second derivative can be expanded into an absolutely and uniformly continuous series* (50) *of the eigenfunctions of the boundary problem* (42) - (43).

Using formula (51) for determining the coefficients in the expansions (48) and (49), we find that

$$A_k = \int_0^l \rho(x)\, f(x)\, X_k(x)\, dx\,, \qquad B_k = \frac{1}{\sqrt{\lambda_k}} \int_0^l \rho(x)\, F(x)\, X_k(x)\, dx\,.$$

When we substitute these values of the coefficients A_k and B_k into the series (47), we shall obviously obtain the solution to the mixed problem (36) - (38), provided the series (47) (and the series that are obtained from it by twice differentiating termwise with respect to x and t) converge uniformly.

Remark: The Fourier method is also applicable in the case of many space variables for hyperbolic equations of a special type (see Chapter XVI)

and also for equations of the elliptic and parabolic types (see Parts II and III).

Problems

1. A homogeneous string, fixed at the ends $x=0$ and $x=l$, has, at the initial instant, the shape of a parabola that is symmetric about the perpendicular drawn through the point $x = \frac{1}{2}l$. Find the displacement of the points of the string from the straight-line equilibrium position, assuming that there is no initial velocity.
Answer:
$$u(x,t) = \frac{32h}{\pi^3} \sum_{k=0}^{\infty} \frac{\cos\dfrac{(2k+1)\pi at}{l} \sin\dfrac{(2k+1)\pi x}{l}}{(2k+1)^3},$$
where h is the initial value of the displacement at the point $x = \frac{1}{2}l$.

2. A homogeneous string with fixed ends is set into vibration by the blow of a hard flat hammer that imparts to it the following initial distribution of velocities:
$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \begin{cases} 0 & 0 \leqslant x \leqslant c-\delta, \\ v_0 & c-\delta \leqslant x \leqslant c+\delta, \\ 0 & c+\delta \leqslant x \leqslant l. \end{cases}$$
Find the vibrations of the string if the initial deviation is equal to zero.
Answer:
$$u(x,t) = \frac{4v_0 l}{\pi^2 a} \sum_{k=1}^{\infty} \frac{1}{k^2} \sin\frac{k\pi c}{l} \sin\frac{k\pi\delta}{l} \sin\frac{k\pi at}{l} \sin\frac{k\pi x}{l}.$$

3. A homogeneous string with fixed ends is set into vibration by the blow of a sharp hammer, which imparts to it an impulse I at the point $x = c$. Find the free vibrations of the string if the initial displacement is equal to zero.
Answer:
$$u(x,t) = \frac{2I}{\pi a\rho} \sum_{k=1}^{\infty} \frac{1}{k} \sin\frac{k\pi c}{l} \sin\frac{k\pi at}{l} \sin\frac{k\pi x}{l}.$$
Method of solution: First we assume the impulse I to be uniformly distributed over the segment $c-\delta \leqslant x \leqslant c+\delta$ of the string. Then, we obtain an expression for $u(x,t)$, defined in the answer to the preceding problem, where $v_0 = I/2\delta\rho$. Taking the limit as δ approaches zero, we obtain the solution of the problem.

4. A homogeneous rod of length $2l$ is compressed by forces applied to its ends in such a way that it is shortened to a length $2l(1-\epsilon)$. At $t = 0$, the load is lifted. Show that the displacement $u(x,t)$ of the section of the rod whose abscissa is x is determined by the formula

$$u(x,t) = \frac{8\epsilon l}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)^2} \sin \frac{(2k+1)\pi x}{2l} \cos \frac{(2k+1)\pi a t}{2l} ,$$

where the point $x = 0$ is in the middle of the rod and a is the velocity of the longitudinal waves in the rod.

Method of solution: The problem is reduced to solving the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (a^2 = E/\rho)$$

with the conditions

$$\left.\frac{\partial u}{\partial t}\right|_{x=-l} = 0 , \quad \left.\frac{\partial u}{\partial x}\right|_{x=l} = 0 ; \qquad u|_{t=0} = -\epsilon x , \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 .$$

5. Examine the free vibrations of a fixed string that is vibrating in a medium whose resistance is proportional to the velocity.

Answer:

$$u(x,t) = e^{-ht} \sum_{k=1}^{\infty} (a_k \cos q_k t + b_k \sin q_k t) \sin \frac{k\pi x}{l} ,$$

where

$$a_k = \frac{2}{l} \int_0^l f(x) \sin \frac{k\pi x}{l} dx , \quad b_k = \frac{h}{q_k} a_k + \frac{2}{l q_k} \int_0^l F(x) \sin \frac{k\pi x}{l} dx ,$$

$$q_k = \sqrt{\frac{k^2\pi^2 a^2}{l^2} - h^2} .$$

Method of solution: Apply the Fourier method to the integration of the equation

$$\frac{\partial^2 u}{\partial t^2} + 2h \frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2} ,$$

where h is a small positive number, with the conditions

$$u|_{x=0} = 0 , \quad u|_{x=l} = 0 ; \qquad u|_{t=0} = f(x) , \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x) .$$

6. A rod AC that is fixed at one end A and free at the other end C consists of two dissimilar parts: AB = l_1 and BC = l_2. Show that the period T of longitudinal oscillations of the rod is determined by equation

$$\tan \frac{2\pi l_1}{a_1 T} \tan \frac{2\pi l_2}{a_2 T} = \frac{E_1 a_2}{E_2 a_1} \qquad (a_1 = \sqrt{E_1/\rho_1},\ a_2 = \sqrt{E_2/\rho_2}) .$$

Method of solution: Single out the oscillations of a single frequency that satisfy the differential equations

$$\frac{\partial^2 u_1}{\partial t^2} = a_1^2 \frac{\partial^2 u_1}{\partial x^2} , \qquad \frac{\partial^2 u_2}{\partial t^2} = a_2^2 \frac{\partial^2 u_2}{\partial x^2}$$

and the conditions

$$u_1\big|_{x=l_1} = u_2\big|_{x=l_1}\,, \qquad E_1 \frac{\partial u_1}{\partial x}\bigg|_{x=l_1} = E_2 \frac{\partial u_2}{\partial x}\bigg|_{x=l_1}\,,$$

$$u_1\big|_{x=0} = 0\,, \qquad \frac{\partial u_2}{\partial x}\bigg|_{x=l_1+l_2} = 0\,.$$

Chapter IX

FORCED VIBRATIONS OF STRINGS AND RODS

1. *Forced vibrations of a string that is fixed at the ends*

Let us consider the forced vibrations of a homogeneous string that is fixed at the ends and that is subject to an external force $p(x, t)$ measured per unit of length. This problem is reduced to solving the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + g(x, t) \qquad \left[g(x, t) = \frac{1}{\rho} p(x, t)\right] \tag{1}$$

with boundary conditions

$$u|_{x=0} = 0 , \qquad u|_{x=l} = 0 \tag{2}$$

and initial conditions

$$u|_{t=0} = f(x) , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x) . \tag{3}$$

Let us seek a solution to this problem in the form of a sum

$$u = v + w , \tag{4}$$

where v is the solution to the *non-homogeneous* equation

$$\frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial x^2} + g(x, t) , \tag{5}$$

satisfying the boundary conditions

$$v|_{x=0} = 0 , \qquad v|_{x=l} = 0 \tag{6}$$

and the initial conditions

$$v|_{t=0} = 0 , \qquad \left.\frac{\partial v}{\partial t}\right|_{t=0} = 0 , \tag{7}$$

and w is the solution to the *homogeneous* equation

$$\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} , \tag{8}$$

satisfying the boundary conditions

$$w|_{x=0} = 0 , \qquad w|_{x=l} = 0 \tag{9}$$

and the initial conditions

$$w|_{t=0} = f(x) , \qquad \left.\frac{\partial w}{\partial t}\right|_{t=0} = F(x) . \tag{10}$$

The solution v represents *forced* vibrations of the string, that is, those vibrations that take place under the effect of an external disturbing force, when there are no initial disturbances.

The solution w represents the *free* vibrations of the string, that is, those vibrations that take place as a result of the initial disturbance only.

The methods of finding the free vibrations w were examined in the preceding chapters, so that here we need only find the forced vibrations v. As in the case of the free vibrations, let us seek a solution v in the form of the series

$$v(x, t) = \sum_{k=1}^{\infty} T_k(t) \sin \frac{k\pi x}{l} , \tag{11}$$

so that the boundary conditions (6) are satisfied automatically (under the assumption that the series converges uniformly).

Let us now determine the functions $T_k(t)$ in such a way that the series (11) satisfies eq. (5) and the initial conditions (7).

Substituting the series (11) into eq. (5), we obtain

$$\sum_{k=1}^{\infty} [T_k''(t) + \omega_k^2 T_k(t)] \sin \frac{k\pi x}{l} = g(x, t) , \tag{12}$$

where

$$\omega_k = k\pi a/l . \tag{13}$$

Let us expand the function $g(x, t)$ in a Fourier sine series over the interval $(0, l)$:

$$g(x, t) = \sum_{k=1}^{\infty} g_k(t) \sin \frac{k\pi x}{l} , \tag{14}$$

where

$$g_k(t) = \frac{2}{l} \int_0^l g(\xi, t) \sin \frac{k\pi\xi}{l} \, d\xi . \tag{15}$$

Equating the expansions (12) and (14) for the same function $g(x, t)$, we obtain the differential equations

$$T_k''(t) + \omega_k^2 T_k(t) = g_k(t) \qquad (k = 1, 2, 3, \ldots) , \tag{16}$$

defining the functions $T_k(t)$.

For the solution v defined by the series (11) to satisfy the initial conditions (7), it is sufficient for the functions $T_k(t)$ to satisfy the conditions

$$T_k(0) = 0 , \qquad T_k'(0) = 0 \qquad (k = 1, 2, 3, \ldots) . \tag{17}$$

The solution to eqs. (16) with the initial conditions (17) is of the form [1]:

$$T_k(t) = \frac{1}{\omega_k} \int_0^l g_k(\tau) \sin \omega_k(t - \tau) \, d\tau ,$$

or, substituting for $g_k(\tau)$ the expression given for it by eq. (15):

$$T_k(t) = \frac{2}{l\omega_k}\int_0^t d\tau \int_0^l g(\xi,\tau)\sin\omega_k(t-\tau)\sin\frac{k\pi\xi}{l}\,d\xi\,. \tag{18}$$

Substituting the above expressions for $T_k(t)$ in eq. (11), we obtain the desired solution $v(x,t)$. From the above, it follows that the solution to the problem (1) - (3) is expressed in the form of the series

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t)\sin\frac{k\pi x}{l} + \sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi at}{l} + b_k\sin\frac{k\pi at}{l}\right)\sin\frac{k\pi x}{l}\,, \tag{19}$$

where the coefficients $T_k(t)$ are determined from formulae (18) and

$$a_k = \frac{2}{l}\int_0^l f(x)\sin\frac{k\pi x}{l}\,dx\,, \qquad b_k = \frac{2}{k\pi a}\int_0^l F(x)\sin\frac{k\pi x}{l}\,dx\,. \tag{20}$$

As an example, let us consider the case in which there are no initial displacements or initial velocities and only a continuously distributed force with linear density

$$p(x,t) = A\rho\sin\omega t$$

acts on the string. In this case, the solution $u(x,t)$ is defined by the series

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t)\sin\frac{k\pi x}{l}\,, \tag{21}$$

where the coefficients $T_k(t)$ are determined from eq. (18) and are equal to

$$T_k(t) = \frac{2Al}{\pi^2k^2a}\left[1-(-1)^k\right]\left\{\frac{\omega_k\sin\omega t}{\omega_k{}^2-\omega^2} - \frac{\omega\sin\omega_k t}{\omega_k{}^2-\omega^2}\right\} \qquad (\omega \neq \omega_k)\,. \tag{22}$$

If $\omega = \omega_k$, eq. (22) for $T_k(t)$ loses its meaning and in this case, we have the following expression for $T_k(t)$:

$$T_k(t) = -\frac{Al}{\pi^2k^2a}\left[1-(-1)^k\right]\frac{\omega_k t\cos\omega_k t - \sin\omega_k t}{\omega_k}\,. \tag{23}$$

Substituting eq. (22) in the series (21), we obtain

$$u(x,t) = \frac{4A}{\pi}\sin\omega t\sum_{k=0}^{\infty}\frac{\sin\frac{(2k+1)\pi x}{l}}{(2k+1)\,(\omega_{2k+1}^2-\omega^2)}$$

$$-\frac{4Al\omega}{a\pi^2}\sum_{k=0}^{\infty}\frac{\sin\frac{(2k+1)\pi at}{l}\sin\frac{(2k+1)\pi x}{l}}{(2k+1)^2\,(\omega_{2k+1}^2-\omega^2)}\,. \tag{24}$$

The first term on the right side of eq. (24), which has the same frequency as the disturbing force, characterizes the "pure" forced vibrations of the string. The second term, which applies to the "free" vibrations of the string, consists of an infinite number of harmonic vibrations of frequency

$$\omega_{2k+1} = \frac{(2k+1)\pi a}{l} .$$

Eq. (24) shows that if the frequency of the external disturbing force ω is close to one of the frequencies ω_{2k+1} of the natural vibrations of the string, there will be a term in the expansion (24) with an especially large amplitude, as a result of which the phenomenon known as *resonance* will occur. On the other hand, when the frequency $\omega = \omega_{2k_1+1}$, eq. (24) becomes meaningless and must be replaced by another equation. This equation is easily obtained if we take eq. (23) into consideration. In the case in question, the solution of the problem is

$$u(x,t) = \frac{2Al^2}{a^2\pi^3(2k_1+1)^3} (\sin \omega_{2k_1+1}t - \omega_{2k_1+1}t \cos \omega_{2k_1+1}t) \sin \frac{(2k_1+1)\pi x}{l}$$

$$+ \frac{4A}{\pi} \sin \omega_{2k_1+1}t \sum_{k=0}^{\infty}{}' \frac{\sin \frac{(2k+1)\pi x}{l}}{(2k+1)\,(\omega_{2k+1}^2 - \omega_{2k_1+1}^2)}$$

$$- \frac{4Al\omega_{2k_1+1}}{a\pi^2} \sum_{k=0}^{\infty}{}' \frac{\sin \frac{(2k+1)\pi at}{l} \sin \frac{(2k+1)\pi x}{l}}{(2k+1)^2\,(\omega_{2k+1}^2 - \omega_{2k_1+1}^2)} , \qquad (25)$$

where the prime in the summation symbol indicates that we should exclude the term corresponding to $k = k_1$.

2. *Forced vibrations of a string under the action of a concentrated force*

Let us now examine the case in which the external disturbing force is concentrated at some one particular point on the string. Let us consider the most interesting case, that in which the force acts periodically with period $2\pi/\omega$. We denote by c the abscissa of the point of the string at which the force is applied:

$$\rho g(c,t) = A\rho \sin \omega t . \qquad (26)$$

Suppose also that $u_1(x,t)$ and $u_2(x,t)$ are the displacements of the string over the segments $(0,c)$ and (c,l), respectively, as shown in fig. 29.

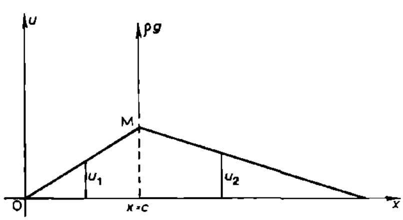

Fig. 29.

The following equations are valid for the functions u_1 and u_2:

$$\begin{aligned} \frac{\partial^2 u_1}{\partial t^2} &= a^2 \frac{\partial^2 u_1}{\partial x^2} \quad (0 < x < c) \, , \\ \frac{\partial^2 u_2}{\partial t^2} &= a^2 \frac{\partial^2 u_2}{\partial t^2} \quad (c < x < l) \, , \end{aligned} \tag{27}$$

because there are no external forces in the intervals $(0, c)$ and (c, l).

Let us find the solutions to eqs. (27) with the conditions

$$u_1|_{x=0} = 0 \, , \qquad u_2|_{x=l} = 0 \, ; \tag{28}$$

$$u_1|_{x=c} = u_2|_{x=c} \, ; \tag{29}$$

$$T_0 \frac{\partial u_1}{\partial x}\bigg|_{x=c} - T_0 \frac{\partial u_2}{\partial x}\bigg|_{x=c} = \rho A \sin \omega t \, . \tag{30}$$

These conditions have the following physical meaning: conditions (28) show that the string is fixed at the ends, condition (29) expresses the continuity of the string at the point $x = c$, and condition (30) has the following interpretation. Let us examine the force $\rho g(c, t)$ as a limiting case of the force $\rho F \mathrm{d}x$ continuously distributed along a segment M_1M_2 of the string (fig. 30); that is, we shall assume

$$\rho g(c, t) = \lim_{M_1 \to M \leftarrow M_2} \rho F \, \mathrm{d}x \, .$$

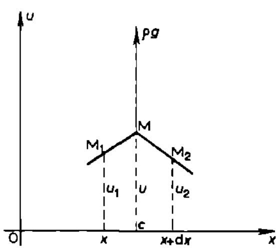

Fig. 30.

From d'Alembert's principle, we can write the following equation of motion for the element M_1M_2:

$$-\rho \, \mathrm{d}x \frac{\partial^2 u}{\partial t^2} + T_0 \frac{\partial u}{\partial x}\bigg|_{M_2} - T_0 \frac{\partial u}{\partial x}\bigg|_{M_1} + \rho F \, \mathrm{d}x = 0 \, .$$

Now let both points M_1 and M_2 approach their limiting position M. Since $\mathrm{d}x$ then approaches zero, the first term of the above equation vanishes, as a result of which this equation is converted into condition (30).

Let us seek solutions to eqs. (27) in the form

$$X(x) \sin \omega t .$$

It is easy to see that solutions of this type that satisfy conditions (28) will be of the following form:

$$u_1 = C_1 \sin \frac{\omega x}{a} \sin \omega t , \quad u_2 = C_2 \sin \frac{\omega(l-x)}{a} \sin \omega t , \tag{31}$$

where C_1 and C_2 are arbitrary constants.

By using conditions (29) and (30), we obtain the following system of two equations for C_1 and C_2:

$$C_1 \sin \frac{\omega c}{a} - C_2 \sin \frac{\omega(l-c)}{a} = 0 ,$$

$$T_0 \frac{\omega}{a} C_1 \cos \frac{\omega c}{a} + T_0 \frac{\omega}{a} C_2 \cos \frac{\omega(l-c)}{a} = \rho A . \tag{32}$$

Recalling that $\rho/T_0 = 1/a^2$, we find from the above system that

$$C_1 = \frac{A}{a\omega} \frac{\sin \omega(l-c)/a}{\sin \omega l/a} , \quad C_2 = \frac{A}{a\omega} \frac{\sin \omega c/a}{\sin \omega l/a} .$$

Then, formulae (31) give the following expressions characterizing the pure forced vibrations of the string:

$$\begin{aligned} u_1 &= \frac{A}{a\omega} \frac{\sin \omega(l-c)/a}{\sin \omega l/a} \sin \frac{\omega x}{a} \sin \omega t \quad (0 < x < c) , \\ u_2 &= \frac{A}{a\omega} \frac{\sin \omega c/a}{\sin \omega l/a} \sin \frac{\omega(l-x)}{a} \sin \omega t \quad (c < x < l) . \end{aligned} \tag{33}$$

3. *Forced vibrations of a heavy rod*

Suppose that we are dealing with a rather heavy and, at the same time, easily stretched rod, whose length in the unstretched state is l. Let us suspend it by the end $x = 0$ and let the end $x = l$ be free. Under the action of gravity, such a rod will begin to vibrate longitudinally. If we denote by u the displacement of the section with abscissa x at the time t, the differential equation for the forced vibrations of this rod will take the form

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + g , \tag{34}$$

where g is the acceleration due to gravity.

Since the initial displacements and the initial velocities are equal to zero, from the physical meaning of the problem, we need to find the solution to eq. (34) that satisfies the boundary conditions

$$u|_{x=0} = 0 , \quad \left.\frac{\partial u}{\partial x}\right|_{x=l} = 0 \tag{35}$$

and the initial conditions

$$u|_{t=0} = 0 , \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 . \tag{36}$$

Let us seek a solution to this problem in the form of the sum

$$u = v + w , \tag{37}$$

where v is the solution to the non-homogeneous eq. (34) satisfying only the initial conditions (35), and w is the solution to the homogeneous equation

$$\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2}, \tag{38}$$

satisfying the boundary conditions

$$w|_{x=0} = 0 , \quad \left.\frac{\partial w}{\partial x}\right|_{x=l} = 0 \tag{39}$$

and the initial conditions

$$w|_{t=0} = f(x) = -v|_{t=0} , \quad \left.\frac{\partial w}{\partial t}\right|_{t=0} = F(x) = -\left.\frac{\partial v}{\partial t}\right|_{t=0} \tag{40}$$

Finding the solution $v(x, t)$ presents no difficulties. For if we take a second-degree polynomial in x

$$\alpha x^2 + \beta x + \gamma$$

and set

$$\alpha = -\frac{g}{2a^2}, \quad \beta = \frac{gl}{a^2}, \quad \gamma = 0 ;$$

then, both eq. (34) and the boundary conditions (35) will obviously be satisfied. Consequently, the solution v is found, namely,

$$v = \frac{gx(2l - x)}{2a^2} . \tag{41}$$

It follows from this that

$$f(x) = \frac{gx(x - 2l)}{2a^2}, \quad F(x) = 0 . \tag{42}$$

We have already examined the problem (38) - (40) in section 4 of Chapter VII. Its solution is given by eqs. (32) and (35). By means of these equations, we obtain

$$a_k = \frac{2}{l}\int_0^l \frac{gx(x-2l)}{2a^2} \sin\frac{(2k+1)\pi x}{2l}\, dx = -\frac{16gl^2}{\pi^3 a^2(2k+1)^3}$$

$$b_k = 0 \quad (k = 0, 1, 2, \ldots) .$$

It follows from the above that the solution to our problem is expressed in the form

$$u(x,t) = \frac{gx(2l-x)}{2a^2} - \frac{16gl^2}{\pi^3 a^2}\sum_{k=0}^{\infty}\frac{\cos\frac{(2k+1)\pi at}{2l}\sin\frac{(2k+1)\pi x}{2l}}{(2k+1)^3} \tag{43}$$

By means of this formula, it is easy to compute, for example, the variations in the length of the entire rod. Setting $x = l$ in eq. (43), we obtain the relative displacement of the terminal section of the rod

$$u|_{x=l} = \frac{gl^2}{2a^2} - \frac{16gl^2}{\pi^3 a^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}\cos\frac{(2k+1)\pi at}{2l}$$

The right side of this equation attains its maximum at $t = 2l/a$, so that

$$u_{\max} = \frac{gl^2}{2a^2} + \frac{16gl^2}{\pi^3 a^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}$$

Recalling that

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32},$$

we find the maximum displacement of the terminal cross section:

$$u_{\max} = gl^2/a^2$$

It follows from this that when the rod is subjected to vibrations of this sort, its length varies between the limits l and $l + gl^2/a^2$

4. *Forced vibrations of a string with moving ends*

Let us examine the forced oscillations of a string of finite length under the action of an external force $p(x,t)$ that is measured per unit of length. The ends of the string are not fixed, but they move according to some stated law. This problem is reduced to solving the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + g(x,t) \tag{44}$$

with boundary conditions

$$u|_{x=0} = \varkappa_1(t), \qquad u|_{x=l} = \varkappa_2(t) \tag{45}$$

and initial conditions

$$u|_{t=0} = f(x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x). \tag{46}$$

We may not apply the Fourier method to solving this problem because the boundary conditions (45) are not homogeneous. But this problem is easily reduced to a problem with zero boundary conditions.

To do so, we introduce the auxiliary function

$$w(x,t) = \varkappa_1(t) + [\varkappa_2(t) - \varkappa_1(t)]\frac{x}{l}. \tag{47}$$

It is clear that

$$w|_{x=0} = \varkappa_1(t)\,, \qquad w|_{x=l} = \varkappa_2(t)\,. \tag{48}$$

We are seeking a solution to the problem in the form of the sum

$$u = v + w\,, \tag{49}$$

where v is a new unknown function.

Because of the boundary conditions (45) and (48) and the initial conditions (46), the function $v(x, t)$ must satisfy the boundary conditions

$$v|_{x=0} = 0\,, \qquad v|_{x=l} = 0 \tag{50}$$

and the initial conditions

$$\begin{aligned} v|_{t=0} &= u|_{t=0} - w|_{t=0} = f(x) - \varkappa_1(0) - [\varkappa_2(0) - \varkappa_1(0)]\,\frac{x}{l} = f_1(x)\,, \\ \left.\frac{\partial v}{\partial t}\right|_{t=0} &= \left.\frac{\partial u}{\partial t}\right|_{t=0} - \left.\frac{\partial w}{\partial t}\right|_{t=0} = F(x) - \varkappa_1'(0) - [\varkappa_2'(0) - \varkappa_1'(0)]\,\frac{x}{l} = F_1(x) \end{aligned} \tag{51}$$

If we now substitute eq. (49) into eq. (44), we obtain

$$\frac{\partial^2 v}{\partial t^2} = a^2\,\frac{\partial^2 v}{\partial x^2} + g(x, t) + a^2\,\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial t^2}$$

or, on the basis of (47),

$$\frac{\partial^2 v}{\partial t^2} = a^2\,\frac{\partial^2 v}{\partial x^2} + g_1(x, t)\,, \tag{52}$$

where

$$g_1(x, t) = g(x, t) - \varkappa_1''(t) - [\varkappa_2''(t) - \varkappa_1''(t)]\Big|\frac{x}{l}\,. \tag{53}$$

Thus, we are confronted with the following problem for the function $v(x, t)$:

$$\frac{\partial^2 v}{\partial t^2} = a^2\,\frac{\partial^2 v}{\partial x^2} + g_1(x, t)\,,$$

$$v|_{x=0} = 0\,, \qquad v|_{x=l} = 0\,,$$

$$v|_{t=0} = f_1(x)\,, \qquad \left.\frac{\partial v}{\partial t}\right|_{t=0} = F_1(x)\,.$$

The method of solving this problem is explained in section 1 of the present chapter.

As an example, let us examine the transverse vibrations of a string of length l that is fixed at the end $x = 0$ and subjected, at the end $x = l$, to the action of a disturbing force that causes a displacement of that end equal to $A \sin \omega t$. Here, we assume that at the time $t = 0$ the initial displacements and the initial velocities are equal to zero.

It is easy to see that this problem is reduced to solving the homogeneous equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{54}$$

with boundary conditions

$$u|_{x=0} = 0 , \quad u|_{x=l} = A \sin \omega t \tag{55}$$

and initial conditions

$$u|_{t=0} = 0 , \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 . \tag{56}$$

Let us seek a solution to the problem in the form of a sum

$$u = v + w , \tag{57}$$

where w is a solution to the homogeneous eq. (54), satisfying only the boundary conditions (55), and v is a solution to the same equation satisfying the boundary conditions

$$v|_{x=0} = 0 , \quad v|_{x=l} = 0 \tag{58}$$

and the initial conditions

$$v|_{t=0} = f(x) = -w|_{t=0} , \quad \left.\frac{\partial v}{\partial t}\right|_{t=0} = F(x) = - \left.\frac{\partial w}{\partial t}\right|_{t=0} . \tag{59}$$

We seek the solution w in the form

$$w = X(x) \sin \omega t . \tag{60}$$

Substituting eq. (60) into eq. (54), we obtain

$$X''(x) + \frac{\omega^2}{a^2} X(x) = 0 . \tag{61}$$

In order to obtain a solution $w(x, t)$ of the form (60) satisfying the boundary conditions (55), we must find a solution to eq. (61) satisfying the boundary conditions

$$X(0) = 0 , \quad X(l) = A . \tag{62}$$

The general solution to eq. (61) is of the form

$$X(x) = C_1 \cos \frac{\omega x}{a} + C_2 \sin \frac{\omega x}{a} .$$

In satisfying the boundary conditions (62), we obtain

$$C_1 = 0 , \quad C_2 \sin \frac{\omega l}{a} = A$$

and, consequently,

$$X(x) = A \frac{\sin (\omega x/a)}{\sin (\omega l/a)} .$$

Therefore, because of eq. (60), we obtain

$$w(x, t) = A \frac{\sin(\omega x/a) \sin \omega t}{\sin(\omega l/a)} . \tag{63}$$

Let us now find the solution $v(x, t)$. From eqs. (59), we easily obtain

$$v|_{t=0} = f(x) = 0 , \quad \left.\frac{\partial v}{\partial t}\right|_{t=0} = F(x) = -\frac{A\omega \sin(\omega x/a)}{\sin(\omega l/a)} . \tag{64}$$

But the solution to the homogeneous equation (54) satisfying the boundary conditions (58) and the initial conditions (64) is given, as we know, by the series

$$v(x, t) = \sum_{k=1}^{\infty} b_k \sin \frac{k\pi at}{l} \sin \frac{k\pi x}{l} ,$$

where

$$b_k = -\frac{2A\omega}{k\pi a \sin(\omega l/a)} \int_0^l \sin \frac{\omega x}{a} \sin \frac{k\pi x}{l}\, dx = (-1)^{k-1} \frac{2Aa\omega}{l[\omega^2 - (k\pi a/l)^2]} ;$$

thus,

$$v = \frac{2A\omega a}{l} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\omega^2 - (k\pi a/l)^2} \sin \frac{k\pi at}{l} \sin \frac{k\pi x}{l} . \tag{65}$$

Taking the sum of eqs. (63) and (65), we obtain the solution to our problem:

$$u(x, t) = A \frac{\sin(\omega x/a)}{\sin(\omega l/a)} \sin \omega t + \frac{2A\omega a}{l} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\omega^2 - (k\pi a/l)^2} \sin \frac{k\pi at}{l} \sin \frac{k\pi x}{l} , \tag{66}$$

where we assume that $\omega \neq k\pi a/l$.

5. *The uniqueness of the solution to a mixed problem*

Let us examine the following mixed problem.

Find the function $u(x, t)$ that is continuous in the rectangle $Q\,[0 \leqslant x \leqslant l,\ 0 \leqslant t \leqslant T]$ and that satisfies, within Q, the equation

$$\rho(x) \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(p(x) \frac{\partial u}{\partial x}\right) - q(x)u + g(x, t) , \tag{67}$$

where

$$p(x) > 0 , \quad q(x) \geqslant 0 , \quad \rho(x) > 0 ,$$

the initial conditions

$$u|_{t=0} = f(x) , \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x) \quad (0 \leqslant x \leqslant l) , \tag{68}$$

and the boundary conditions

$$u|_{x=0} = \varkappa_1(t) , \quad u|_{x=l} = \varkappa_2(t) \quad (0 \leqslant t \leqslant T) . \tag{69}$$

Let us show the uniqueness of the solution to the mixed problem (67) - (69),

under the assumption that the solution $u(x, t)$ has continuous partial derivatives up to the second order inclusive, within Q.

Suppose that u_1 and u_2 are two solutions to the problem. Then, the difference

$$v(x, t) = u_1(x, t) - u_2(x, t)$$

will satisfy the homogeneous equation

$$\rho(x) \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x} \left(p(x) \frac{\partial v}{\partial x}\right) - q(x)v \ , \tag{70}$$

the zero initial conditions

$$v|_{t=0} = 0 \ , \qquad \left.\frac{\partial v}{\partial t}\right|_{t=0} = 0 \ , \tag{71}$$

and the homogeneous boundary conditions

$$v|_{x=0} = 0 \ , \qquad v|_{x=l} = 0 \ . \tag{72}$$

Let us show that $v(x, t)$ is identically equal to zero in Q.

Let us examine the energy integral

$$E(t) = \frac{1}{2} \int_0^l \left[\rho(x) \left(\frac{\partial v}{\partial t}\right)^2 + p(x) \left(\frac{\partial v}{\partial x}\right)^2 + q(x)v^2\right] dx \tag{73}$$

and show that it is independent of t. Differentiating $E(t)$ with respect to t, we obtain

$$\frac{dE(t)}{dt} = \int_0^l \left[\rho(x) \frac{\partial v}{\partial t} \frac{\partial^2 v}{\partial t^2} + p(x) \frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x\, \partial t} + q(x)v \frac{\partial v}{\partial t}\right] dx \ . \tag{74}$$

Differentiation under the integral sign is possible because of the continuity of the second derivatives. Integrating the first term on the right by parts, we obtain

$$\frac{dE(t)}{dt} = \int_0^l \left[\rho(x) \frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x} \left(p(x) \frac{\partial v}{\partial x}\right) + q(x)v\right] \frac{\partial v}{\partial t}\, dx + p(x) \left.\frac{\partial v}{\partial x} \frac{\partial v}{\partial t}\right|_{x=0}^{x=l} .$$

Hence, because of eq. (70) and the boundary conditions (72), it follows that

$$\frac{dE(t)}{dt} = 0 \ , \qquad \text{that is} \qquad E(t) = \text{constant}$$

Taking the initial conditions (71) into consideration, we obtain

$$E(t) = \text{constant} = E(0) = \frac{1}{2} \int_0^l \left.\left[\rho(x) \left(\frac{\partial v}{\partial t}\right)^2 + p(x) \left(\frac{\partial v}{\partial x}\right)^2 + q(x)v^2\right]\right|_{t=0} dx = 0 \ .$$

Then, it follows from (73) and from the initial conditions (71) that $v(x, t)$ is identically equal to zero in Q, that is, that $u_1 = u_2$, which was to be proved.

Remark: The solution to the mixed problem (67) will remain unique if the boundary conditions (69) are replaced with the following more complicated conditions

$$\frac{\partial u}{\partial x} - h_1 u|_{x=0} = \varkappa_1(t) , \qquad \frac{\partial u}{\partial x} + h_2 u|_{x=l} = \varkappa_2(t) ,$$

where h_1 and h_2 are non-negative constants.

Problems

1. Find the solution to the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + bx(x - l)$$

with the zero initial and boundary conditions

$$u(0, t) = 0 , \qquad u(l, t) = 0 .$$

Answer:

$$u(x, t) = -\frac{bx}{12}(x^3 - 2x^2 l + l^3) + \frac{8l^4}{\pi^5}\sum_{n=0}^{\infty}\frac{\cos\frac{(2n+1)\pi at}{l}\sin\frac{(2n+1)\pi x}{l}}{(2n+1)^5}$$

2. A rod of length l, with the end $x = 0$ fixed, is at rest. At the time $t = 0$, a force Q (measured per unit of area) is applied to the free end and is directed along the rod. Find the displacement $u(x, t)$ of the rod at an arbitrary subsequent instant.

Answer:

$$u(x, t) = \frac{Qx}{E} - \frac{8Ql}{\pi^2 E}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\cos\frac{(2k+1)\pi at}{2l}\sin\frac{(2k+1)\pi x}{2l} ,$$

where E is the modulus of elasticity.

Chapter X

TORSIONAL VIBRATIONS OF A HOMOGENEOUS ROD

1. *The differential equation for torsional vibrations of a cylindrical rod*

Let us examine a homogeneous circular cylindrical rod of length l. Let us suppose that for some reason or other this rod undergoes *torsional vibrations*, that is, vibrations during which its cross sections remain plane and are rotated about the axis of the rod with no distortion of one cross section with respect to another. In the case of a circular cylindrical rod, the cross sections are not displaced parallel to the axis of the rod, as the twisting takes place. We shall consider only vibrations of small amplitude.

Let us show that, in this case, the angles of rotation of any cross section of the rod will satisfy the wave equation. We take as the origin of our coordinate system one of the ends of the rod, and we direct the x-axis along the axis of the rod.

Suppose that mn and m_1n_1 are two cross sections separated by a distance dx. For the section mn to rotate relative to the section m_1n_1 through an angle θ, a torque M must be applied to it.

We compute this torque in the following manner: Let us take an infinitesimally thin cylinder with cross section $d\sigma$ (fig. 31). Let us suppose that, under the action of the torque that is applied to this section, the end A of the generator AA_1 is displaced through an extremely small distance

$$AB = r\, d\theta \tag{1}$$

Let us denote by τ the value of the stress, caused by the displacement of the generator AA_1 to the position BA_1.

If we apply Hooke's law, we find that

$$\tau = G\varphi \ ,$$

where φ is the angle AA_1B and G is a constant known as the shear modulus.

It follows from this that the force applied to the cross section $d\sigma$ is expressed by the product

$$\tau\, d\sigma = G\varphi\, d\sigma \tag{2}$$

Furthermore, because of the very small dimensions of the triangle AA_1B, we may assume that

$$AB = \varphi\, dx \ , \tag{3}$$

and by comparing eqs. (1) and (3), we see that

$$\varphi = r\frac{\partial\theta}{\partial x} \ ;$$

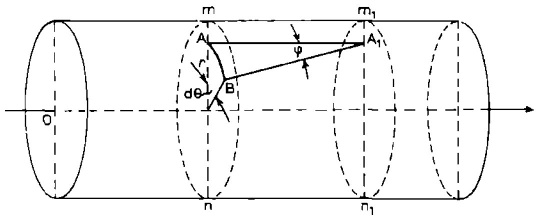

Fig. 31.

consequently,

$$\tau \, d\sigma = G \frac{\partial \theta}{\partial x} r \, d\sigma \, .$$

If we now denote by dM the element of the torque applied to the cross section $d\sigma$, we obtain

$$dM = r\tau \, d\sigma = G \frac{\partial \theta}{\partial x} r^2 \, d\sigma \, .$$

In order to find the total torque M, we need to integrate this equation over the entire area of the cross section mn; we then obtain

$$M = G \frac{\partial \theta}{\partial x} \int\int r^2 \, d\sigma \, .$$

The integral

$$\int\int r^2 \, d\sigma$$

is the polar moment of inertia of the cross section mn. Therefore, if we denote this quantity by J, our expression for the torque is

$$M = GJ \frac{\partial \theta}{\partial x} \, . \tag{4}$$

Let us now derive the differential equation for the torsional vibrations of the rod.

Let us consider that portion of the rod between the two cross sections mn and m_1n_1 whose abscissae are x and $x + dx$. Thr torque for the section whose abscissa is x is equal to $GJd\theta/dx$; the torque for the section with abscissa $x + dx$ is equal to

$$GJ \frac{\partial \theta}{\partial x} + GJ \frac{\partial^2 \theta}{\partial x^2} \, dx \, .$$

To obtain the equation for the torsional vibrations, we need to equate the resultant torque $GJ(\partial^2\theta/\partial x^2)dx$ with the product of the angular acceleration $\partial^2\theta/\partial t^2$ and the moment of inertia of the element mnm_1n_1 about the axis of the rod. Thus, we obtain

$$GJ \frac{\partial^2\theta}{\partial x^2}\,dx = \frac{\partial^2\theta}{\partial t^2}\,K\,dx \ ,$$

where K denotes the moment of inertia of a unit length of the rod.

If we now divide by $K\,dx$, we obtain

$$\frac{\partial^2\theta}{\partial t^2} = a^2\frac{\partial^2\theta}{\partial x^2}\,, \qquad \text{where} \qquad a = \sqrt{GJ/K}\ . \tag{5}$$

This is the differential equation for the torsional vibrations of a cylindrical rod.

If the rod is not a circular cylindrical one, the cross sections will not remain plane, but will be warped as the twisting takes place. On the basis of the theory, the torque M is determined from the formula

$$M = C\frac{\partial\theta}{\partial x}\,,$$

where C represents the resistance to torsion.

The differential equation for torsional vibrations of a cylindrical rod has the same form as eq. (5) except that GJ is replaced by C.

2. *The vibrations of a rod with fastened disk*

Let us investigate the torsional vibrations of a homogeneous rod in the case in which one of its ends $(x = 0)$ is fixed and the other end $(x = l)$ is fastened to a massive disk with moment of inertia K_1 about the axis of the rod. If we equate the moment of inertia of the disk with the torque at the cross section $x = l$, we obtain the following boundary condition at the end $x = l$:

$$K_1\left.\frac{\partial^2\theta}{\partial t^2}\right|_{x=l} = -\,GJ\left.\frac{\partial\theta}{\partial x}\right|_{x=l}.$$

Our problem is thus reduced to solving eq. (5) with the boundary conditions

$$\theta\big|_{x=0} = 0\,, \qquad \left.\frac{\partial^2\theta}{\partial t^2}\right|_{x=l} = -\,c^2\left.\frac{\partial\theta}{\partial x}\right|_{=} \qquad (c = \sqrt{GJ/K_1}) \tag{6}$$

and the initial conditions

$$\theta\big|_{t=0} = f(x)\,, \qquad \left.\frac{\partial\theta}{\partial x}\right|_{t=0} = F(x)\,. \tag{7}$$

Following the Fourier method, we seek particular solutions to eq. (5) in the form

$$\theta(x,t) = T(t)\,X(x)\ ; \tag{8}$$

then, we obtain the equations

$$T''(t) + a^2\lambda^2\,T(t) = 0\,, \tag{9}$$

$$X''(x) + \lambda^2 X(x) = 0\,. \tag{10}$$

For the function (8) to satisfy the boundary conditions (6) and not be identically equal to zero, the following conditions must be satisfied:

$$X(0) = 0 , \qquad c^2X'(l) - a^2\lambda^2X(l) = 0 . \tag{11}$$

Thus, we are confronted with the eigenvalue problem for eq. (10) with the boundary conditions (11).

Integrating eq. (10), we obtain

$$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x .$$

From the boundary conditions (11), we find

$$C_1 = 0 , \qquad (c^2\lambda \cos \lambda l - a^2\lambda^2 \sin \lambda l)\, C_2 = 0 .$$

Setting $C_2 \neq 0$, we obtain the transcendental equation

$$a^2\lambda \sin \lambda l - c^2 \cos \lambda l = 0 , \tag{12}$$

which determines the eigenvalues for the problem (10)-(11). Let us examine eq. (12). If we set

$$l\lambda = \mu , \qquad p = \frac{lc^2}{a^2} = \frac{lK}{K_1} , \tag{13}$$

eq. (12) will take the form

$$\mu \sin \mu - p \cos \mu = 0 \qquad (p > 0) . \tag{14}$$

To find the real roots of this equation, we need only construct the graph of the function

$$y = \cot \mu , \qquad y = \mu/p$$

and then determine the abscissae of the points of intersection of the two curves (fig. 32).

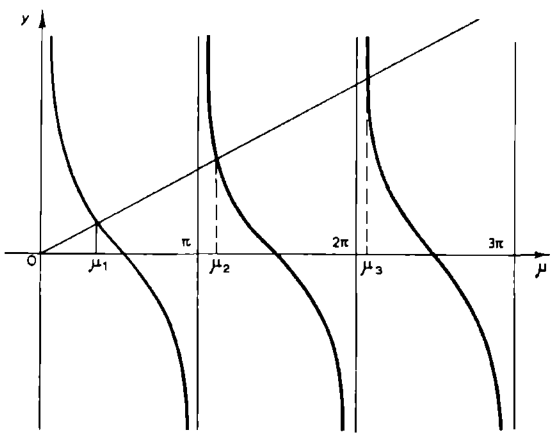

Fig. 32.

It is clear from the drawing that the root μ_k to eq. (14) increases without bound in absolute value with increase in the index k. At the same time, the difference $\mu_k - (k-1)\pi$ tends to zero. It follows from this that for sufficiently large k, we may set

$$\mu_k \approx (k-1)\pi\ . \tag{15}$$

If, from the conditions of the problem, the value of p is small, the approximating eq. (15) will give a sufficiently accurate result even if k is small. If p is not especially small, we resort to the method of iteration to find the roots

$$\mu_1, \mu_2, \mu_3, \dots\ .$$

In this method, we set

$$\mu_k = (k-1)\pi + \epsilon_k \tag{16}$$

and reduce eq. (14) to the following form:

$$\cot \epsilon_k = \frac{(k-1)\pi}{p} + \frac{\epsilon_k}{p}\ . \tag{17}$$

We then take the familiar expansion

$$\cot \epsilon_k = \frac{1}{\epsilon_k} - \tfrac{1}{3}\epsilon_k - \tfrac{1}{45}\epsilon_k{}^3 + \dots\ . \tag{18}$$

Under the assumption that $k > 1$, eq. (17) can be rewritten in the form

$$\epsilon_k = \frac{p}{(k-1)\pi} - \left(\frac{1}{p} + \frac{1}{3}\right)\frac{p}{(k-1)\pi}\epsilon_k{}^2 + \frac{p}{45(k-1)\pi}\epsilon_k{}^4 + \dots\ . \tag{19}$$

We take

$$\epsilon_k{}^{(1)} = \frac{p}{(k-1)\pi} \tag{20}$$

as the first approximation of the root of eq. (19), and we substitute it into the right side of the same equation. Then, if we confine ourselves to the first two terms, we obtain as our second approximation

$$\epsilon_k{}^{(2)} = \frac{p}{(k-1)\pi} - \left(\frac{1}{3} + \frac{1}{p}\right)\left(\frac{p}{(k-1)\pi}\right)^3\ . \tag{21}$$

If we put both the expressions found for ϵ_k successively in formulae (16), we obtain approximate values of the roots $\mu_2, \mu_3, \dots$, and then, as a first approximation, we have

$$\mu_2 = \pi + \frac{p}{\pi}\ , \qquad \mu_3 = 2\pi + \frac{p}{2\pi}\ ,$$

and as a second,

$$\mu_2 = \pi + \frac{p}{\pi} - \left(\frac{1}{3} + \frac{1}{p}\right)\left(\frac{p}{\pi}\right)^3\ , \qquad \mu_3 = 2\pi + \frac{p}{2\pi} - \left(\frac{1}{3} + \frac{1}{p}\right)\left(\frac{p}{2\pi}\right)^3\ .$$

With regard to the first root

$$\mu_1 = \epsilon_1\ ,$$

eq. (17) takes on the following form:

$$\cot \epsilon_1 = \epsilon_1/p \ .$$

Expanding $\cot \epsilon_1$ in a series in accordance with formula (18), we obtain

$$\epsilon_1{}^2 = \frac{3p}{3+p} - \frac{1}{15}\,\frac{p}{3+p}\,\epsilon_1{}^4 + \ldots \tag{22}$$

For a first approximation of the root μ_1, we obtain

$$\mu_1 = \epsilon_1{}^{(1)} = \sqrt{\frac{3p}{3+p}} \,. \tag{23}$$

To obtain a second approximation of the root μ_1, we need to substitute (23) into the right side of (22) and confine ourselves to the first two terms.

We then substitute the second approximation of the root μ_1 that we have found in this manner into the right side of eq. (22), and so on. By repeating this process a sufficient number of times, we can compute the value of the root μ_1 with a high degree of accuracy.

Eq. (14) cannot have purely imaginary roots. To prove this, let us suppose the opposite. If $\mu = i\nu$, where ν is a real number, we have

$$i\nu \sin i\nu - p \cos i\nu = 0$$

or

$$\nu \sinh \nu + p \cosh \nu = 0 \ ,$$

which is impossible because both terms are non-negative for all ν.

We shall prove below that eq. (14) cannot have complex roots (that is, roots with non-zero imaginary parts).

Thus, eq. (14) has only real roots. They occur in pairs that are equal in absolute value but opposite in sign, so that it is sufficient to examine only the positive roots. We denote by $\mu_1, \mu_2, \mu_3, \ldots$ the positive roots to eq. (14). Then, in accordance with (13), the eigenvalues will be

$$\lambda_k{}^2 = (\mu_k/l)^2 \qquad (k = 1, 2, 3, \ldots) \ . \tag{24}$$

To each eigenvalue $\lambda_k{}^2$ corresponds the eigenfunction

$$X_k(x) = \sin \frac{\mu_k x}{l} \qquad (k = 1, 2, 3, \ldots) \ . \tag{25}$$

It is easy to show that the eigenfunctions (25) are not orthogonal on the interval $(0, l)$.

For $\lambda = \lambda_k$, the general solution to eq. (9) will be of the form

$$T_k(t) = a_k \cos \frac{\mu_k a t}{l} + b_k \sin \frac{\mu_k a t}{l} \ ,$$

where a_k and b_k are arbitrary constants.

We see from (8) that the function

$$\theta_k(x, t) = \left(a_k \cos \frac{\mu_k a t}{l} + b_k \sin \frac{\mu_k a t}{l}\right) \sin \frac{\mu_k x}{l}$$

satisfies eq. (5) and the boundary conditions (6) for all values of a_k and b_k.

Let us define

$$\theta(x,t) = \sum_{k=1}^{\infty} \left(a_k \cos\frac{\mu_k a t}{l} + b_k \sin\frac{\mu_k a t}{l}\right) \sin\frac{\mu_k x}{l} . \tag{26}$$

For the initial conditions (7) to be satisfied, it is necessary that

$$\theta(x,0) = \sum_{k=1}^{\infty} a_k \sin\frac{\mu_k x}{l} = f(x) , \tag{27}$$

$$\frac{\partial\theta(x,0)}{\partial t} = \sum_{k=1}^{\infty} \frac{a\mu_k}{l} b_k \sin\frac{\mu_k x}{l} = F(x) . \tag{28}$$

These equations show that to find the coefficients a_k and b_k, we need to expand the functions $f(x)$ and $F(x)$ in a Fourier series of the eigenfunctions (25). It has been shown that these functions are not orthogonal in the interval $(0,l)$. But it is easy to show that the functions

$$\cos\frac{\mu_k x}{l} \qquad (k = 1, 2, \dots) \tag{29}$$

form an orthogonal system of functions in the interval $(0,l)$.

It is easy to show that

$$\int_0^l \cos\frac{\mu_k x}{l} \cos\frac{\mu_n x}{l}\, dx = l \cos\mu_k \cos\mu_n \frac{\mu_k \tan\mu_k - \mu_n \tan\mu_n}{\mu_k^2 - \mu_n^2}$$

from which it is clear that if μ_k and μ_n are roots to eq. (14),

$$\int_0^l \cos\frac{\mu_k x}{l} \cos\frac{\mu_n x}{l}\, dx = \begin{cases} 0 & \text{for } k \neq n , \\ \dfrac{l}{4\mu_k}(2\mu_k + \sin 2\mu_k) & \text{for } k = n . \end{cases} \tag{30}$$

Let us also suppose that the series (27) and (28) can be differentiated termwise with respect to x. Then, by taking eq. (30) into account, we easily find the values of the coefficients a_k and b_k:

$$a_k = \frac{4}{2\mu_k + \sin 2\mu_k} \int_0^l f'(x) \cos\frac{\mu_k x}{l}\, dx$$

$$b_k = \frac{4l}{a\mu_k} \frac{1}{2\mu_k + \sin 2\mu_k} \int_0^l F'(x) \cos\frac{\mu_k x}{l}\, dx .$$

Substituting these values for the coefficients in the series (26), we obtain the solution to the problem of the torsional vibrations of a homogeneous rod.

We stated above that eq. (14)

$$\mu \sin\mu - p \cos\mu = 0$$

can have only real roots. Let us suppose the opposite. Suppose that $\mu = a + ib$ is a root of eq. (14). Since p is a real number, eq. (14) will also have the

conjugate root $\mu = a - ib$. To these roots correspond the two eigenfunctions

$$X(x) = \sin\frac{(a+ib)x}{l}, \qquad \overline{X(x)} = \sin\frac{(a-ib)x}{l}.$$

From the condition of orthogonality (30), we have

$$\int_0^l \cos\frac{(a+ib)x}{l}\cos\frac{(a-ib)x}{l}\,dx = 0$$

or

$$\int_0^l \left(\cos^2\frac{ax}{l}\cosh^2\frac{bx}{l} + \sin^2\frac{ax}{l}\sinh^2\frac{bx}{l}\right)dx = 0$$

and we arrive at a contradiction.

Problems

1. Discuss the torsional vibrations of a homogeneous rod with one end ($x = 0$) free and the other ($x = l$) attached to a disk whose moment of inertia is k_1.
   Answer:

$$\theta(x,t) = \sum_{k=1}^{\infty}\left(a_k\cos\frac{\mu_k at}{l} + b_k\sin\frac{\mu_k at}{l}\right)\cos\frac{\mu_k x}{l}$$

$$a_k = \frac{2}{\mu_k}\,\frac{p^2+\mu_k^2}{p(p+1)+\mu_k^2}\int_0^l f(x)\sin\frac{\mu_k x}{l}\,dx \quad,$$

$$b_k = \frac{2l}{a\mu_k^2}\,\frac{p^2+\mu_k^2}{p(p+1)+\mu_k^2}\int_0^l F(x)\sin\frac{\mu_k x}{l}\,dx\,,$$

   where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the equation

$$\mu\cos\mu + p\sin\mu = 0 \qquad \left(p = \frac{lK}{K_1} > 0\right).$$

2. Discuss the torsional vibrations of a rod, both ends of which are attached to identical disks.
   Answer:

$$\theta(x,t) = \sum_{k=1}^{\infty}\left(a_k\cos\frac{\mu_k at}{l} + b_k\sin\frac{\mu_k at}{l}\right)\left(\sin\frac{\mu_k x}{l} - \frac{p}{\mu_k}\cos\frac{\mu_k x}{l}\right),$$

$$a_k = \frac{2\mu_k}{\mu_k^2 + p(p+2)}\int_0^l f(x)\left(\cos\frac{\mu_k x}{l} + \frac{p}{\mu_k}\sin\frac{\mu_k x}{l}\right)dx \quad,$$

$$b_k = \frac{2l}{a[\mu_k^2 + p(p+2)]}\int_0^l F(x)\left(\cos\frac{\mu_k x}{l} + \frac{p}{\mu_k}\sin\frac{\mu_k x}{l}\right)dx\,,$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are positive roots of the equation

$$2 \cot \mu = \frac{\mu}{p} - \frac{p}{\mu} \qquad (p = lK/K_1)$$

3. Suppose that a rod is fixed at one end ($x = 0$) and that a weight P is hung at the other end ($x = l$). Discuss the longitudinal vibrations of the rod under the assumption that an external force $\rho g(x, t)$ is acting on it.
Method of solution: The problem is reduced to integrating the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + g(x, t)$$

with boundary conditions

$$u|_{x=0} = 0 , \qquad \left.\frac{\partial^2 u}{\partial t^2}\right|_{x=l} = - c^2 \left.\frac{\partial u}{\partial x}\right|_{x=l} \qquad (c = \sqrt{gE/P})$$

and initial conditions

$$u|_{t=0} = f(x) , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x) .$$

Answer: the displacement of a cross section of the rod is expressed by the sum

$$u = u_1 + u_2 ,$$

where u_1 denotes the free vibrations of the rod (which are determined from formula (26) if we replace $\theta(x, t)$ with u_1) and where u_2 denotes the forced vibrations of the rod, which are determined by means of the series

$$u_2(x, t) = \sum_{k=1}^{\infty} T_k(t) \sin \frac{\mu_k x}{l} ,$$

where

$$T_k(t) = \frac{2}{a\mu_k^2} \frac{p^2 + \mu_k^2}{p(p+1) + \mu_k^2} \int_0^t d\tau \int_0^l \frac{\partial g(\xi, \tau)}{\partial \xi} \sin \frac{a\mu_k(t-\tau)}{l} \cos \frac{\mu_k \xi}{l} \, d\xi ,$$

and $\mu_1, \mu_2, \mu_3, \ldots$ are positive roots of the equation

$$\mu \tan \mu = p \qquad (p = gl\rho/P)$$

Chapter XI

ELECTRIC OSCILLATIONS IN LINES

1. *Transient phenomena in electric lines*

Suppose that we have an electric line in which oscillations have been set up as a result of some external influence. Suppose further, that, at the initial instant $t = 0$, the steady state of the line is suddenly changed. Such changes can take place for various reasons. For example, in a circuit with constant current and constant voltage, the resistance may suddenly be changed from R_a to R_b, or an aerial may suddenly receive a charge of atmospheric origin.

As a result of such a change, the electric line will pass from its original state into a new state; this change will not take place instantaneously, but will require a more or less extended period of time, which theoretically may be infinite (though, in actuality, it is finite). During this time, oscillations characterized by the values of the voltage v_f and the current i_f will take place. We shall assume that during the time of the transient process the state of the line is determined by the following resultant values of the current and voltage:

$$i = i_2 + i_f , \qquad v = v_2 + v_f , \tag{1}$$

where i_2 and v_2 are the current and voltage in the steady state of the line.

For $t = 0$, that is, at the beginning of the transient process, these sums must be equal to i_1 and v_1 (i_1 and v_1 being the current and the voltage in the original state). It is obvious that the functions i_f and v_f must satisfy the system of differential equations

$$\frac{\partial v}{\partial x} + L\frac{\partial i}{\partial t} + Ri = 0 , \qquad \frac{\partial i}{\partial x} + C\frac{\partial v}{\partial t} + Gv = 0 , \tag{2}$$

which were derived in Chapter V, and that the functions i_2 and v_2 must satisfy the system of equations

$$\frac{dv_2}{dx} + Ri_2 = 0 , \qquad \frac{di_2}{dx} + Gv = 0 .$$

2. *Steady-state processes following the application of a voltage*

Let us examine, as an example, the following process of steady-state oscillations in a line. Let us suppose that we have a line of length l that, at some initial instant of time, is characterized by the following values of voltage and current:

$$v_1 = 0\,, \qquad i_1 = 0\,. \tag{3}$$

Let us now suppose that, at the time $t = 0$, one end ($x = 0$) of the line is connected with a source of constant voltage E, while the other end ($x = l$) remains open. Thus, a change in the circuit will then take place, and its final *steady state* will be characterized by the quantities v_2 and i_2, which must satisfy the system of equations

$$\frac{\mathrm{d}v_2}{\mathrm{d}x} + Ri_2 = 0\,, \qquad \frac{\mathrm{d}i_2}{\mathrm{d}x} + Gv_2 = 0 \tag{4}$$

and the boundary conditions

$$v_2|_{x=0} = E\,, \qquad i_2|_{x=l} = 0\,. \tag{5}$$

The general solution to the system (4) is of the form

$$v_2 = A_1\,\mathrm{e}^{-bx} + A_2\,\mathrm{e}^{bx}\,, \qquad i_2 = \frac{b}{R}\,(A_1\,\mathrm{e}^{-bx} - A_2\,\mathrm{e}^{bx})\,, \tag{6}$$

where

$$b = \sqrt{RG}\,.$$

From the boundary conditions (5), we have

$$A_1 + A_2 = E\,, \qquad A_1\,\mathrm{e}^{-bl} - A_2\,\mathrm{e}^{bl} = 0\,,$$

from which

$$A_1 = \frac{E\,\mathrm{e}^{bl}}{2\sinh bl}\,, \qquad A_2 = \frac{E\,\mathrm{e}^{-bl}}{2\cosh bl}\,.$$

Substituting in eq. (6), we obtain

$$v_2 = E\,\frac{\cosh b(l-x)}{\cosh bl}\,, \qquad i_2 = \frac{Eb}{R}\,\frac{\sinh b(l-x)}{\cosh bl}\,. \tag{7}$$

We know that, at the time of the transient process, the state of the line is characterized by the quantities

$$v = v_2 + v_f\,, \qquad i = i_2 + i_f\,. \tag{8}$$

Obviously, if we find the values of v_f and i_f, we can also determine the transient state of the line.

At the first end ($x = 0$), where the voltage E is connected, the voltage v immediately takes the value E. Therefore, it follows from eqs. (8) and the boundary conditions (5) that

$$v_f|_{x=0} = 0\,.$$

Thus, the fact that the other end ($x = l$) of the line is open implies that

$$i_f|_{x=l} = 0\,.$$

Also, the functions v_f and i_f must satisfy the initial conditions

$$v_f|_{t=0} = v|_{t=0} - v_2|_{t=0} = v_1 - v_2|_{t=0}\,,$$

$$i_f|_{t=0} = i|_{t=0} - i_2|_{t=0} = i_1 - i_2|_{t=0}\,,$$

or, on the basis of (3) and (7),

$$v_f|_{t=0} = -E\frac{\cosh b(l-x)}{\cosh bl}, \qquad i_f|_{t=0} = -E\frac{b}{R}\frac{\sinh b(l-x)}{\cosh bl} \tag{9}$$

Thus, to determine the functions v_f and i_f, we need to solve the system of differential equations

$$\frac{\partial v}{\partial x} + L\frac{\partial i}{\partial t} + Ri = 0, \qquad \frac{\partial i}{\partial x} + C\frac{\partial v}{\partial t} + Gv = 0, \tag{10}$$

satisfying the boundary conditions

$$v_f|_{x=0} = 0, \qquad i_f|_{x=l} = 0 \tag{11}$$

and the initial conditions (9).

To simplify calculations, we set $G = 0$; that is, we assume that the conductor is completely insulated. Then, eqs. (10) and the initial conditions (9) take the simpler form

$$\frac{\partial v_f}{\partial x} + L\frac{\partial i_f}{\partial t} + Ri_f = 0, \qquad \frac{\partial i_f}{\partial x} + C\frac{\partial v_f}{\partial t} = 0, \tag{12}$$

$$v_f|_{t=0} = -E, \qquad i_f|_{t=0} = 0. \tag{13}$$

With the boundary conditions (11) in mind, we seek a solution of the form

$$v_f = \sum_{k=0}^{\infty} T_k(t)\sin\frac{(2k+1)\pi x}{2l}, \qquad i_f = \sum_{k=0}^{\infty}\tau_k(t)\cos\frac{(2k+1)\pi x}{2l} \tag{14}$$

Substituting (14) into the system (12), we obtain

$$\sum_{k=0}^{\infty}\left[\frac{(2k+1)\pi}{2l}T_k(t) + L\tau_k'(t) + R\tau_k(t)\right]\cos\frac{(2k+1)\pi x}{2} = 0,$$

$$\sum_{k=0}^{\infty}\left[-\frac{(2k+1)\pi}{l}\tau_k(t) + CT_k'(t)\right]\sin\frac{(2k+1)\pi x}{2l} = 0,$$

from which,

$$L\tau_k'(t) + R\tau_k(t) + \frac{(2k+1)\pi}{2l}T_k(t) = 0, \tag{15}$$

$$CT_k'(t) - \frac{(2k+1)\pi}{2l}\tau_k(t) = 0 \qquad (k = 0,1,2,\ldots). \tag{16}$$

Differentiating eq. (15) and replacing $T_k'(t)$ with the value given by eq. (16), we obtain

$$\tau_k''(t) + \frac{R}{L}\tau_k'(t) + \frac{(2k+1)^2\pi^2}{4l^2LC}\tau_k(t) = 0. \tag{17}$$

The general solution to eq. (17) is of the form

$$\tau_k(t) = e^{-\alpha t}(A_k\cos\omega_k t + B_k\sin\omega_k t), \tag{18}$$

where A_k and B_k are arbitrary constants and

$$\alpha = \frac{R}{2L}\,, \qquad \omega_k = \sqrt{\frac{(2k+1)^2\,\pi^2}{4l^2LC} - \frac{R^2}{4L^2}} \tag{19}$$

Substituting eq. (18) into eq. (15), we obtain

$$T_k(t) = -\frac{2lL}{(2k+1)\pi}\,e^{-\alpha t}\,[(\alpha A_k + \omega_k B_k)\cos\omega_k t + (\alpha B_k - \omega_k A_k)\sin\omega_k t]\,. \tag{20}$$

It now remains to determine the constants A_k and B_k that will satisfy the initial conditions (13). Setting $t = 0$ in the solution (14), we obtain, as a result of (13),

$$-E = \sum_{k=0}^{\infty} T_k(0)\sin\frac{(2k+1)\pi x}{2l}\,, \qquad 0 = \sum_{k=0}^{\infty} \tau_k(0)\cos\frac{(2k+1)\pi x}{2l}\,,$$

from which,

$$T_k(0) = -\frac{2}{l}\int_0^l E\sin\frac{(2k+1)\pi x}{2l}\,dx = -\frac{4E}{(2k+1)\pi}\,, \qquad \tau_k(0) = 0 \tag{21}$$

$$(k = 0, 1, 2, \ldots)\,.$$

If we now set $t = 0$ in eqs. (18) and (20), and if we take eq. (21) into account, we obtain

$$A_k = 0\,, \qquad B_k = 2E/lL\omega_k$$

and, consequently,

$$\begin{aligned} T_k(t) &= -\frac{4E}{(2k+1)\pi\omega_k}\,e^{-\alpha t}\,(\omega_k\cos\omega_k t + \alpha\sin\omega_k t)\,, \\ \tau_k(t) &= \frac{2E}{lL\omega_k}\,e^{-\alpha t}\sin\omega_k t\,. \end{aligned} \tag{22}$$

Substituting the functions $T_k(t)$ and $\tau_k(t)$ into the series (14), we obtain

$$\begin{aligned} v_f &= -\frac{4E}{\pi}\,e^{-\alpha t}\sum_{k=0}^{\infty}\frac{\omega_k\cos\omega_k t + \alpha\sin\omega_k t}{(2k+1)\omega_k}\sin\frac{(2k+1)\pi x}{2l}\,, \\ i_f &= \frac{2E}{lL}\,e^{-\alpha t}\sum_{k=0}^{\infty}\frac{\sin\omega_k t}{\omega_k}\cos\frac{(2k+1)\pi x}{2l}\,. \end{aligned} \tag{23}$$

From (19) and (23), it is easy to see that if $R < \pi(L/C)^{\frac{1}{2}}/l$, the free oscillations in the line are made up of the damped oscillations. If $R > \pi(L/C)^{\frac{1}{2}}/l$, a finite number of first terms in the series (23) describes the *non-periodic* motion. In an arbitrary case, the solution (23) will be damped with increasing t. Formulae (7), (8), and (23) give us the final expressions for the voltage and current:

$$v = E - \frac{4E}{\pi} e^{-\alpha t} \sum_{k=0}^{\infty} \frac{\omega_k \cos \omega_k t + \alpha \sin \omega_k t}{(2k+1)\omega_k} \sin \frac{(2k+1)\pi x}{2l} ,$$

$$i = \frac{2E}{lL} e^{-\alpha t} \sum_{k=0}^{\infty} \frac{\sin \omega_k t}{\omega_k} \cos \frac{(2k+1)\pi x}{2l} .$$

These determine the transient state of the line. We recall that we assumed that $G = 0$. Consequently, $b = 0$.

Problems

1. A line of length l is free of distortion ($R/L = G/C$). Suppose that a constant potential E is applied to it. Suppose that the end $x = l$ is insulated, and that at the instant $t = 0$ the other end ($x = 0$) is grounded. Show that the potential at any point x is equal to

$$v = \frac{4E}{\pi} e^{-(R/L)t} \sum_{k=0}^{\infty} \frac{\sin \frac{(2k+1)\pi x}{2l} \cos \frac{(2k+1)\pi a t}{2l}}{2k+1} \qquad (a^2 = 1/LC) .$$

2. Suppose that a potential E is applied to a line of length l in which there are no losses ($R = G = 0$), and that this line is open at both ends. Determine the current at every point of the line if, at the instant $t = 0$, the end $x = l$ is connected to a coil with self-inductance L_l, whose end is grounded.
Method of solution: The initial and boundary conditions for the free vibrations are the following:

$$v_f|_{t=0} = E , \qquad i_f|_{t=0} = 0 ,$$
$$i_f|_{x=0} = 0 , \qquad v_f|_{x=l} = L_l \left.\frac{\partial i_f}{\partial t}\right|_{x=l} .$$

Answer:

$$i_f = 2\alpha E \sqrt{\frac{C}{L}} \sum_{k=1}^{\infty} \frac{\sin (\mu_k a t/l) \sin (\mu_k x/l)}{[\mu_k^2 + \alpha(1+\alpha)] \cos \mu_k} \qquad (a^2 = 1/LC) ,$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are positive roots of the equation

$$\mu \tan \mu = \alpha \qquad (\alpha = lL/L_l) .$$

Chapter XII

BESSEL FUNCTIONS

1. *Bessel's equation*

In solving many of the problems of mathematical physics, we are led to the linear differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 , \tag{1}$$

where ν is a constant. This equation is encountered in problems in physics, mechanics, astronomy, and elsewhere. Eq. (1) is called *Bessel's* equation. Since eq. (1) has a singular point $x = 0$, we seek a particular solution to it in the form of a generalized power series:

$$y = x^\rho \sum_{k=0}^{\infty} a_k x^k \quad (a_0 \neq 0) . \tag{2}$$

Substituting the series (2) into eq. (1), we obtain

$$(\rho^2 - \nu^2)\, a_0\, x^\rho + [(\rho+1)^2 - \nu^2]\, a_1\, x^{\rho+1} + \sum_{k=2}^{\infty} \{[(\rho+k)^2 - \nu^2]\, a_k + a_{k-2}\}\, x^{\rho+k} = 0 . \tag{3}$$

If we set the coefficients of the various powers of x equal to zero, we obtain

$$\rho^2 - \nu^2 = 0 , \tag{4}$$

$$[(\rho+1)^2 - \nu^2]\, a_1 = 0 , \tag{5}$$

$$[(\rho+k)^2 - \nu^2]\, a_k + a_{k-2} = 0 . \tag{6}$$

From the first equation, we find two values for ρ:

$$\rho_1 = \nu \quad \text{and} \quad \rho_2 = -\nu .$$

If we take the first root $\rho = \nu$, we then obtain from eqs. (5) and (6)

$$a_1 = 0 \quad \text{and} \quad a_k = -\frac{a_{k-2}}{k(2\nu + k)} \quad (k = 2,3,4,\ldots) .$$

It then follows that

$$a_{2k+1} = 0 \quad (k = 0,1,2,\ldots) ,$$

and the coefficients with even subscripts are obviously determined by the formulae

$$a_2 = -\frac{a_0}{2^2(\nu+1)\times 1!}, \quad a_4 = \frac{a_0}{2^4(\nu+1)(\nu+2)\times 2!} \quad \text{etc.},$$

from which it is clear that the general expression for the coefficients a_{2k} is of the form

$$a_{2k} = (-1)^k \frac{a_0}{2^{2k}(\nu+1)(\nu+2)\dots(\nu+k)\times k!} \quad (k = 1, 2, 3, \dots) .$$

For the coefficient a_0, which up to now has been completely arbitrary, we choose the value

$$a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}, \tag{7}$$

where $\Gamma(\nu)$ is the gamma function, which is defined for all positive values of ν (and also for all complex values with positive real parts) by

$$\Gamma(\nu) = \int_0^\infty e^{-x} x^{\nu-1}\, dx . \tag{8}$$

With this choice of a_0, the coefficient a_{2k} can be written in the form

$$a_{2k} = (-1)^k \frac{a_0}{2^{2k+\nu}\, k!\, (\nu+1)(\nu+2)\dots(\nu+k)\,\Gamma(\nu+1)} . \tag{9}$$

This expression can be simplified if we use one of the basic properties of the gamma function. If we integrate the right side of eq. (8) by parts we obtain

$$\Gamma(\nu+1) = \nu\Gamma(\nu) . \tag{10}$$

We note that formula (10) makes it possible to define the gamma function for negative values of ν and also for all complex values.

Suppose that k is some positive integer. By repeated application of formula (10), we obtain

$$\Gamma(\nu+k+1) = (\nu+1)(\nu+2)\dots(\nu+k)\,\Gamma(\nu+1) . \tag{11}$$

Setting $\nu = 0$ in this formula, we find, on the basis of the equation

$$\Gamma(1) = \int_0^\infty e^{-x}\, dx = 1 ,$$

another important property of the gamma function, which is expressed by the equation

$$\Gamma(k+1) = k! . \tag{12}$$

By applying eq. (11) to the expression on the right side of eq. (9) for the coefficient a_{2k}, we obtain

$$a_{2k} = \frac{(-1)^k}{2^{2k+\nu}\; k!\, \Gamma(\nu+k+1)} . \tag{13}$$

If we substitute the values found above for the coefficients a_{2k+1} and a_{2k} into the series (2), we obtain a particular solution to eq. (1). This solution is known as the Bessel function of the first kind of ν-th order and is ordinarily denoted by $J_\nu(x)$. Thus,

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2}x)^{2k+\nu}}{k!\,\Gamma(\nu+k+1)} . \tag{14}$$

The series (14) converges for all values of x, which is easily verified by use of d'Alembert's test.

By using the second root $\rho_2 = -\nu$, we may obtain a second particular solution to eq. (1). Obviously, it may be obtained from eq. (14) simply by replacing ν by $-\nu$, since eq. (1) contains only ν^2 and is not changed when we make this substitution:

$$J_{-\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2}x)^{-\nu+2k}}{k!\,\Gamma(-\nu+k+1)} . \tag{15}$$

If ν is not equal to an integer, the particular solutions $J_\nu(x)$ and $J_{-\nu}(x)$ to Bessel's equation (1) will be linearly independent, since the expansions on the right sides of eqs. (14) and (15) will begin with different powers of x. However, if ν is a positive integer n, we can easily show that the solutions $J_n(x)$ and $J_{-n}(x)$ are linearly dependent. To show this, we note that for integral values of ν and for $k = 0, 1, 2, \ldots, n-1$, the value of $-\nu + k + 1$ will be a non-positive integer. For these values of k, we have $\Gamma(-\nu+k+1) = \infty$, which follows from the formula

$$\Gamma(m) = \frac{\Gamma(m+1)}{m} .$$

Thus, the first n terms in the expansion (15) vanish and we have

$$J_{-n}(x) = \sum_{k=n}^{\infty} \frac{(-1)^k (\frac{1}{2}x)^{-n+2k}}{\Gamma(k+1)\,\Gamma(-n+k+1)}$$

or, setting $k = n + l$,

$$J_{-n}(x) = (-1)^n \sum_{l=0}^{\infty} \frac{(-1)^l (\frac{1}{2}x)^{n+2l}}{\Gamma(l+1)\,\Gamma(n+l+1)} ,$$

or

$$J_{-n}(x) = (-1)^n J_n(x) \qquad (n \text{ is an integer}) . \tag{16}$$

From this it follows that for integral values of n, the functions $J_n(x)$ and $J_{-n}(x)$ are linearly dependent. In order to find the general solution to eq. (1) when ν is an integer, we need to find a second particular solution that is linearly independent of $J_\nu(x)$. To do this, we introduce the new function $Y_\nu(x)$, defined by

$$Y_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi} . \tag{17}$$

Obviously, this function will also be a solution to eq. (1), since it is a linear combination of the particular solutions $J_\nu(x)$ and $J_{-\nu}(x)$. It is then

easy, on the basis of eq. (16), to verify that when ν is a whole number, the right side of eq. (17) takes the indeterminate form 0/0. If we evaluate this form by l'Hopital's rule, we shall obtain, after a number of calculations (which we do not reproduce here because of their complexity), the following representation of the function $Y_n(x)$ for positive integral values of n:

$$Y_n(x) = \frac{2}{\pi} J_n(x) \ln \tfrac{1}{2}x - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} (\tfrac{1}{2}x)^{-n+2k}$$
$$- \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2}x)^{n+2k}}{k!\,(k+n)} \left[\frac{\Gamma'(k+1)}{\Gamma(k+1)} + \frac{\Gamma'(n+k+1)}{\Gamma(n+k+1)}\right]. \tag{18}$$

In the special case of $n = 0$, the function $Y_0(x)$ is represented as

$$Y_0(x) = \frac{2}{\pi} J_0(x) \ln \tfrac{1}{2}x - \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2}x)^{2k}}{(k!)^2} \frac{\Gamma'(k+1)}{\Gamma(k+1)}. \tag{19}$$

The function $Y_\nu(x)$ that we have just defined is called Bessel's function of ν-th order of the second kind, or Weber's function. Weber's function $Y_\nu(x)$ is also a solution to Bessel's equation, even when ν is an integer.

Obviously, $J_\nu(x)$ and $Y_\nu(x)$ are linearly independent; consequently, for every value of ν (integral or not), these functions form a fundamental system of solutions. It follows from this that the general solution to eq. (1) can be represented in the form

$$y = C_1 J_\nu(x) + C_2 Y_\nu(x), \tag{20}$$

where C_1 and C_2 are arbitrary constants.

We conclude this section by noting that the following recursion formulae are valid for Bessel's and Weber's functions:

$$J'_\nu(x) = J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x), \qquad Y'_\nu(x) = Y_{\nu-1}(x) - \frac{\nu}{x} Y_\nu(x), \tag{21}$$

$$J'_\nu(x) = -J_{\nu+1}(x) + \frac{\nu}{x} J_\nu(x), \qquad Y'_\nu(x) = -Y_{\nu+1}(x) + \frac{\nu}{x} Y_\nu(x), \tag{22}$$

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x), \qquad Y_{\nu+1}(x) = \frac{2\nu}{x} Y_\nu(x) - Y_{\nu-1}(x). \tag{23}$$

Formulae (21) and (22) can be verified directly by differentiating the series for Bessel functions. For example, let us prove formula (22). We have

$$\frac{d}{dx}\left[\frac{J_\nu(x)}{x^\nu}\right] = \frac{d}{dx}\left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{\nu+2k}\, k!\, \Gamma(\nu+k+1)}\right]$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^k\, 2k x^{2k-1}}{2^{\nu+2k}\, k!\, \Gamma(\nu+k+1)} = \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k-1}}{(k-1)!\, \Gamma(\nu+k+1) 2^{\nu+2k-1}}.$$

If we replace the index of summation k by $k+1$, we obtain

$$\frac{d}{dx}\left[\frac{J_\nu(x)}{x^\nu}\right] = -\sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2k+1}}{k!\,\Gamma(\nu+k+2)\,2^{\nu+2k+1}} = -\frac{1}{x^\nu}\sum_{k=0}^{\infty} \frac{(-1)^k\,(\frac{1}{2}x)^{\nu+1+2k}}{k!\,\Gamma(\nu+1+k+1)},$$

and, if we compare this expression with the right side of eq. (14), with ν replaced by $\nu+1$, we obtain

$$\frac{d}{dx}\left[\frac{J_\nu(x)}{x^\nu}\right] = -\frac{J_{\nu+1}(x)}{x^\nu}.$$

By performing the indicated differentiation on the left side, we can prove formula (22). Formula (21) can be verified in an analogous fashion.

2. *Certain particular cases of Bessel functions*

The most frequently encountered Bessel functions in mathematical physics are

$$J_0(x), \quad J_1(x), \quad Y_0(x) \quad \text{and} \quad J_{\pm n+\frac{1}{2}}(x),$$

where n is an integer.

The first two of these are represented by the series

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\times 4^2} - \frac{x^6}{2^2\times 4^2\times 6^2} + \dots, \tag{24}$$

$$J_1(x) = \frac{x}{2}\left(1 - \frac{x^2}{2\times 4} + \frac{x^4}{2\times 4^2\times 6} - \frac{x^6}{2\times 4^2\times 6^2\times 8} + \dots\right). \tag{25}$$

Detailed tables of these have been made. The graphs of the functions $J_0(x)$, $J_1(x)$, and $Y_0(x)$ are shown in figs. 33 and 34.

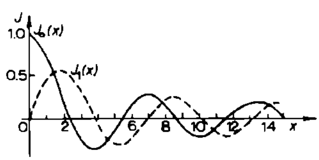

Fig. 33.

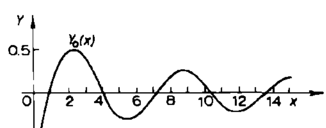

Fig. 34.

It is clear from formula (23) that calculation of the functions $J_2(x)$, $J_3(x)$, and so on, reduces to calculating the corresponding values of the functions $J_0(x)$ and $J_1(x)$.

Let us now turn to the function $J_{n+\frac{1}{2}}(x)$, where n is an integer. Let us first find the values of the functions $J_{\frac{1}{2}}(x)$ and $J_{-\frac{1}{2}}(x)$. From the expansion (14), we see that

$$J_{\frac{1}{2}}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\,(\frac{1}{2}x)^{\frac{1}{2}+2k}}{k!\,\Gamma(\frac{3}{2}+k)}.$$

But it follows immediately from formula (11) that

$$\Gamma(\tfrac{3}{2}+k) = \frac{1\times 3\times 5\ldots(2k+1)}{2^{k+1}}\,\Gamma(\tfrac{1}{2}) ,$$

where

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} .$$

Thus,

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} .$$

This last sum is the expansion of $\sin x$ in a power series, so that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x . \tag{26}$$

In an analogous way, it follows from the expansion (15) that

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x . \tag{27}$$

If we now apply formula (23), we easily see that

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(-\cos x + \frac{\sin x}{x}\right) = \sqrt{\frac{2}{\pi x}}\left[\sin(x-\tfrac{1}{2}\pi) + \frac{1}{x}\cos(x-\tfrac{1}{2}\pi)\right] ,$$

$$J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left\{-\sin x + \frac{3}{x}\left[\sin(x-\tfrac{1}{2}\pi) + \frac{1}{x}\cos(x-\tfrac{1}{2}\pi)\right]\right\}$$

$$= \sqrt{\frac{2}{\pi x}}\left[\left(1-\frac{3}{x^2}\right)\sin(x-\pi) + \frac{3}{x}\cos(x-\pi)\right].$$

In general, the function $J_{n+\frac{1}{2}}(x)$ for an integral value of n is expressed in terms of the elementary functions; specifically,

$$J_{n+\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[P_n\left(\frac{1}{x}\right)\sin(x-\tfrac{1}{2}n\pi) + Q_{n-1}\left(\frac{1}{x}\right)\cos(x-\tfrac{1}{2}n\pi)\right], \tag{28}$$

where $P_n(1/x)$ is a polynomial of degree n in $1/x$ and $Q_{n-1}(1/x)$ is a polynomial of degree $n-1$. $P_n(0) = 1$, and $Q_{n-1}(0) = 0$. It follows from this that for large values of x, we obtain the asymptotic representation of the Bessel function

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\left[\cos(x-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi) + O(x^{-1})\right] \quad (x>0) , \tag{29}$$

where $O(x^{-1})$ denotes a magnitude of the order of $1/x$.

We note that the asymptotic formula (29) is valid not only for $\nu = n + \frac{1}{2}$, but for *all* values of ν.

3. *The orthogonality of the Bessel functions and the roots of these functions*

Let us examine the equation

$$x^2y'' + xy' + (k^2x^2 - \nu^2)y = 0 \ , \tag{30}$$

where k is some non-zero constant.

We introduce, instead of x, the new independent variable $t = kx$. Then, eq. (30) becomes

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0 \ ,$$

which is Bessel's equation. Consequently, the function $y = J_\nu(kx)$ will be the solution to the equation

$$x^2 \frac{d^2J_\nu(kx)}{dx^2} + x\frac{dJ_\nu(kx)}{dx} + (k^2x^2 - \nu^2)\ J_\nu(kx) = 0$$

or, dividing by x, we may rewrite the given equation in the form

$$\frac{d}{dx}\left[x\frac{dJ_\nu(kx)}{dx}\right] + \left(k^2x - \frac{\nu^2}{x}\right) J_\nu(kx) = 0 \ . \tag{31}$$

Let us take two distinct values of k and write the corresponding differential equations

$$\frac{d}{dx}\left[x\frac{dJ_\nu(k_1x)}{dx}\right] + \left(k_1{}^2x - \frac{\nu^2}{x}\right) J_\nu(k_1x) = 0 \ ,$$

$$\frac{d}{dx}\left[x\frac{dJ_\nu(k_2x)}{dx}\right] + \left(k_2{}^2x - \frac{\nu^2}{x}\right) J_\nu(k_2x) = 0 \ .$$

Let us multiply the first of these equations by $J_\nu(k_2x)$ and the second by $J_\nu(k_1x)$, and subtract the first from the second. After some manipulation, we obtain

$$(k_2{}^2 - k_1{}^2)\ xJ_\nu(k_1x)\ J_\nu(k_2x) = \frac{d}{dx}\left[xJ_\nu(k_2x)\ \frac{dJ_\nu(k_1x)}{dx} - xJ_\nu(k_1x)\ \frac{dJ_\nu(k_2x)}{dx}\right] \tag{32}$$

If we now apply formula (14), we can easily see that the expression in the square brackets can be expanded in powers of x; in such an expansion, the lowest power of x will be $2(\nu + 1)$. It is clear from this that the above expression will vanish for $x = 0$ if $\nu > -1$. With this in mind, let us integrate eq. (32) over the finite interval $[0, l]$. We then obtain

$$(k_2{}^2 - k_1{}^2) \int_0^l xJ_\nu(k_1x)\, J_\nu(k_2x)\, dx = l[k_1J_\nu'(k_1l)\, J_\nu(k_2l) - k_2J_\nu'(k_2l)\, J_\nu(k_1l)] \tag{33}$$

where the prime followed by parentheses denotes, as usual, differentiation with respect to the argument. For $l = 1$, this formula becomes

$$(k_2{}^2 - k_1{}^2) \int_0^1 xJ_\nu(k_1x)\ J_\nu(k_2x)\ dx = k_1J_\nu'(k_1)\ J_\nu(k_2) - k_2J_\nu'(k_2)\ J_\nu(k_1) \ . \tag{34}$$

Let us now show that if $\nu > -1$ the Bessel function $J_\nu(x)$ cannot have other than real roots. Suppose that it has a root of the form $a + ib$, where

$a \neq 0$. In the expansion (14), all coefficients in the expansion are real and, consequently, the function $J_\nu(x)$ must have, in addition to the root $a + ib$, the conjugate root $a - ib$. Let us set $k_1 = a + ib$ and $k_2 = a - ib$ in formula (34). Here, $k_1{}^2 \neq k_2{}^2$ and the formula gives

$$\int_0^1 xJ_\nu(k_1x)\, J_\nu(k_2x)\, dx = 0 .$$

The quantities $J_\nu(k_1x)$ and $J_\nu(k_2x)$ will be complex conjugates; consequently, in the preceding formula, the integrand will be positive and the equation cannot be valid. Let us now show that the function $J_\nu(x)$ cannot have purely imaginary roots. If we substitute $\pm\, ib$ in formula (14), we obtain an expansion containing only positive terms:

$$J_\nu(ib) = (ib)^\nu \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(\nu+k+1)}\, \frac{b^{2k}}{2^{\nu+2k}} ,$$

since, as is indicated by eq. (8), the gamma function has only positive values for positive values of x.

Let us now show that the function $J_\nu(x)$ does have real roots. To do this, we use the asymptotic expansion of a Bessel function (29):

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\left[\cos\left(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right) + O(x^{-1})\right] \qquad (x > 0) .$$

It is clear from this formula that, as x increases without bound, the second term in the square brackets will approach zero and the first term will oscillate infinitely between -1 and +1. It follows immediately that the function $J_\nu(x)$ has an infinite number of real roots.

Thus, we arrive at the following result: if $\nu > -1$, the function $J_\nu(x)$ has all real roots.

We further note that it follows immediately from the expansion (14), which contains only even powers, that pairs of roots of $J_\nu(x)$ will be identical in absolute value and opposite in sign. Thus, we need only examine the positive roots. Suppose that $k_1 = \mu_i/l$ and $k_2 = \mu_j/l$, where μ_i and μ_j are two distinct positive roots of the equation

$$J_\nu(x) = 0 . \tag{35}$$

Then formula (33) directly yields the following *property of orthogonality of the Bessel functions:*

$$\int_0^l xJ_\nu\left(\mu_i \frac{x}{l}\right) J_\nu\left(\mu_j \frac{x}{l}\right) dx = 0 \qquad (i \neq j) . \tag{36}$$

Now suppose that $k = \mu/l$, where μ is a positive root of eq. (35). Let us set $k_1 = k$ in formula (33), and let us assume that k_2 is a variable that approaches k. We then obtain

$$\int_0^l xJ_\nu(kx)\, J_\nu(k_2x)\, dx = \frac{lkJ_\nu'(kl)\, J_\nu(k_2l)}{k_2{}^2 - k^2} .$$

As k_2 approaches k, the right side of this equation becomes indeterminate, since the numerator and denominator both approach zero. By using l'Hopital's rule, we obtain

$$\int_0^l xJ_\nu{}^2\left(\mu\frac{x}{l}\right)\,\mathrm{d}x = \tfrac{1}{2}l^2 J_\nu'^2(\mu)\ . \tag{37}$$

Setting $x = \mu$ in formula (22) and remembering that μ is a root of eq. (35), we obtain

$$J_\nu'(\mu) = -\ J_{\nu+1}(\mu)\ ,$$

and formula (37) can be rewritten in the following form:

$$\int_0^l xJ_\nu{}^2\left(\mu\frac{x}{l}\right)\,\mathrm{d}x = \tfrac{1}{2}l^2 J_{\nu+1}^2(\mu)\ . \tag{38}$$

We thus obtain

$$\int_0^l xJ_\nu\left(\mu_i\frac{x}{l}\right)J_\nu\left(\mu_j\frac{x}{l}\right)\mathrm{d}x = \begin{cases} 0 & \text{for} \quad j \neq i\ , \\ \tfrac{1}{2}l^2 J_\nu'^2(\mu_i) = \tfrac{1}{2}l^2 J_{\nu+1}^2(\mu_i) & \text{for} \quad j = i\ , \end{cases} \tag{39}$$

$$(\nu > -1)\ ,$$

where μ_i and μ_j are positive roots of the equation $J_\nu(x) = 0$.

Let us now examine the more general equation

$$\alpha J_\nu(x) + \beta xJ_\nu'(x) = 0 \qquad (\nu > -1)\ , \tag{40}$$

where α and β are real constants.

Suppose that $k_1 = \mu_i/l$ and $k_2 = \mu_j/l$, where μ_i and μ_j are two distinct roots of eq. (40); that is,

$$\alpha J_\nu(k_1 l) + \beta k_1 lJ_\nu'(k_1 l) = 0\ , \qquad \alpha J_\nu(k_2 l) + \beta k_2 lJ_\nu'(k_2 l) = 0\ .$$

It immediately follows that

$$k_1 J_\nu'(k_1 l)\, J_\nu(k_2 l) - k_2 J_\nu'(k_2 l)\, J_\nu(k_1 l) = 0\ ,$$

and, consequently, in this case, the right side of formula (33) is equal to zero and we have, as before, the condition of orthogonality (36).

Just as above, it follows immediately from the condition of orthogonality that eq. (40) cannot have complex roots of the form $a + ib$, where $a \neq 0$. Eq. (40) also cannot have purely imaginary roots $\pm\, ib$ (with the exception of the case in which $\alpha/\beta + \nu < 0$, when it does have two purely imaginary roots).

It is easy to show that eq. (40) does have real roots. Let us set

$$y = J_\nu'^2(x) + \left(1 - \frac{\nu^2}{x^2}\right) J_\nu^2(x)\ .$$

Then, after certain simple calculations, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{xJ_\nu'(x)}{J_\nu(x)}\right] = -\ \frac{xy}{J_\nu^2(x)}\ .$$

It follows from this that between any two consecutive roots μ_i and μ_{i+1} (where $\mu_{i+1} > \mu_i > \nu$) of the function $J_\nu(x)$ the derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{xJ_\nu'(x)}{J_\nu(x)}\right] < 0 .$$

Consequently, the function $xJ_\nu'(x)/J_\nu(x)$ decreases monotonically from $+\infty$ to $-\infty$ as x is increased from μ_i to μ_{i+1}, and therefore, it takes on every value once and only once. It follows from this that eq. (40) has one and only one root in the interval (μ_i, μ_{i+1}). Thus, we have the following result:

If $\nu > -1$ and $\alpha/\beta + \nu$ is non-negative, all roots of eq. (40) will be real. Suppose now that $k = \mu/l$, where μ is a positive root of eq. (40). Let us set $k_1 = k$ in formula (33) and let us assume that k_2 is a variable that approaches k. We then obtain

$$\int_0^l xJ_\nu(kx)\, J_\nu(k_2x)\, \mathrm{d}x = \frac{l[kJ_\nu'(\mu)\, J_\nu(k_2l) - k_2J_\nu'(k_2l)\, J_\nu(\mu)]}{k_2{}^2 - k^2}$$

As k_2 approaches k, the right side of this equation becomes indeterminate. By using l'Hôpital's rule, we obtain

$$\int_0^l xJ_\nu^2\left(\mu\frac{x}{l}\right)\mathrm{d}x = \frac{l[\mu J_\nu'^2(\mu) - J_\nu'(\mu)\, J_\nu(\mu) - \mu J_\nu''(\mu)\, J_\nu(\mu)]}{2k}$$

or, by using the equation

$$J_\nu''(\mu) + \frac{1}{\mu}J_\nu'(\mu) + \left(1 - \frac{\nu^2}{\mu^2}\right) J_\nu(\mu) = 0 ,$$

we arrive after certain simple manipulations, at the formula

$$\int_0^l xJ_\nu{}^2\left(\mu\frac{x}{l}\right)\mathrm{d}x = \tfrac{1}{2}l^2\left[J_\nu'^2(\mu) + \left(1 - \frac{\nu^2}{\mu^2}\right)J_\nu^2(\mu)\right],$$

and, by remembering that

$$J_\nu'(\mu) = -\frac{\alpha}{\beta\mu}J_\nu(\mu) ,$$

we finally obtain

$$\int_0^l xJ_\nu^2\left(\mu\frac{x}{l}\right)\mathrm{d}x = \tfrac{1}{2}l^2\left(1 + \frac{\alpha^2 - \beta^2\nu^2}{\beta^2\mu^2}\right)J_\nu^2(\mu) , \tag{41}$$

where μ is a positive root of eq. (40).

4. *The expansion of an arbitrary function in a series of Bessel functions*

Suppose that an arbitrary function $f(x)$ is represented in the form of a series

$$f(x) = \sum_{i=1}^{\infty} a_i J_\nu\left(\mu_i \frac{x}{l}\right) \qquad (\nu > -1) \,, \tag{42}$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are positive roots of the equation $J_\nu(x) = 0$, numbered in increasing order.

To determine the coefficients a_i, we multiply both sides of eq. (42) by $xJ_\nu(\mu_i x/l)$ and integrate the result over the interval $[0, l]$, assuming such termwise integration is possible. Then, taking formula (39) into consideration, we obtain

$$a_i = \frac{2}{l^2 J_{\nu+1}^2(\mu_i)} \int_0^l xf(x)\, J_\nu\left(\mu_i \frac{x}{l}\right) \mathrm{d}x \,. \tag{43}$$

The expansion (42), in which the coefficients a_i are determined in accordance with the formula (43), is called an expansion of the function $f(x)$ in a Fourier-Bessel series.

In mathematical physics, one frequently encounters the following series in Bessel functions:

$$f(x) = \sum_{i=1}^{\infty} b_i J_\nu\left(\mu_i \frac{x}{l}\right) \,, \tag{44}$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the equation

$$\alpha J_\nu(x) + \beta x J_\nu'(x) = 0 \,, \tag{40}$$

numbered in increasing order, where $\alpha/\beta + \nu > 0$.

On the basis of the orthogonality of the Bessel functions and of formula (41), the coefficients b_i are determined by

$$b_i = \frac{2}{l^2\left(1 + \dfrac{\alpha^2 - \beta^2\nu^2}{\beta^2\mu_i^2}\right) J^2(\mu_i)} \int_0^l xf(x)\, J_\nu\left(\mu_i \frac{x}{l}\right) \mathrm{d}x \,. \tag{45}$$

The expansion (44), in which the coefficients b_i are determined by the formula (45), is called the expansion of the function $f(x)$ in a Dini-Bessel series.

If $\alpha/\beta + \nu = 0$, then, as will be shown below (see eq. (49)), x^ν will be orthogonal to the functions $J_\nu(\mu_i x/l)$, with weight x, over the interval $[0, l]$, and, therefore, expansion (44) must be replaced by

$$f(x) = b_0 x^\nu \sum_{i=1}^{\infty} b_i J_\nu\left(\mu_i \frac{x}{l}\right) \,. \tag{46}$$

In this case, eq. (40) can be rewritten as

$$J_\nu'(x) = \frac{\nu}{x} J_\nu(x)$$

or, on the basis of formula (22),

$$J'_\nu(x) = -J_{\nu+1}(x) + \frac{\nu J_\nu(x)}{x},$$

we obtain

$$J_{\nu+1}(x) = 0 ; \tag{47}$$

that is, $\mu_1, \mu_2, \mu_3, \ldots$ will be roots of eq. (47).

To determine the coefficient b_0, we multiply both sides of expansion (46) by $x^{\nu+1}$ and integrate with respect to x from 0 to l, assuming that termwise integration is possible. Then, we obtair

$$\int_0^l x^{\nu+1} f(x)\,\mathrm{d}x = \frac{b_0 l^{2\nu+2}}{2\nu+2} + \sum_{i=1}^{\infty} b_i \int_0^l x^{\nu+1} J_\nu\left(\mu_i \frac{x}{l}\right) \mathrm{d}x . \tag{48}$$

We already have the formula

$$x^{\nu+1} J_\nu(x) = \frac{\mathrm{d}}{\mathrm{d}x}\,[x^{\nu+1} J_{\nu+1}(x)]$$

or

$$x^{\nu+1} J_\nu(xt) = \frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}x}\,[x^{\nu+1} J_{\nu+1}(xt)] .$$

Integrating this identity, we obtain

$$\int_0^l x^{\nu+1} J_\nu(xt)\,\mathrm{d}x = \frac{l^{\nu+1}}{t} J_{\nu+1}(tl) .$$

If we now set $t = \mu_i/l$, where μ_i is a root of eq. (47), we obtain

$$\int_0^l x^{\nu+1} J_\nu\left(\mu_i \frac{x}{l}\right) \mathrm{d}x = 0 ; \tag{49}$$

then, it follows from eqs. (48) and (49) that

$$b_0 = \frac{2(\nu+1)}{l^{2(\nu+1)}} \int_0^l x^{\nu+1} f(x)\,\mathrm{d}x . \tag{50}$$

The coefficients b_i $(i = 1, 2, 3, \ldots)$ are determined from ɪormula (45), which follows immediately from eq. (49).

5. *Some integral representations of the Bessel functions*

Bessel functions admit various representations in the form of definite integrals and line integrals. One of the simplest integral representations was given by Poisson. It can be obtained in the following manner:

We have

$$J_\nu(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(\frac{1}{2}x)^{\nu+2k}}{\Gamma(k+1)\,\Gamma(\nu+k+1)} . \tag{51}$$

We multiply the numerator and denominator of the general term of this series by $\Gamma(\nu+\frac{1}{2})\Gamma(k+\frac{1}{2})$ and, since

$$\Gamma(k+1)\,\Gamma(k+\tfrac{1}{2}) = \sqrt{\pi}\;2^{-2k}\,(2k)!\;,$$

we obtain

$$\frac{(-1)^k\,(\frac{1}{2}x)^{\nu+2k}}{\Gamma(k+1)\,\Gamma(\nu+k+1)} = \frac{(\frac{1}{2}x)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\;\frac{(-1)^k\,x^{2k}}{(2k)!}\;\frac{\Gamma(\nu+\frac{1}{2})\,\Gamma(k+\frac{1}{2})}{\Gamma(\nu+k+1)}$$

or, on the basis of a well-known formula,

$$\int_0^{\frac{1}{2}\pi}\cos^m\varphi\,\sin^n\varphi\,d\varphi = \frac{1}{2}\,\frac{\Gamma(\frac{1}{2}m+\frac{1}{2})\,\Gamma(\frac{1}{2}n+\frac{1}{2})}{\Gamma(\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2})}\;,$$

$$\frac{(-1)^k\,(\frac{1}{2}x)^{\nu+2k}}{\Gamma(k+1)\,\Gamma(\nu+k+1)} = \frac{2(\frac{1}{2}x)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\;\frac{(-1)^k\,x^{2k}}{(2k)!}\int_0^{\frac{1}{2}\pi}\cos^{2\nu}\varphi\,\sin^{2k}\varphi\,d\varphi\;. \tag{52}$$

As a consequence of eq. (52), the series (51) takes the form

$$J_\nu(x) = \frac{2(\frac{1}{2}x)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\sum_{k=0}^{\infty}\frac{(-1)^k\,x^{2k}}{(2k)!}\int_0^{\frac{1}{2}\pi}\cos^{2\nu}\varphi\,\sin^{2k}\varphi\,d\varphi\;.$$

Interchanging the summation and the integral, we obtain

$$J_\nu(x) = \frac{2(\frac{1}{2}x)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{1}{2}\pi}\cos^{2\nu}\varphi\sum_{k=0}^{\infty}\frac{(-1)^k\,x^{2k}\sin^{2k}\varphi}{(2k)!}\,d\varphi\;. \tag{53}$$

This last manipulation is permissible because of the uniform convergence of the series under the integral sign. This series is easily summed: it is equal to $\cos(x\sin\varphi)$. Thus, we finally obtain Poisson's formula

$$J_\nu(x) = \frac{2(\frac{1}{2}x)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{1}{2}\pi}\cos(x\sin\varphi)\,\cos^{2\nu}\varphi\,d\varphi\;, \tag{54}$$

where the real part of ν must be greater than $-\frac{1}{2}$ for the integral to converge; in this case, x can take any real or complex value.

Introducing a new variable of integration (making the substitution $t = \sin\varphi$), we may rewrite eq. (54) in the form

$$J_\nu(x) = \frac{2(\frac{1}{2}x)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^1(1-t^2)^{\nu-\frac{1}{2}}\cos xt\;dt\;.$$

Since the integrand here is even and the function $(1-t^2)^{\nu-\frac{1}{2}}\sin xt$ is odd, we may also write

$$J_\nu(x) = \frac{(\frac{1}{2}x)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_{-1}^1(1-t^2)^{\nu-\frac{1}{2}}\,e^{ixt}\;dt\;. \tag{55}$$

With Poisson's formula, it is easy to evaluate $J_\nu(x)$ for any real x. For, recalling that $|\cos(x\sin\varphi)| \leqslant 1$, we obtain from (54)

$$|J_\nu(x)| \leqslant \frac{2|\frac{1}{2}x|^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^{\frac{1}{2}\pi} \cos^{2\nu}\varphi\, d\varphi .$$

The right side of the last inequality, as can be seen from (52), is nothing more than the absolute value of the first term of the expansion of $J_\nu(x)$. Thus, we obtain a simple inequality that is valid for every real x and for every ν that exceeds $-\frac{1}{2}$:

$$|J_\nu(x)| \leqslant \frac{|\frac{1}{2}x|^\nu}{\Gamma(\nu+1)} .$$

Among the other representations of Bessel functions, let us consider one that is suitable only for functions $J_n(x)$ of integral order. Let us multiply the two series

$$e^{\frac{1}{2}xt} = \sum_{s=0}^{\infty} \frac{(\frac{1}{2}x)^s}{s!} t^s , \qquad e^{-x/2t} = \sum_{k=0}^{\infty} (-1)^k \frac{(\frac{1}{2}x)^k}{k!} t^{-k} ,$$

which converge absolutely for $|t| > 0$, so that the multiplication is permissible. If we group the result in powers of t, we obtain

$$\exp\left[\tfrac{1}{2}x\left(t - \frac{1}{t}\right)\right] = \sum_{m=-\infty}^{\infty} a_m\, t^m , \tag{56}$$

where, for non-negative m,

$$a_m = \sum_{k=0}^{\infty} (-1)^k \frac{(\frac{1}{2}x)^{m+2k}}{k!\,(m+k)!} = J_m(x) ,$$

and for negative m, if we set $-m = n$,

$$a_m = a_{-n} = \sum_{k=n}^{\infty} (-1)^k \frac{(\frac{1}{2}x)^{-n+2k}}{k!\,(k-n)!} = \sum_{s=0}^{\infty} (-1)^{s+n} \frac{(\frac{1}{2}x)^{n+2s}}{(s+n)!\,s!} = (-1)^n J_n(x) = J_m(x)$$

Now, the expression (56) can be rewritten as

$$\exp\left[\tfrac{1}{2}x\left(t - \frac{1}{t}\right)\right] = \sum_{m=-\infty}^{\infty} J_m(x)\, t^m . \tag{57}$$

The function on the left side of this equation is called the generating function for Bessel functions of integral order.

Since $J_{-m}(x) = (-1)^m J_m(x)$, we may rewrite eq. (57) in the form

$$\exp\left[\tfrac{1}{2}x\left(t - \frac{1}{t}\right)\right] = J_0(x) + \sum_{m=1}^{\infty} J_m(x)\,[t^m + (-1)^m t^{-m}] .$$

If we now set $t = e^{i\varphi}$, we obtain

$$\exp[ix \sin\varphi] = J_0(x) + \sum_{m=1}^{\infty} J_m(x)\,[e^{im\varphi} + (-1)^m e^{-im\varphi}] .$$

Let us multiply both sides of this equation by $e^{-in\varphi}$, where n is some positive number, and let us integrate with respect to φ from $-\pi$ to π. Then, since

$$\int_{-\pi}^{\pi} \exp\left[i(m-n)\varphi\right] d\varphi = \begin{cases} 0 & \text{for} \quad m \neq n, \\ 2\pi & \text{for} \quad m = n, \end{cases}$$

we obtain

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left[i(x \sin\varphi - n\varphi)\right] d\varphi . \tag{58}$$

Separating the real and imaginary parts on the right side, and making use of the properties of the integrals of even and odd functions, we easily obtain

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\varphi - n\varphi)\, d\varphi . \tag{59}$$

We note that formula (59) is not valid if n is not an integer. In such a case, we have a more complicated expression, namely,

$$J_\nu(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\varphi - \nu\varphi)\, d\varphi - \frac{\sin \nu\pi}{\pi} \int_0^{\infty} \exp\left[-\nu\varphi - x \sinh\varphi\right] d\varphi , \tag{60}$$

which is valid for arbitrary ν and any x whose real part is positive.

6. *Hankel's functions*

Besides the Bessel functions of the type $J_\nu(x)$, other particular solutions to Bessel's equation are of great significance in applications. Among these are Hankel's functions of the first and second kinds, denoted respectively by $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$, and defined by

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x) , \qquad H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x) . \tag{61}$$

For real values of x and ν, Hankel's functions have complex conjugate values:

$$H_\nu^{(2)}(x) = \overline{H_\nu^{(1)}(x)} .$$

If ν is not an integer, we replace the function $Y_\nu(x)$ in the definitive equations (61) with the expression for it given in eq. (17). We then obtain

$$H_\nu^{(1)}(x) = i \frac{J_\nu(x)\, e^{-i\nu\pi} - J_{-\nu}(x)}{\sin \nu\pi} , \qquad H_\nu^{(2)}(x) = -i \frac{J_\nu(x)\, e^{i\nu\pi} - J_{-\nu}(x)}{\sin \nu\pi} . \tag{62}$$

Formulae (62) remain valid for integral values of $\nu = n$, if we interpret the right members of these equations as the limits to which they tend as ν approaches n. If $\nu = n + \frac{1}{2}$, Hankel's functions can be expressed in finite form in terms of elementary functions. In particular, if $\nu = \frac{1}{2}$, we have

$$H_{\frac{1}{2}}^{(1)}(x) = i\left[-iJ_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)\right] = -i \sqrt{\frac{2}{\pi x}} (\cos x + i \sin x) = -i \sqrt{\frac{2}{\pi x}}\, e^{ix} .$$

Similarly,

$$H_{\frac{1}{2}}^{(2)}(x) = \mathrm{i}\sqrt{\frac{2}{\pi x}}\,\mathrm{e}^{-\mathrm{i}x} .$$

From formulae (62), it immediately follows that for pairs of Hankel functions with subscripts of opposite sign,

$$H_{-\nu}^{(1)}(x) = \mathrm{e}^{\mathrm{i}\nu\pi}\, H_\nu^{(1)}(x) , \qquad H_{-\nu}^{(2)}(x) = \mathrm{e}^{-\mathrm{i}\nu\pi}\, H_\nu^{(2)}(x) . \tag{63}$$

Furthermore, since Hankel's functions are linear combinations of $J_\nu(x)$ and $Y_\nu(x)$, they satisfy the same recursion formulae as do these functions; specifically,

$$\frac{\mathrm{d}H_\nu^{(1)}(x)}{\mathrm{d}x} = -\, H_{\nu+1}^{(1)}(x) + \frac{\nu}{x} H_\nu^{(1)}(x) ,$$

$$\frac{\mathrm{d}H_\nu^{(2)}(x)}{\mathrm{d}x} = -\, H_{\nu+1}^{(2)}(x) + \frac{\nu}{x} H_\nu^{(2)}(x) ,$$

$$\frac{\mathrm{d}H_\nu^{(1)}(x)}{\mathrm{d}x} = H_{\nu-1}^{(1)}(x) - \frac{\nu}{x} H_\nu^{(1)}(x) ,$$

$$\frac{\mathrm{d}H_\nu^{(2)}(x)}{\mathrm{d}x} = H_{\nu-1}^{(2)}(x) - \frac{\nu}{x} H_\nu^{(2)}(x) ,$$

$$H_{\nu+1}^{(1)}(x) = \frac{2\nu}{x} H_\nu^{(1)}(x) - H_{\nu-1}^{(1)}(x) ,$$

$$H_{\nu+1}^{(2)}(x) = \frac{2\nu}{x} H_\nu^{(2)}(x) - H_{\nu-1}^{(2)}(x) .$$

We conclude this section with the asymptotic representations of Hankel's functions, which we give without proof:

$$\left.\begin{aligned} H_\nu^{(1)}(x) &= \sqrt{\frac{2}{\pi x}}\, \exp\left[\mathrm{i}(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)\right]\left[1 + \mathrm{O}(x^{-1})\right] \\ H_\nu^{(2)}(x) &= \sqrt{\frac{2}{\pi}}\, \exp\left[-\,\mathrm{i}(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)\right]\left[1 + \mathrm{O}(x^{-1})\right] \end{aligned}\right\} \quad (x > 0) . \tag{64}$$

7. *Bessel's functions with imaginary argument*

In many problems of mathematical physics, we encounter the equation

$$x^2 y'' + xy' - (x^2 + \nu^2) y = 0 . \tag{65}$$

It is easy to show that this equation is obtained from Bessel's equation if we replace x by $\mathrm{i}x$. Consequently, the function $J_\nu(\mathrm{i}x)$ is a particular solution to eq. (65). Since eq. (65) is homogeneous, the product of $J_\nu(\mathrm{i}x)$ and an arbitrary constant is also a solution to that equation. We let the constant be $i^{-\nu}$ and we define

$$I_\nu(x) = i^{-\nu}\, J_\nu(\mathrm{i}x) . \tag{66}$$

With this choice of constant, the particular solution to eq. (65) that we are examining can be expressed by the series

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{(\frac{1}{2}x)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)} \,. \tag{67}$$

The function $I_{-\nu}(x)$ is also a solution to eq. (65), and if ν is not an integer, then $I_\nu(x)$ and $I_{-\nu}(x)$ are two linearly independent solutions to eq. (65). If $\nu = n$ is an integer, the functions $I_\nu(x)$ and $I_{-\nu}(x)$ are linearly dependent because

$$I_\nu(x) = I_{-\nu}(x) \,, \tag{68}$$

which follows immediately from eqs. (66) and (16).

To obtain the general solution of eq. (65), we need to find another particular solution, one that is linearly independent of $I_\nu(x)$. Such a particular solution, which is called Macdonald's function, is

$$K_\nu(x) = \frac{\pi}{2}\,\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi} \tag{69}$$

If $\nu = n$, the right side of eq. (69) is indeterminate, which follows easily from the relation (68). By using l'Hôpital's rule, we obtain the following expression for the function $K_n(x)$ for arbitrary n :

$$K_n(x) = -\,I_n(x)\ln \tfrac{1}{2}x + \tfrac{1}{2}\sum_{k=0}^{n-1} (-1)^k \frac{(n-k-1)!}{k!} (\tfrac{1}{2}x)^{-n+2k}$$
$$+ \tfrac{1}{2}\sum_{k=0}^{\infty} \frac{(\frac{1}{2}x)^{n+2k}}{k!\,(k+n)!}\left[\frac{\Gamma'(k+1)}{\Gamma(k+1)} + \frac{\Gamma'(k+n+1)}{\Gamma(k+n+1)}\right]. \tag{70}$$

In particular,

$$K_0(x) = -\,I_0(x)\ln \tfrac{1}{2}x + \tfrac{1}{2}\sum_{k=0}^{\infty} \frac{(\frac{1}{2}x)^{2k}}{(k!)^2}\,\frac{\Gamma'(k+1)}{\Gamma(k+1)} \,. \tag{71}$$

We note that $K_n(x)$ approaches $+\infty$ as x approaches zero.

Since $I_\nu(x)$ and $K_\nu(x)$ are two linearly independent solutions to eq. (65) for arbitrary ν, the general solution to this equation can be written in the form

$$y = C_1 I_\nu(x) + C_2 K_\nu(x) \,, \tag{72}$$

where C_1 and C_2 are arbitrary constants.

In conclusion, we note that $I_\nu(x)$ increases without bound as x tends to $+\infty$, and that $K_\nu(x)$ tends to zero as x tends to $+\infty$. This is clear from the asymptotic representations of the two functions, which we give here without proof:

$$\left.\begin{aligned} I_\nu(x) &= \frac{e^x}{\sqrt{2\pi x}}\,[1 + O(x^{-1})] \\ K_\nu(x) &= \sqrt{\frac{\pi}{2x}}\,e^{-x}\,[1 + O(x^{-1})] \end{aligned}\right\} \quad (x > 0) \,. \tag{73}$$

Problems

1. Prove the formulae

$$\frac{d}{dx}\left[x^{\frac{1}{2}\nu} J_\nu(2\sqrt{x})\right] = x^{\frac{1}{2}(\nu-1)} J_{\nu-1}(2\sqrt{x}) ,$$

$$\frac{d}{dx}\left[x^{-\frac{1}{2}\nu} J_\nu(2\sqrt{x})\right] = - x^{-\frac{1}{2}(\nu-1)} J_{\nu+1}(2\sqrt{x}) .$$

2. Find the general solution to the equation

$$y'' + \frac{5}{x} y' + y = 0 .$$

Method of proof: Introduce a new function u by setting $u = x^2 y$.
Answer:

$$y = \frac{C_1 J_2(x) + C_2 Y_2(x)}{x^2} .$$

3. Expand the function $f(x) = x^\nu$ (for $0 < x < 1$) in a Fourier-Bessel series.
Answer:

$$x^\nu = 2 \sum_{k=1}^{\infty} \frac{J_\nu(\mu_k x)}{\mu_k J_{\nu+1}(\mu_k)} \qquad (\nu > -1) .$$

4. Expand the function $f(x) = 1$ in a series of functions

$$J_0(\mu_1 x), J_0(\mu_2 x), \dots ,$$

where $\mu_1, \mu_2, \mu_3, \dots$ are the positive roots of the equation

$$\mu J_1(\mu) - \rho J_0(\mu) = 0 \qquad (\rho > 0) ,$$

in the interval $(0, l)$.
Answer:

$$f(x) = \sum_{k=1}^{\infty} \frac{2\rho}{\mu_k^2 + \rho^2} \frac{J_0(\mu_k x)}{J_0(\mu_k)} .$$

5. Prove that

$$\exp[ix \sin\varphi] = J_0(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x) \cos 2n\varphi + 2i \sum_{n=0}^{\infty} J_{2n+1}(x) \sin(2n+1)\varphi$$

and then derive Bessel's formulae

$$\left.\begin{aligned} J_{2n}(x) &= \frac{1}{\pi}\int_0^\pi \cos(x \sin\varphi) \cos 2n\varphi \, d\varphi \\ J_{2n+1}(x) &= \frac{1}{\pi}\int_0^\pi \sin(x \sin\varphi) \sin(2n+1)\varphi \, d\varphi \end{aligned}\right\} \qquad (n = 0, 1, 2, \dots) .$$

Method of solution: Use formula (57).

6. Prove that

$$\int_0^\infty e^{-ax} J_\nu(bx)\, x^{\nu+1}\, dx = \frac{2a(2b)^\nu\, \Gamma(\nu+\frac{3}{2})}{\sqrt{\pi}\,(a^2+b^2)^{\nu+\frac{3}{2}}}$$

$$\int_0^\infty e^{-a^2x^2}\, x^{\nu+1} J_\nu(bx)\, dx = \frac{b^\nu}{(2a^2)^{\nu+1}}\, e^{-b^2/4a^2} \qquad (\nu > -1)\ .$$

7. Prove that

$$\int_0^x xJ_0(x)\, dx = xJ_1(x)\ ,$$

$$\int_0^x x^3J_0(x)\, dx = 2x^2J_0(x) + (x^3 - 4x)\, J_1(x)\ .$$

Method of solution: Use the differential equations for the function $J_0(x)$.

Chapter XIII

SMALL-AMPLITUDE VIBRATIONS OF A THREAD SUSPENDED FROM ONE END

1. *The free vibrations of a suspended thread*

Let us examine a heavy, homogeneous, pliable thread of length l. The thread is fastened at the upper end (at the point $x = l$) and it vibrates under the action of gravity. The maximum displacement of its lower end $x=0$ from the vertical is equal to h. We take our x-axis in the vertical direction, so that it coincides with the initial position of the thread, which, under the action of its own weight, hangs straight downward. We denote by $u = u(x, t)$ the displacement of the points of the thread from this equilibrium position at the time t (fig. 35).

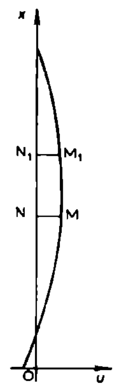

Fig. 35.

Let us consider vibrations of small amplitude, so that we may neglect the square of the derivative $\partial u/\partial x$ in comparison with unity. Then,

$$\sin \alpha(x) = \frac{\tan \alpha(x)}{\sqrt{1 + \tan^2\alpha(x)}} = \frac{\partial u/\partial x}{\sqrt{1 + (\partial u/\partial x)^2}} \approx \frac{\partial u}{\partial x},$$

where $\alpha(x)$ is the angle between the positive direction of the x-axis and the tangent to the thread at the point whose abscissa is x at the time t.

The tension T of the thread, at a point N whose abscissa is x, is equal to the weight of that portion of the thread that is lower than N; that is, $T = g\rho x$, where ρ is the linear density of the thread and g is the accelera-

tion due to gravity. Let us take an arbitrary element of the thread MM_1 of length dx, which, at equilibrium, occupies a position NN_1 (fig. 35). The horizontal component of the resultant of the forces of tension that are acting on the ends of the element MM_1 is expressed by the difference

$$\left(g\rho x \frac{\partial u}{\partial x}\right)_{M_1} - \left(g\rho x \frac{\partial u}{\partial x}\right)_{M},$$

which, with accuracy up to infinitesimally small terms of higher order, is equal to the expression

$$g\rho \frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x}\right) dx. \tag{1}$$

The vertical component is equal to

$$(g\rho x \cos\alpha(x))_{M_1} - (g\rho x \cos\alpha(x))_M \approx g\rho\, dx,$$

since, due to the smallness of the amplitudes of the vibrations of the thread,

$$\cos\alpha(x) = \frac{1}{\sqrt{1 + (\partial u/\partial x)^2}} \approx 1.$$

The motion of the element MM_1 can be regarded as free, provided we retain the forces of tension acting at the points M and M_1 and compute the force of gravity (which is directed downward and is equal to $-g\rho\, dx$). The vertical component of the resultant of the tension is exactly counterbalanced by the force of gravity. Therefore, we may assume that the element of the thread MM_1 moves under the action of the horizontal component of the force (1). If we equate this force with the product of the mass $\rho\, dx$ of the element of thread and its acceleration $\partial^2 u/\partial t^2$, we obtain the desired differential equation for small-amplitude vibrations of a suspended thread

$$\frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x}\right) = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}, \tag{2}$$

where $a = \sqrt{g}$.

The problem of the vibrations of the suspended thread is reduced to integrating eq. (2) with the boundary condition

$$u|_{x=l} = 0 \tag{3}$$

and initial conditions

$$u|_{t=0} = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x). \tag{4}$$

Since we wish to apply the Fourier method to the solution of problem (2) - (4), let us transform eq. (2) by introducing a new variable

$$\xi = \sqrt{x},$$

so that the transformed equation is now

$$\frac{1}{4\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial u}{\partial\xi}\right)=\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}\,. \tag{5}$$

Let us seek a solution to this equation in the form

$$u = w(\xi)\;T(t)\;. \tag{6}$$

Substituting in eq. (5), we have

$$\frac{1}{\xi w(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}w}{\mathrm{d}\xi}\right)=\left(\frac{2}{a}\right)^2\frac{T''(t)}{T(t)}\,.$$

Denoting the common value of the two sides of this equation by $-\lambda^2$, we obtain the two equations

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}w}{\mathrm{d}\xi}\right)+\lambda^2\xi w=0\;, \tag{7}$$

$$T''(t)+(\tfrac{1}{2}a\lambda)^2\,T(t)=0\;. \tag{8}$$

The general solution to eq. (7) is of the form (see Chapter XII, section 1):

$$w(\xi)=C_1 J_0(\lambda\xi)+C_2 Y_0(\lambda\xi)\;. \tag{9}$$

Since

$$Y_0(\lambda\xi)\to\infty\quad\text{as}\quad\xi\to 0\;,$$

C_2 must be zero. The boundary condition (3) gives

$$J_0(\lambda\sqrt{l})=0\;.$$

We showed in Chapter XII that the transcendental equation

$$J_0(\mu)=0$$

has an infinite number of real roots $\mu_1,\mu_2,\mu_3,\ldots$. From this, it follows that the eigenvalues of the problem are determined by the equation

$$\lambda_k{}^2=\mu_k{}^2/l\qquad(k=1,2,3,\ldots)\;. \tag{10}$$

The eigenfunctions corresponding to these eigenvalues are of the form

$$w_k(x)=J_0(\mu_k\sqrt{x/l})\;. \tag{11}$$

If we now turn to eq. (8), we see that its general solution is of the form

$$T_k(t)=A_k\cos\frac{a\mu_k t}{2\sqrt{l}}+B_k\sin\frac{a\mu_k t}{2\sqrt{l}}$$

and, consequently, the series

$$u(x,t)=\sum_{k=1}^{\infty}\left(A_k\cos\frac{a\mu_k t}{2\sqrt{l}}+B_k\sin\frac{a\mu_k t}{2\sqrt{l}}\right)J_0(\mu_k\sqrt{x/l}) \tag{12}$$

gives the solution to eq. (2) with the boundary condition (3).

It now remains to determine the constants A_k and B_k that satisfy the initial conditions (4). If we set $t = 0$ in the expansion (12), we find that

$$f(x) = \sum_{k=1}^{\infty} A_k J_0(\mu_k \sqrt{x/l}) \ . \tag{13}$$

By comparing this expansion with eqs. (42) and (43) of the preceding chapter, we easily see that

$$A_k = \frac{1}{l J_1{}^2(\mu_k)} \int_0^l f(x)\, J_0(\mu_k \sqrt{x/l})\, dx \ . \tag{14}$$

By a similar reasoning, we find the expression for the coefficients B_k, namely,

$$B_k = \frac{2}{a\sqrt{l}\mu_k J_1{}^2(\mu_k)} \int_0^l F(x)\, J_0(\mu_k \sqrt{x/l})\, dx \ . \tag{15}$$

If we now define N_k and φ_k by

$$A_k = N_k \sin \varphi_k \ , \qquad B_k = N_k \cos \varphi_k \ ,$$

we can rewrite the solution (12) in the form

$$u(x,t) = \sum_{k=1}^{\infty} N_k J_0(\mu_k \sqrt{x/l}) \sin\left(\frac{a\mu_k t}{2\sqrt{l}} + \varphi_k\right) , \tag{16}$$

from which it is clear that the small-amplitude vibrations of the suspended thread may be regarded as a composite motion of an infinite number of harmonic vibrations.

The period of the fundamental frequency of these vibrations is expressed by the formula

$$T_1 = \frac{4\pi}{\mu_1} \sqrt{l/g} \ , \tag{17}$$

where

$$\mu_1 = 2.40483 \ .$$

Eq. (16) also shows that the amplitude of the k-th harmonic vanishes at those points at which

$$J_0(\mu_k \sqrt{x/l}) = 0 \ ,$$

from which it is clear that we have k nodes:

$$x_1 = (\mu_1/\mu_k)^2\, l \ , \quad x_2 = (\mu_2/\mu_k)^2\, l \ , \quad \ldots , \quad x_{k-1} = (\mu_{k-1}/\mu_k)^2\, l \ , \quad x_k = l \ .$$

2. *Forced vibrations of a suspended thread*

Let us now suppose that a horizontal force $\Phi(x,t)$, distributed uniformly per unit length, is applied to a suspended thread. Then, the equation for the forced vibrations takes the form

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x}\right) + Y(x, t) \ , \tag{18}$$

where

$$Y(x, t) = \frac{\Phi(x, t)}{\rho} \ .$$

Let us combine this equation with the boundary and initial conditions

$$u|_{x=l} = 0 \ , \tag{19}$$

$$u|_{t=0} = f(x) \ , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x) \ . \tag{20}$$

To solve this problem, we apply the method explained in section 1 of Chapter IX; in other words, let us seek a solution to the problem (18) - (20) in the form of a sum

$$u = u_1 + u_2 \ , \tag{21}$$

where $u_1(x, t)$ is the solution of the *non-homogeneous* eq. (18), satisfying the boundary condition (19) and the zero initial conditions

$$u_1|_{t=0} = 0 \ , \qquad \left.\frac{\partial u_1}{\partial t}\right|_{t=0} = 0 \ , \tag{22}$$

and the function $u_2(x, t)$ is the solution to the homogeneous equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x}\right) , \tag{23}$$

satisfying the boundary condition (19) and the initial conditions (20). The problem stated by eqs. (23), (19), and (20) was examined in section 1, and its solution was obtained in the form of the series (12).

Let us seek a solution $u_1(x, t)$ in the form of the series

$$u_1 = \sum_{k=1}^{\infty} T_k(t) \ J_0(\mu_k \sqrt{x/l}) \ , \tag{24}$$

such that the boundary condition (19) will be satisfied. Substituting the series (24) into eq. (18) and remembering the equation

$$\frac{d}{dx}\left[x \frac{dJ_0(\mu_k \sqrt{x/l})}{dx}\right] = -\frac{1}{4} \frac{\mu_k^2}{l} J_0(\mu_k \sqrt{x/l}) \ ,$$

which was a consequence of relations (7) and (10), we obtain

$$\sum_{k=1}^{\infty} [T_k''(t) + \omega_k^2 T(t)] \ J_0(\mu_k \sqrt{x/l}) = Y(x, t) \ , \tag{25}$$

where

$$\omega_k = \frac{\mu_k a}{2\sqrt{l}} \ . \tag{26}$$

Let us now expand the function $Y(x, t)$ in a series of the eigenfunctions $J_0(\mu_k \sqrt{x/l})$; that is, we set

$$Y(x, t) = \sum_{k=1}^{\infty} H_k(t)\, J_0(\mu_k\sqrt{x/l})\ . \tag{27}$$

This expansion coincides in form with the expansion (13) and, consequently, the coefficients $H_k(t)$ are determined from formula (14):

$$H_k(t) = \frac{1}{lJ_1{}^2(\mu_k)} \int_0^l Y(\xi, t)\, J_0(\mu_k\sqrt{\xi/l})\, d\xi\ . \tag{28}$$

Equating eqs. (25) and (27), we obtain

$$T_k''(t) + \omega_k{}^2\, T_k(t) = H_k(t)\ , \tag{29}$$

which the coefficients $T_k(t)$ must satisfy. When the coefficients $T_k(t)$ are thus determined, the function (24) will satisfy the differential eq. (18) and the boundary condition (19). For the initial conditions (22) to be satisfied, it will be sufficient to impose on the functions $T_k(t)$ the conditions

$$T_k(0) = 0\ , \qquad T_k'(0) = 0\ . \tag{30}$$

The solution to eq. (29) that satisfies the initial conditions (30) is given by

$$T_k(t) = \frac{1}{\omega_k} \int_0^t H_k(\tau)\, \sin \omega_k(t-\tau)\, d\tau\ .$$

Substituting, in this equation, the expression for $H_k(\tau)$ given by eq. (28), we obtain

$$T_k(t) = \frac{1}{l\omega_k J_1{}^2(\mu_k)} \int_0^t d\tau \int_0^l Y(\xi, \tau)\, J_0(\mu_k\sqrt{\xi/l})\, \sin \omega_k(t-\tau)\, d\xi\ . \tag{31}$$

It now follows that the displacement of the suspended thread from its vertical equilibrium position is expressed by the formula

$$u(x, t) = \sum_{k=1}^{\infty} T_k(t)\, J_0(\mu_k\sqrt{x/l}) + \sum_{k=1}^{\infty} \left(A_k \cos \frac{a\mu_k t}{2\sqrt{l}} + B_k \sin \frac{a\mu_k t}{2\sqrt{l}}\right) J_0(\mu_k\sqrt{x/l})\ , \tag{32}$$

where the coefficients $T_k(t)$, A_k, and B_k are determined by eqs. (31), (14), and (15) and $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the equation $J_0(\mu) = 0$.

Let us examine in greater detail the case in which the external force is harmonic, that is, in which

$$Y(x, t) = A \sin \omega t\ .$$

In this case, the coefficients $T_k(t)$ are determined from the formula

$$T_k(t) = \frac{A}{l\omega_k J_1{}^2(\mu_k)} \int_0^t \sin \omega_k(t-\tau)\, \sin \omega\tau\, d\tau \int_0^l J_0(\mu_k\sqrt{\xi/l})\, d\xi\ .$$

Let us now take the formula

$$\int_0^x J_0(\sqrt{x})\,dx = 2\sqrt{x}\,J_1(\sqrt{x}) ,$$

which is easily derived from the expansions of the functions $J_0(x)$ and $J_1(x)$ in power series. By means of these formulae, we see that

$$\int_0^l J_0(\mu_k\sqrt{\xi/l})\,d\xi = \frac{2l}{\mu_k} J_1(\mu_k) ,$$

and since

$$\int_0^t \sin\omega_k(t-\tau)\sin\omega\tau\,d\tau = \frac{\omega_k \sin\omega t}{\omega_k^2-\omega^2} - \frac{\omega\sin\omega_k t}{\omega_k^2-\omega^2} ,$$

we have

$$T_k(t) = 4\sqrt{\frac{l}{g}}\frac{A}{\mu_k^2 J_1(\mu_k)}\left[\frac{\omega_k\sin\omega t}{\omega_k^2-\omega^2} - \frac{\omega\sin\omega_k t}{\omega_k^2-\omega^2}\right] . \tag{33}$$

Let us suppose that there are no initial displacements or velocities in the case in question, and that the thread is vibrating only as a result of the disturbing force. Then, it follows from eqs. (32) and (33) that the deviation of the thread from the vertical equilibrium position will be expressed by the formula

$$u(x,t) = 4\sqrt{\frac{l}{g}}A\sin\omega t\sum_{k=1}^{\infty}\frac{\omega_k J_0(\mu_k\sqrt{x/l})}{(\omega_k^2-\omega^2)\,\mu_k^2 J_1(\mu_k)}$$
$$-\,4A\omega\sqrt{\frac{l}{g}}\sum_{k=1}^{\infty}\frac{J_0(\mu_k\sqrt{x/l})\sin\omega_k t}{(\omega_k^2-\omega^2)\,\mu_k^2 J_1(\mu_k)} . \tag{34}$$

The first term on the right side of this equation can be simplified.

Let us seek a solution to the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) + A\sin\omega t , \tag{35}$$

satisfying the conditions

$$u|_{x=0} = \text{finite value} , \quad u|_{x=l} = 0 , \tag{36}$$

in the form of a product

$$u = X(x)\sin\omega t . \tag{37}$$

Substituting eq. (37) into eq. (35), we obtain

$$\frac{d}{dx}\left(x\frac{dX}{dx}\right) + \left(\frac{\omega}{a}\right)^2 X + \frac{A}{a^2} = 0 . \tag{38}$$

Its general solution will be of the form

$$X(x) = C_1 J_0\left(\frac{2\omega}{a}\sqrt{x}\right) + C_2 Y_0\left(\frac{2\omega}{a}\sqrt{x}\right) - \frac{A}{\omega^2}. \tag{39}$$

On the basis of the boundary conditions (36), we have

$$C_1 = \frac{A}{\omega^2}\,\frac{1}{J_0(2\omega\sqrt{l/g})}, \qquad C_2 = 0 .$$

It follows from these results that the solution (34) can be put in the form

$$u(x,t) = \frac{A}{\omega^2}\left[\frac{J_0(2\omega\sqrt{x/g})}{J_0(2\omega\sqrt{l/g})} - 1\right]\sin\omega t - 4A\omega\sqrt{\frac{l}{g}}\sum_{k=1}^{\infty}\frac{J_0(\mu_k\sqrt{x/l})\sin\omega_k t}{(\omega_k^2-\omega^2)\,\mu_k^2 J_1(\mu_k)}. \tag{40}$$

We note, in conclusion, that when the frequency ω of the external disturbing force is close to one of the frequencies of the natural oscillations of the thread, the phenomenon of *resonance* will be observed. We note also that a comparison of formulae (34) and (40) will give the following expansion of the function $J_0(tx)/J_0(t)$ in rational fractions:

$$\frac{J_0(tx)}{J_0(t)} = 1 + 2\sum_{k=1}^{\infty}\frac{t^2}{\mu_k^2 - t^2}\,\frac{J_0(\mu_k x)}{\mu_k J_1(\mu_k)}, \tag{41}$$

where the summation is made over all positive roots of the equation $J_0(x) = 0$

Problems

1. A completely pliable thread of length l is suspended from the end ($x = l$) and a weight P is fastened to the other end ($x = 0$). The linear density of the thread varies according to the equation

$$\rho = \frac{A}{\sqrt{l_1 + x}},$$

where the constants A and l_1 are related to the mass m of the weight by the relation

$$m = 2A\sqrt{l_1}.$$

Show that the equation for small-amplitude vibrations of the thread around its vertical equilibrium position will be of the form

$$\frac{\partial^2 u}{\partial\theta^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \qquad (a = \sqrt{\tfrac{1}{2}g}),$$

where

$$\theta = \sqrt{l_1 + l} - \sqrt{l_1 + x}.$$

Method of solution: The tension in the thread at the point M(x, u) is determined by the formula

$$T = mg + \int_0^x \frac{Ag\,dx}{\sqrt{l_1 + x}} = 2Ag\sqrt{l_1 + x} .$$

2. The linear density of a suspended thread varies according to the equation

$$\rho = \alpha x^m \qquad (m > -1) .$$

Derive the equation for the free vibrations of such a thread, and show that its displacement from its vertical equilibrium position is given by

$$u(x, t) = \sum_{k=1}^{\infty} N_k \frac{J_m(\mu_k\sqrt{x/l})}{x^{\frac{1}{2}m}} \sin\left(\tfrac{1}{2}\sqrt{\frac{g}{l(m+1)}}\,\mu_k t + \varphi_k\right) ,$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the equation

$$J_m(x) = 0 \qquad (m > -1) .$$

Method of solution: The problem is reduced to solving the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{m+1}{x}\frac{\partial u}{\partial x} = \frac{m+1}{gx}\frac{\partial^2 u}{\partial t^2}$$

under the conditions

$$u|_{x=l} = 0 , \quad u|_{t=0} = f(x) , \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x) .$$

When we apply the Fourier method, we need to seek a solution to this equation in the form

$$u = \frac{w(\xi)\ T(t)}{\xi^m} ,$$

where $\xi = \sqrt{x}$.

3. A heavy homogeneous thread of length l, fastened at the upper end ($x = l$) to the vertical axis, rotates around this axis with a constant angular velocity ω. Derive the equations for small-amplitude vibrations of the thread, and show that its displacement from the equilibrium position is expressed by

$$u(x, t) = \sum_{k=1}^{\infty} (A_k \cos a\lambda_k t + B_k \sin a\lambda_k t)\, J_0(\mu_k\sqrt{x/l}) ,$$

where

$$A_k = \frac{1}{lJ_1^2(\mu_k)} \int_0^l f(x)\, J_0(\mu_k\sqrt{x/l})\, dx ,$$

$$B_k = \frac{1}{a\lambda_k l J_1^2(\mu_k)} \int_0^l F(x)\, J_0(\mu_k\sqrt{x/l})\, dx , \quad \lambda_k = \sqrt{\frac{\mu_k^2}{4l} - \left(\frac{\omega}{a}\right)^2} ,$$

and $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the equation $J_0(\mu) = 0$.

Method of solution: The problem is reduced to solving the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x}\right) + \omega^2 u , \quad \text{where} \quad a = \sqrt{g} ,$$

and where

$$u|_{x=l} = 0 , \quad u|_{t=0} = f(x) , \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x) .$$

Chapter XIV

SMALL-AMPLITUDE RADIAL VIBRATIONS OF A GAS

1. *Radial vibrations of a gas in a sphere*

Suppose that we are dealing with a gas contained in a rigid impenetrable spherical container. Let us examine the small-amplitude vibrations of the gas around its equilibrium position.

It was shown in Chapter VI that the velocity potential satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{1}$$

In this section, we shall examine the so-called *radial* vibrations of a gas that take place when the initial conditions are expressed by the equations

$$u|_{t=0} = f(r) \ , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(r) \ , \tag{2}$$

where r is the distance from the vibrating particle of the gas to the center of the sphere.

Since the surface of the sphere is rigid, the normal component of the velocity is equal to zero, which leads to the boundary condition

$$\left.\frac{\partial u}{\partial r}\right|_{r=R} = 0 \ , \tag{3}$$

where R is the radius of the sphere.

Since the velocity potential u, in the case of radial vibrations, depends only on r and t, we can use the expression for the Laplacian operator in spherical coordinates (Chapter XVII, section 7) and can write eq. (1) as

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \ . \tag{4}$$

Thus, our problem is reduced to solving eq. (4) with the initial conditions (2) and the boundary conditions (3). Let us seek particular solutions to eq. (4) in the form

$$u = T(t)\ w(r) \ . \tag{5}$$

Substituting this expression into eq. (4), we obtain

$$\frac{T''(t)}{a^2 T(t)} = \frac{w''(r) + \dfrac{2}{r}w'(r)}{w(r)} \ .$$

Denoting the common value of the two sides of this equation by $-\lambda^2$, we obtain the two equations

$$T''(t) + \lambda^2 a^2 T(t) = 0 \,, \tag{6}$$

$$w''(r) + \frac{2}{r} w'(r) + \lambda^2(r) = 0 \,. \tag{7}$$

If the function (5) is not to be identically equal to zero, and is to satisfy the boundary condition (3), it is obviously necessary that the condition

$$\left.\frac{dw}{dr}\right|_{r=R} = 0 \tag{8}$$

be fulfilled. The general solution to eq. (7) is of the form

$$w(r) = C_1 \frac{\sin \lambda r}{r} + C_2 \frac{\cos \lambda r}{r} \,, \tag{9}$$

where C_1 and C_2 are arbitrary constants.

Since, by the nature of the problem, the function $u(r, t)$ that we are seeking must be bounded at all points within the sphere, including the center (that is, for $r = 0$), we must set $C_2 = 0$ in eq. (9). With no loss of generality, we may assume that $C_1 = 1$. Thus,

$$w(r) = \frac{\sin \lambda r}{r} \,. \tag{10}$$

Substituting this value into the boundary condition (8), we obtain the equation

$$\lambda R \cos \lambda R - \sin \lambda R = 0 \tag{11}$$

for determining the eigenvalues of eq. (7) under the boundary conditions (8) with $w(0)$ finite. If we set

$$\lambda R = \mu \,, \tag{12}$$

then eq. (11) can be rewritten in the form

$$\tan \mu = \mu \,. \tag{13}$$

To find the real roots of this equation, we construct graphs of the functions

$$y = \tan \mu \,, \qquad y = \mu \,.$$

Obviously, the abscissae of the points of intersection of these curves will give the desired roots (fig. 36).

It is clear from the figure that the absolute values of the roots μ_k of eq. (13) increase without bound with increasing k. Also, the difference $\mu_k - (k+\frac{1}{2})\pi$ tends to zero. It then follows that for sufficiently large k, we may set

$$\mu_k = (k+\tfrac{1}{2})\pi \,. \tag{14}$$

We may compute the first roots as follows. Let us set

$$\mu_k = (k+\tfrac{1}{2})\pi - \epsilon_k \,. \tag{15}$$

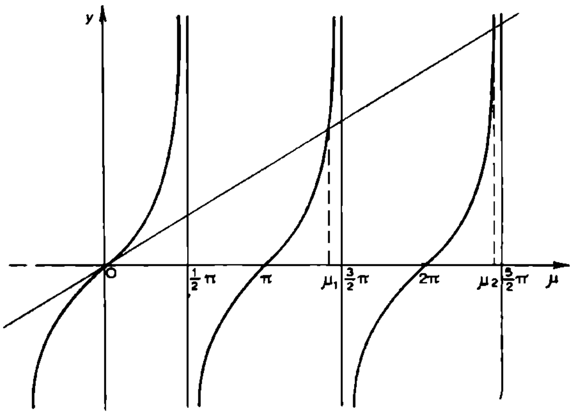

Fig. 36.

Substituting this into eq. (13), we obtain

$$\cot \epsilon_k = (k+\tfrac{1}{2})\pi - \epsilon_k \tag{16}$$

Let us take the first two terms of the expansion

$$\cot \epsilon_k = \frac{1}{\epsilon_k} - \tfrac{1}{3}\epsilon_k - \tfrac{1}{45}\epsilon_k{}^2 + \dots .$$

Then, eq. (16) can be rewritten in the form

$$\epsilon_k = \frac{2}{(2k+1)\pi} + \frac{4\epsilon_k{}^2}{3(2k+1)\pi} . \tag{17}$$

By applying the method of iteration to this last equation, we find an approximate value for ϵ_k and, consequently, from formula (15), an approximate value for the roots μ_k (where $k = 1, 2, \dots$).

For example, we obtain, correct to four decimal places,

$$\mu_1 = 4.4935 , \quad \mu_2 = 7.7250 , \quad \mu_3 = 10.9044 .$$

We denote by $\mu_1, \mu_2, \mu_3, \dots$ the positive roots of eq. (13). Then, by eq. (12), the eigenvalues will be

$$\lambda_k{}^2 = (\mu_k/R)^2 \quad (k = 1, 2, 3, \dots) . \tag{18}$$

To each eigenvalue $\lambda_k{}^2$ corresponds the eigenfunction

$$w_k(r) = \frac{\sin(\mu_k r/R)}{r} . \tag{19}$$

We note that $\lambda_0 = 0$ is also an eigenvalue of the problem (7) - (8), to which corresponds the eigenfunction $w_0(r) = \text{constant}$.

For $\lambda = \lambda_k$, the general solution to eq. (6) is of the form

$$T_k(t) = a_k \cos \frac{\mu_k at}{R} + b_k \sin \frac{\mu_k at}{R} ,$$

where a_k and b_k are arbitrary constants.

For $\lambda_0 = 0$, we have

$$T_0(t) = a_0 + b_0 t .$$

On the basis of eq. (5), we see that the functions

$$u_k(r, t) = \left(a_k \cos \frac{\mu_k at}{R} + b_k \sin \frac{\mu_k at}{R}\right) \frac{\sin (\mu_k r/R)}{r} ,$$

$$u_0(r, t) = a_0 + b_0 t$$

satisfy eq. (4) and the boundary condition (3) for arbitrary values of a_0, b_0, a_k, and b_k.

Let us now set up the series

$$u(r, t) = a_0 + b_0 t + \sum_{k=1}^{\infty} \left(a_k \cos \frac{\mu_k at}{R} + b_k \sin \frac{\mu_k at}{R}\right) \frac{\sin (\mu_k r/R)}{r} \tag{20}$$

For the initial conditions (2) to be satisfied, it is necessary that

$$f(r) = a_0 + \sum_{k=1}^{\infty} a_k \frac{\sin (\mu_k r/R)}{r} \tag{21}$$

$$F(r) = b_0 + \sum_{k=1}^{\infty} \frac{\mu_k a}{R} b_k \frac{\sin (\mu_k r/R)}{r} . \tag{22}$$

Assuming that the series (21) converges uniformly, we may determine the coefficients a_k by multiplying both sides of eq. (21) by $r \sin (\mu_n r/R)$ and integrating with respect to r from 0 to R. We then obtain

$$\int_0^R rf(r) \sin \frac{\mu_n r}{R} dr = a_0 \int_0^R r \sin \frac{\mu_n r}{R} dr + \sum_{k=1}^{\infty} a_k \int_0^R \sin \frac{\mu_k r}{R} \sin \frac{\mu_n r}{R} dr . \tag{23}$$

Let us show that

$$\int_0^R r \sin \frac{\mu_n r}{R} dr = 0 . \tag{24}$$

We integrate by parts, obtaining

$$\int_0^R r \sin \frac{\mu_n r}{R} dr = - \frac{R^2}{\mu_n^2} (\mu_n \cos \mu_n - \sin \mu_n) = \frac{R^2 \cos \mu_n}{\mu_n^2} (\tan \mu_n - \mu_n) ,$$

from which eq. (24) follows on the basis of eq. (13). Furthermore, from the formula

$$\int_0^R \sin \frac{\mu_n r}{R} \sin \frac{\mu_k r}{R} dr = \tfrac{1}{2} R \left[\frac{\sin (\mu_n - \mu_k)}{\mu_n - \mu_k} - \frac{\sin (\mu_n + \mu_k)}{\mu_n + \mu_k}\right],$$

it follows that

$$\int_0^R \sin\frac{\mu_n r}{R}\sin\frac{\mu_k r}{R}\,dr = \frac{R}{2}\,\frac{\cos\mu_k\,\sin\mu_n\,(\mu_k\tan\mu_n - \mu_n\tan\mu_k)}{\mu_n^2-\mu_k^2},$$

but, since μ_k and μ_n are roots of eq. (13), we obtain

$$\int_0^R \sin\frac{\mu_n r}{R}\sin\frac{\mu_k r}{R}\,dr = 0 \qquad (k \neq n)\;; \tag{25}$$

that is, the functions $\sin(\mu_k r/R)$ are orthogonal in the interval $(0, R)$. However, if $n = k$,

$$\int_0^R \sin^2\frac{\mu_k r}{R}\,dr = \tfrac{1}{2}R\left(1 - \frac{\sin^2\mu_k}{\mu_k^2}\right),$$

and since

$$\sin^2\mu_k = \frac{\tan^2\mu_k}{1+\tan^2\mu_k} = \frac{\mu_k^2}{1+\mu_k^2},$$

we have

$$\int_0^R \sin^2\frac{\mu_k r}{R}\,dr = \frac{R}{2}\,\frac{\mu_k^2}{1+\mu_k^2}. \tag{26}$$

Taking (24), (25), and (26) into consideration, we obtain from eq. (23)

$$a_k = \frac{2}{R}\left(1+\frac{1}{\mu_k^2}\right)\int_0^R rf(r)\sin\frac{\mu_k r}{R}\,dr. \tag{27}$$

To determine the coefficient a_0, we multiply both sides of eq. (21) by r^2 and integrate with respect to r from 0 to R. Then,

$$\int_0^R r^2 f(r)\,dr = \tfrac{1}{3}R^3 a_0 + \sum_{k=1}^{\infty} a_k \int_0^R r\sin\frac{\mu_k r}{R}\,dr.$$

It follows from eq. (24) that the integral under the summation sign in this expression is equal to zero. Consequently,

$$a_0 = \frac{3}{R^3}\int_0^R r^2 f(r)\,dr \tag{28}$$

In an analogous manner, we obtain

$$b_k = \frac{2}{a\mu_k}\left(1+\frac{1}{\mu_k^2}\right)\int_0^R rF(r)\sin\frac{\mu_k r}{R}\,dr, \tag{29}$$

$$b_0 = \frac{3}{R^3}\int_0^R r^2 F(r)\,dr \tag{30}$$

Thus, all the constants appearing in the solution (20) are found. We now show that the term

$$a_0 + b_0 t \tag{31}$$

in this solution can be discarded. In determining the process of motion of the gas, we must first of all determine the velocity v with which the particles vibrate. The projections of the components v_x, v_y, and v_z of this velocity onto the coordinate axes are computed from the formulae

$$v_x = -\frac{\partial u}{\partial x}, \qquad v_y = -\frac{\partial u}{\partial y}, \qquad v_z = -\frac{\partial u}{\partial z},$$

where the potential u is expressed by the series (20). But the term (31) appearing in this series is independent of x, y, and z. Therefore, the distribution of velocities in the vibrating gas does not change if we discard the term (31) in the series (20).

Let us now set

$$a_k = A_k \sin\varphi_k, \qquad b_k = A_k \cos\varphi_k.$$

Then, expression (20) for the velocity potential can be rewritten as

$$u(r,t) = \sum_{k=1}^{\infty} A_k \frac{\sin(\mu_k r/R)}{r} \sin\left(\frac{a\mu_k t}{R} + \varphi_k\right). \tag{32}$$

This formula shows that the general radial vibration of a gas can be considered as consisting of an infinite number of natural *harmonic* vibrations whose periods are

$$T_k = \frac{2\pi R}{\mu_k}\sqrt{\frac{\rho_0}{\gamma p_0}}$$

The first term of the series (32) gives the *fundamental frequency* of the radial vibrations of the gas, and its period is determined by the formula

$$T_1 = \frac{2\pi R}{\mu_1}\sqrt{\frac{\rho_0}{\gamma p_0}},$$

where μ_1 is the smallest root of eq. (13).

2. *The radial vibrations of a gas in an infinite cylindrical tube*

Suppose that we have an immovable tube that is so long that we may consider it as extending infinitely far in both directions. We denote the radius of any cross section of this tube by R.

Let us suppose that this tube is filled with a gas that performs small-amplitude vibrations about its equilibrium position. Let us investigate these small vibrations, confining ourselves to *radial* vibrations, where the velocity potential u depends only on the distance r of the vibrating particle of gas from the z-axis (which is the axis of the cylinder) and on the time t.

In this case, the wave equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2},$$

written in cylindrical coordinates r, φ, and z, acquires a simpler form:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}. \tag{33}$$

Obviously, we can solve this problem of the small-amplitude vibrations of the gas if we find the solution to eq. (33) satisfying the initial conditions

$$u|_{t=0} = f(r), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(r) \tag{34}$$

and the boundary condition

$$\left.\frac{\partial u}{\partial r}\right|_{r=R} = 0. \tag{35}$$

Using the Fourier method, we seek particular solutions to eq. (33) in the form

$$u(r, t) = T(t)\, w(r). \tag{36}$$

Substituting this equation into (33), we obtain

$$\frac{w''(r) + \frac{1}{r}w'(r)}{w(r)} = \frac{T''(t)}{a^2 T(t)} = -\lambda^2$$

and, consequently,

$$T''(t) + a^2\lambda^2 T(t) = 0, \tag{37}$$

$$w''(r) + \frac{1}{r}w'(r) + \lambda^2 w(r) = 0. \tag{38}$$

If the function (36) is not to be identically equal to zero and if it is to satisfy the boundary condition (35), we must obviously require that

$$\left.\frac{dw}{dr}\right|_{r=R} = 0. \tag{39}$$

The general solution to eq. (38) is of the form (see section 1 of Chapter XII)

$$w(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r), \tag{40}$$

where C_1 and C_2 are arbitrary constants.

The second solution $Y_0(\lambda r)$ to Bessel's equation becomes infinite at $r = 0$. Since, by the nature of the problem, the desired solution must remain bounded at all points of the cylinder (including those points on the axis where $r = 0$), we must set $C_2 = 0$ in eq. (40). With no loss of generality, we may assume that $C_1 = 1$; that is, we may set

$$w(r) = J_0(\lambda r)$$

so that the boundary condition (39) gives

$$J_0'(\lambda R) = 0 \tag{41}$$

or, by using the equation $J_0'(x) = -J_1(x)$, eq. (41) can be replaced by

$$J_1(\lambda R) = 0\ . \tag{42}$$

This equation determines the eigenvalues for eq. (38) subject to the boundary condition (39) and with $w(0)$ finite.

It was shown in Chapter XII that the equation

$$J_1(\mu) = 0 \tag{43}$$

has an infinite number of positive roots: $\mu_1, \mu_2, \mu_3, \ldots$. It then follows that the eigenvalues of the problem are determined by the formula

$$\lambda_k^2 = (\mu_k/R)^2\ . \tag{44}$$

To each eigenvalue λ_k^2 corresponds the eigenfunction

$$w_k(r) = J_0(\mu_k r/R)\ . \tag{45}$$

We note that $\lambda^2 = 0$ is also an eigenvalue to the problem (38) - (39), to which the eigenfunction $w_0(r)$ = constant corresponds.

For $\lambda = \lambda_k$, the general solution to eq. (37) is of the form

$$T_k(t) = a_k \cos\frac{\mu_k at}{R} + b_k \sin\frac{\mu_k at}{R}\ ,$$

where a_k and b_k are arbitrary constants.

For $\lambda = 0$, we have

$$T_0(t) = a_0 + b_0 t\ .$$

On the basis of eq. (36), we see that the functions

$$u_0 = a_0 + b_0 t\ , \qquad u_k(r, t) = \left(a_k \cos\frac{\mu_k at}{R} + b_k \sin\frac{\mu_k at}{R}\right) J_0\left(\frac{\mu_k r}{R}\right)$$

satisfy eq. (33) and the boundary condition (35) for arbitrary values of a_0, b_0, a_k, and b_k. We now seek a solution to the problem in the form

$$u(r, t) = a_0 + b_0 t + \sum_{k=1}^{\infty} \left(a_k \cos\frac{\mu_k at}{R} + b_k \sin\frac{\mu_k at}{R}\right) J_0\left(\frac{\mu_k r}{R}\right)\ . \tag{46}$$

For the initial conditions (34) to be satisfied, it is necessary that

$$f(r) = a_0 + \sum_{k=1}^{\infty} a_k J_0\left(\frac{\mu_k r}{R}\right)\ , \tag{47}$$

$$F(r) = b_0 + \sum_{k=1}^{\infty} \frac{\mu_k a}{R} b_k J_0\left(\frac{\mu_k r}{R}\right)\ . \tag{48}$$

These series are expansions of the given functions $f(r)$ and $F(r)$ in Bessel functions $J_0(\mu_k r/R)$ in the interval $(0,R)$, where the μ_k are positive roots of eq. (43). However, we studied expansions of this kind at the end of Chapter XII. Here, we have the case in which $\alpha = \nu = 0$. Applying formulae

(45), (46), and (50) of Chapter XII to the present case, we find the values of the coefficients a_0, b_0, a_k, and b_k:

$$a_0 = \frac{2}{R^2}\int_0^R rf(r)\,dr\,, \qquad a_k = \frac{2}{R^2 J_0{}^2(\mu_k)}\int_0^R rf(r)\,J_0\left(\frac{\mu_k r}{R}\right) dr\,, \tag{49}$$

$$b_0 = \frac{2}{R^2}\int_0^R rF(r)\,dr\,, \qquad b_k = \frac{2}{aR\mu_k J_0{}^2(\mu_k)}\int_0^R rF(r)\,J_0\left(\frac{\mu_k r}{R}\right) dr\,. \tag{50}$$

Thus, all the constants appearing in eq. (46) are determined. Noting, now, that u is the velocity potential, we may discard the term $a_0 + b_0 t$, since the velocity distribution in the vibrating gas it not changed by this term. If, in place of the coefficients a_k and b_k, we substitute new constants A_k and φ_k, defined by

$$a_k = A_k \sin\varphi_k\,, \qquad b_k = A_k \cos\varphi_k\,,$$

we rewrite the series (46) in the form

$$u(r,t) = \sum_{k=1}^{\infty} A_k J_0\left(\frac{\mu_k r}{R}\right) \sin\left(\frac{\mu_k a t}{R} + \varphi_k\right), \tag{51}$$

from which it is clear that the radial vibrations of the gas are of a *harmonic* nature. Here, the period of the fundamental frequency is determined by

$$T_1 = \frac{2\pi R}{\mu_1}\sqrt{\frac{\rho_0}{\gamma p_0}}\,,$$

where $\mu_1 = 3.83171$ is the smallest root of eq. (43).

Problems

1. An ideal gas is contained between two motionless concentric spheres of radius R_1 and R_2 (where $R_1 < R_2$). Find the small-amplitude vibrations of the gas between the spheres caused by an initial radial disturbance in the density given by

$$\rho(r,0) - \rho_0 = f(r) \qquad (R_1 < r < R_2)\,.$$

Answer:

$$u(r,t) = \sum_{k=1}^{\infty} A_n \frac{\cos\lambda_n r + \gamma_n \sin\lambda_n r}{r} \sin a\lambda_n t\,,$$

where

$$\gamma_n = \frac{\lambda_n R_2 \sin\lambda_n R_2 + \cos\lambda_n R_2}{\lambda_n R_2 \cos\lambda_n R_2 - \sin\lambda_n R_2}\,,$$

and $\lambda_1, \lambda_2, \ldots$ are the positive roots of the equation

$$\tan \lambda(R_2 - R_1) = \frac{\lambda(R_2 - R_1)}{1 + \lambda^2 R_1 R_2},$$

$$A_n = \frac{a}{\rho_0 \lambda_n \delta_n^2} \int_{R_1}^{R_2} r f(r) \, (\cos \lambda_n r + \gamma_n \sin \lambda_n r) \, dr ,$$

$$\delta_n^2 = \int_{R_1}^{R_2} (\cos \lambda_n r + \gamma_n \sin \lambda_n r)^2 \, dr .$$

Method of solution: The problem is reduced to solving the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} \right)$$

with the conditions

$$\left. \frac{\partial u}{\partial r} \right|_{r=R_1} = 0 , \quad \left. \frac{\partial u}{\partial r} \right|_{r=R_2} = 0 ,$$

$$u(r, 0) = 0 , \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = \frac{a^2}{\rho_0} f(r) \quad (R_1 < r < R_2) .$$

2. An ideal gas is contained between two concentric spheres s_{R_1} and s_{R_2}. The radius of the internal sphere s_{R_1} varies according to the equation

$$R(t) = R_1 + \epsilon \sin \omega t \quad (0 < \epsilon < R_1) ,$$

and the external sphere remains motionless. Find the steady-state vibrations of the gas between the spheres.
Answer:

$$u(r, t) = \left[\frac{R_1 R_2 \omega \cos \frac{\omega R_2}{a} - a R_1 \sin \frac{\omega R_2}{a}}{(R_2 - R_1) \cos \frac{\omega(R_2 - R_1)}{a}} \frac{\cos \frac{\omega r}{a}}{r} \right.$$
$$\left. + \frac{\omega R_1 R_2 \sin \frac{\omega R_2}{a} + a R_1 \cos \frac{\omega R_2}{a}}{(R_2 - R_1) \cos \frac{\omega(R_2 - R_1)}{a}} \frac{\sin \frac{\omega r}{a}}{r} \right] 2\epsilon \cos \omega t .$$

Method of solution: The problem is reduced to solving the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2}$$

with the boundary conditions

$$\left. \frac{\partial u}{\partial r} \right|_{r=R_1} = \epsilon \omega \cos \omega t , \quad \left. \frac{\partial u}{\partial r} \right|_{r=R_2} = 0 .$$

3. A homogeneous gas fills an infinitely long hollow tube whose internal radius is equal to R_1 and whose external radius is equal to R_2. Find the small-amplitude vibrations of the gas if the initial disturbances are radially symmetric.
Answer:

$$u(r, t) = \sum_{k=1}^{\infty} (a_k \cos a\lambda_k t + b_k \sin a\lambda_k t)\, R_k(r) ,$$

where

$$R_k(r) = J_0(\lambda_k r) H_0^{(1)'}(\lambda_k R_2) - J_0'(\lambda_k R_2)\, H_0^{(1)}(\lambda_k r)$$

and where the λ_k are the positive roots of the equation

$$J_1(\lambda_k R_1)\, H_0^{(1)'}(\lambda_k R_2) - J_0(\lambda_k R_2)\, H_0^{(1)'}(\lambda_k R_1) = 0 ,$$

$$a_k = \frac{1}{N_k} \int_{R_1}^{R_2} r f(r)\, R_k(r)\, dr , \qquad b_k = \frac{1}{a\lambda_k N_k} \int_{R_1}^{R_2} r F(r)\, R_k(r)\, dr ,$$

$$N_k = \int_{R_1}^{R_2} r\, R_k^2(r)\, dr$$

4. Find the vibrations of a gas in a round closed cylinder of radius R and height l that are caused by transverse vibrations of the upper face that begin at the time $t = 0$. The velocities of the particles at this face are equal to $f(r) \cos \omega t$. The lower face and the lateral surface of the cylinder are motionless.
Answer: The velocity potential is equal to

$$u(r, z, t) = \sum_{m=0}^{\infty} A_m \cosh z \sqrt{\frac{\mu_m^2}{R^2} - \frac{\omega^2}{a^2}}\, J_0\left(\frac{\mu_m r}{R}\right) \cos \omega t$$
$$+ \sum_{n,m=0}^{\infty} B_{nm} \cos \frac{n\pi z}{l}\, J_0\left(\frac{\mu_m r}{R}\right) \cos \sqrt{\frac{\mu_m^2}{R^2} + \frac{n^2\pi^2}{l^2}}\, at ,$$

where

$$A_m = \frac{2}{R^2 J_0^2(\mu_m) \sqrt{\frac{\mu_m^2}{R^2} - \frac{\omega^2}{a^2}} \sinh \sqrt{\frac{\mu_m^2}{R^2} - \frac{\omega^2}{a^2}}} \int_0^R r f(r)\, J_0\left(\frac{\mu_m r}{R}\right) dr ,$$

$$B_{nm} = -\frac{2A_m}{l} \int_0^l \cosh z \sqrt{\frac{\mu_m^2}{R^2} - \frac{\omega^2}{a^2}} \cos \frac{n\pi z}{l}\, dz ,$$

$$B_{0m} = -\frac{A_m}{l} \int_0^l \cosh z \sqrt{\frac{\mu_m^2}{R^2} - \frac{\omega^2}{a^2}}\, dz ,$$

and $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the equation $J_1(\mu) = 0$.

Chapter XV

LEGENDRE POLYNOMIALS

1. *Legendre's differential equation*

Legendre's equation is an equation of the form

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \lambda y = 0 , \tag{1}$$

where λ is some parameter. This equation has singular points $x = -1$ and $x = 1$.

Let us consider the following boundary problem. Find the values of the parameter λ for which there exists, in the interval $[-1, 1]$, a non-trivial solution to eq. (1) that is bounded at the singular points $x = -1$ and $x = 1$.

Let us seek a solution to Legendre's equation in the form of a power series

$$y = \sum_{n=0}^{\infty} a_n x^n . \tag{2}$$

Substituting eq. (2) into eq. (1), we obtain

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + \lambda a_n] x^n = 0 .$$

It then follows that

$$(n+2)(n+1)a_{n+2} - [n(n+1) - \lambda]a_n = 0$$

or

$$a_{n+2} = \frac{n(n+1) - \lambda}{(n+1)(n+2)} a_n . \tag{3}$$

The coefficients a_0 and a_1 are arbitrary. For $a_0 \neq 0$ and $a_1 = 0$, we have a particular solution to eq. (1) that contains only even powers of x; for $a_0 = 0$ and $a_1 \neq 0$, we have a particular solution containing only odd powers of x.

For $\lambda = n(n+1)$, eq. (1) has a solution in the form of a polynomial of degree n that is bounded at the two singular points. Let us now find the corresponding solutions to the equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0 , \tag{4}$$

that are of the form of polynomials of degree n.

Let us examine the polynomial of degree $2n$:

$$z = (x^2 - 1)^n .$$

It is easy to see that this polynomial satisfies the following differential equation

$$(x^2 - 1)\frac{dz}{dx} - 2nxz = 0 .$$

If we differentiate both sides of this equation n times with respect to x, we obtain

$$(1 - x^2)\frac{dz^{(n)}}{dx} + n(n+1)z^{(n-1)} = 0 .$$

If we differentiate this equation once more with respect to x, we see that $z^{(n)}$ satisfies eq. (4).

Thus, eq. (4) has the solution

$$y = Cz^{(n)} = C\frac{d^n(x^2 - 1)^n}{dx^n} ,$$

where C is a constant. Setting

$$C = \frac{1}{2^n\, n!} ,$$

we obtain

$$y = P_n(x) = \frac{1}{2^n\, n!}\,\frac{d^n(x^2 - 1)^n}{dx^n} \quad (n = 0, 1, 2, \ldots) . \tag{5}$$

These are the *Legendre polynomials* which are solutions to eq. (1) for $\lambda = n(n+1)$.

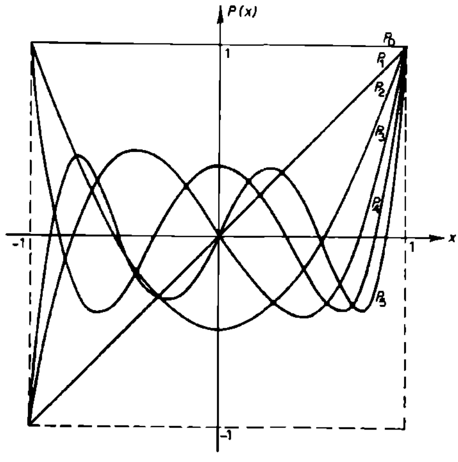

Fig. 37.

Eq. (5) is called Rodrigues' formula.

Thus, the Legendre polynomials are the eigenfunctions of the problem in question, corresponding to the eigenvalues

$$\lambda_n = n(n+1) \qquad (n = 0, 1, 2, \dots) .$$

By using Rodrigues' formula (5), we obtain

$$P_0(x) = 1 , \quad P_1(x) = x , \quad P_2(x) = \tfrac{1}{2}(3x^2 - 1) , \quad P_3(x) = \tfrac{1}{2}(5x^3 - 3x) ,$$

$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3) , \qquad P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x) ,$$

and so on.

The graphs of the Legendre polynomials of the first six orders are shown in fig. 37.

2. *The orthogonality of the Legendre polynomials and their norm*

Let us show that Legendre polynomials of different orders are orthogonal in the interval $(-1,1)$. Let us write eq. (1) for two distinct Legendre polynomials:

$$\left.\begin{aligned} \frac{d}{dx}\left[(1-x^2)\, P_m'(x)\right] + \lambda_m P_m(x) &= 0 \\ \frac{d}{dx}\left[(1-x^2)\, P_n'(x)\right] + \lambda_n P_n(x) &= 0 \end{aligned}\right. \qquad (m \neq n) .$$

Multiplying the first of these equations by $P_n(x)$ and the second by $P_m(x)$, subtracting, and integrating over the interval $(-1,1)$, we obtain

$$\begin{aligned} &(\lambda_m - \lambda_n) \int_{-1}^{1} P_n(x)\, P_m(x)\, dx \\ &= \int_{-1}^{1} \left\{P_m(x) \frac{d}{dx}\left[(1-x^2)\, P_n'(x)\right] - P_n(x) \frac{d}{dx}\left[(1-x^2)\, P_m'(x)\right]\right\} dx \\ &= \int_{-1}^{1} \frac{d}{dx}\left\{(1-x^2)\left[P_m(x)\, P_n'(x) - P_n(x)\, P_m'(x)\right]\right\} dx \\ &= (1-x^2)\left[P_m(x)\, P_n'(x) - P_n(x)\, P_m'(x)\right]\Big|_{x=-1}^{x=+1} = 0 . \end{aligned}$$

Thus,

$$(\lambda_m - \lambda_n) \int_{-1}^{1} P_n(x)\, P_m(x)\, dx = 0$$

or

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = 0 \qquad (m \neq n) ;$$

that is, the Legendre polynomials are orthogonal in the interval (-1, 1).

Let us evaluate the square of the norm of the Legendre polynomials

$$J_n = \int_{-1}^{1} P_n^2(x)\ \mathrm{d}x \ .$$

By using Rodrigues' formula, we rewrite this integral in the form

$$J_n = \frac{1}{2^{2n}\,(n!)^2} \int_{-1}^{1} \frac{\mathrm{d}^n(x^2-1)^n}{\mathrm{d}x^n}\ \frac{\mathrm{d}^n(x^2-1)^n}{\mathrm{d}x^n}\ \mathrm{d}x \ .$$

Integrating n times by parts and noting that the term outside the integral is each time equal to zero, we obtain

$$\int_{-1}^{1} P_n^2(x)\ \mathrm{d}x = \frac{(-1)^n}{2^{2n}\,(n!)^2} \int_{-1}^{1} (x^2-1)^n\ \frac{\mathrm{d}^{2n}(x^2-1)^n}{\mathrm{d}x^{2n}}\ \mathrm{d}x$$

or

$$\int_{-1}^{1} P_n^2(x)\ \mathrm{d}x = \frac{(-1)^n\,(2n)!}{2^{2n}\,(n!)^2} \int_{-1}^{1} (x^2-1)^n\ \mathrm{d}x \ .$$

We know that

$$\int_{-1}^{1} (x^2-1)^n\ \mathrm{d}x = (-1)^n\ 2\ \frac{2\times 4\ldots 2n}{3\times 5\ldots (2n+1)} \ ,$$

so that the preceding formula finally gives

$$\int_{-1}^{1} P_n^2(x)\ \mathrm{d}x = \frac{2}{2n+1} \ .$$

Thus, we obtain

$$\int_{-1}^{1} P_n(x)\ P_m(x)\ \mathrm{d}x = \begin{cases} 0 \ , & m \neq n \ , \\ \dfrac{2}{2n+1} \ , & m = n \ . \end{cases} \tag{6}$$

Suppose that an arbitrary function $f(x)$ is represented in the form of a series of Legendre polynomials

$$f(x) = \sum_{n=0}^{\infty} a_n\ P_n(x) \ . \tag{7}$$

The coefficients a_n of this expansion can be formally determined on the basis of the property of orthogonality of the Legendre polynomials. Specifically, by multiplying the series (7) by $P_m(x)$ and integrating over the interval $[-1, 1]$, we obtain, on the basis of eq. (6),

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)\ P_n(x)\ \mathrm{d}x \ . \tag{8}$$

Let us now show that the system of orthogonal Legendre polynomials (5) in the interval $(-1, 1)$ is a *closed system*. The system (5) contains polynomials of all degrees. Therefore, an arbitrary polynomial $Q_n(x)$ of degree n can be represented in the form of a linear combination of Legendre polynomials of orders from zero to n:

$$Q_n(x) = \sum_{k=0}^{n} C_k \, P_k(x) \, .$$

On the other hand, the Weierstrass theorem states that an arbitrary continuous function defined on the closed interval $[-1, 1]$ can be uniformly approximated by a polynomial $Q_n(x)$ to any desired degree of accuracy.

Consequently, for an arbitrary $\epsilon > 0$, it is possible to find a linear combination of Legendre polynomials such that

$$\left| f(x) - \sum_{k=0}^{n} C_k \, P_k(x) \right| < \tfrac{1}{2}\sqrt{\epsilon} \, ,$$

from which it follows immediately that

$$\int_{-1}^{1} \left[f(x) - \sum_{k=0}^{n} C_k \, P_k(x) \right]^2 \mathrm{d}x < \epsilon \, .$$

If, instead of the coefficients C_k, we take the Fourier coefficients of the function $f(x)$ with respect to the system of Legendre polynomials (5), this inequality will still be satisfied. Since $\epsilon > 0$ is arbitrary, we may assert that the mean square error in the representation of the function by a partial sum of its Fourier series in Legendre polynomials, approaches zero; that is, the Legendre polynomials do form a *closed system*, hence, a complete system. From this, we easily conclude that eq. (1) does not have bounded solutions at the singular points $x = -1$ and $x = 1$ other than the Legendre polynomials. For if there were such a solution, it would be orthogonal to all the Legendre polynomials $P_n(x)$, which is impossible since the system $\{P_n(x)\}$ is complete.

3. *Certain properties of Legendre polynomials*

1) The n-th degree Legendre polynomial is an even function if n is even and an odd function if n is odd:

$$P_n(-x) = (-1)^n \, P_n(x) \, . \tag{9}$$

This assertion follows immediately from Rodrigues' formula if we note that $(x^2 - 1)^n$ is an even function and that each differentiation changes it from even to odd or vice versa.

2)

$$P_{2n-1}(0) = 0 \, , \qquad P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n} \, (n!)^2} \, . \tag{10}$$

The first of these equations follows immediately from eq. (9). To prove the second, we note that the value of a polynomial for $x = 0$ is its constant term. Since the degree of each term is lowered by n when the expression is differentiated n times, the constant term $P_{2n}(0)$ is obtained upon differentiation of the term containing x^{2n} in the polynomial $(x^2 - 1)^{2n}$. Obviously, this term is equal to

$$(-1)^n \frac{(2n)!}{(n!)^2} x^{2n}$$

Differentiating $2n$ times and multiplying by $1/2^{2n}(2n)!$, we obtain the second of formulae (10).

3)

$$P_n(1) = 1 \,, \qquad P_n(-1) = (-1)^n \,. \tag{11}$$

To prove this, we rewrite Rodrigues' formula (5) in the following form:

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n} [(x+1)^n (x-1)^n]$$

and by using Leibnitz' formula, we obtain

$$P_n(x) = \frac{1}{2^n\, n!} \left[(x+1)^n \frac{d^n (x-1)^n}{dx^n} + n \frac{d(x+1)^n}{dx} \frac{d^{n-1}(x-1)^n}{dx^{n-1}} + \ldots \right]$$

Then, obviously,

$$\frac{d^n (x-1)^n}{dx^n} = n! \quad \text{and} \quad \left. \frac{d^{n-k}(x-1)^n}{dx^{n-k}} \right|_{x=1} = 0 \quad (k = 1, 2, \ldots, n) \,,$$

from which it immediately follows that

$$P_n(1) = 1 \,.$$

The second of eqs. (11) is obtained from the first by means of eq. (9).

4) All zeros of a Legendre polynomial $P_n(x)$ are real, distinct, and less than unity in absolute value.

This assertion follows easily from Rodrigues' formula (5) and Rolle's theorem. For the polynomial $d(x^2 - 1)^n/dx$ of degree $2n - 1$ has zeros $x = \pm 1$ of multiplicity $n - 1$ and, by Rolle's theorem, has still another zero $x = \xi_1$ within the interval $[-1, 1]$. These are all of its roots. The polynomial $d^2(x^2 - 1)^n/dx^2$ of degree $2n - 2$ has zeros $x = \pm 1$ of multiplicty $n - 2$ and, by Rolle's theorem, it has two other real roots: one within the interval $[-1, \xi_1]$ and the other within the interval $[\xi_1, 1]$. By continuing in this fashion, we see that $P_n(x)$ has n distinct zeros within the interval $[-1, 1]$.

4. *Integral representations of Legendre polynomials*

Besides the differential formula of Rodrigues (5), there are a number of integral representations of the Legendre polynomials. Thus, Schläfli represented the Legendre polynomials in the form of complex integrals:

$$P_n(x) = \frac{1}{2\pi i} \int_L \frac{1}{2^n} \frac{(z^2-1)^n}{(z-x)^{n+1}} \, dz \, , \tag{12}$$

where L is an arbitrary closed curve drawn around the point x.

To prove eq. (12), we note that Cauchy's theorem implies that the integral is equal to the residue of the integrand corresponding to the unique pole $z = x$. The coefficient of $(z-x)^n$ in the expansion of the polynomial $(z^2-1)^n$ in powers of $z-x$ is equal to

$$\frac{1}{n!} \frac{d^n}{dx^n} (x^2-1)^n$$

and, therefore, the desired residue is

$$\frac{1}{2^n \, n!} \frac{d^n(x^2-1)^n}{dx^n} \, ,$$

which is $P_n(x)$.

From Schläfli's formula, we may obtain *Laplace's formula:*

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} (x + \sqrt{x^2-1} \cos\varphi)^n \, d\varphi \, . \tag{13}$$

Suppose that x is a real number greater than unity. Suppose that the curve L in eq. (12) is a circle with center x and radius $\sqrt{(x^2-1)}$. Then, we may make the change of variables

$$z = x + \sqrt{x^2-1} \, e^{i\varphi} \, ,$$

where φ runs from 0 to 2π.

We thus have

$$\begin{aligned} z^2 - 1 = (x + \sqrt{x^2-1}\, e^{i\varphi})^2 - 1 &= (x^2-1)(1+e^{2i\varphi}) + 2x\sqrt{x^2-1}\, e^{i\varphi} \\ &= 2\sqrt{x^2-1}\, e^{i\varphi}(x + \sqrt{x^2-1}\cos\varphi) \, . \end{aligned}$$

Substituting this into eq. (12), we obtain

$$\begin{aligned} P_n(x) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{2^n} \frac{2^n(\sqrt{x^2-1}\, e^{i\varphi})^n (x + \sqrt{x^2-1}\cos\varphi)^n}{(\sqrt{x^2-1}\, e^{i\varphi})^{n+1}} \, i\sqrt{x^2-1}\, e^{i\varphi} \, d\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} (x + \sqrt{x^2-1}\cos\varphi)^n \, d\varphi = \frac{1}{\pi} \int_0^{\pi} (x + \sqrt{x^2-1}\cos\varphi)^n \, d\varphi \, . \end{aligned}$$

Formula (13) is thus verified for values of x that are greater than unity, but since $P_n(x)$ is a polynomial, it must be valid for *all* values of x. The choice of sign in front of the radical is completely immaterial, since, when we expand the integrand by the binomial theorem and then perform the integration, the terms containing radicals drop out.

From Laplace's integral formula (13), we may obtain the following inequality:

$$|P_n(x)| \leq 1 \quad \text{for} \quad -1 \leq x \leq 1 \tag{14}$$

In fact,

$$|P_n(x)| \leqslant \frac{1}{\pi}\int_0^\pi |x + i\sqrt{1-x^2}\cos\varphi|^n \,d\varphi$$

$$= \frac{1}{\pi}\int_0^\pi (\sqrt{x^2 + (1-x^2)\cos^2\varphi})^n \,d\varphi$$

$$= \frac{1}{\pi}\int_0^\pi (\sqrt{x^2\sin^2\varphi + \cos^2\varphi})^n \,d\varphi \leqslant \frac{1}{\pi}\int_0^\pi d\varphi = 1 .$$

We note that we cannot strengthen the inequality (14) for the entire interval $[-1, 1]$ because $P_n(1) = 1$.

5. *The generating function*

The function

$$\frac{1}{\sqrt{1-2xz+z^2}}$$

is the generating function for the Legendre polynomials; that is, these polynomials are the coefficients in the expansion of this expression in a series of positive powers of z:

$$\frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} P_n(x)\, z^n \tag{15}$$

for arbitrary values of x and for values of z that are sufficiently small: $|z| < |x \pm \sqrt{(x^2-1)}|$.

Under the above assumptions, if we use Laplace's formula (13), we obtain

$$\sum_{n=0}^{\infty} P_n(x)\, z^n = \frac{1}{\pi}\int_0^\pi \sum_{n=0}^{\infty} [(x+\sqrt{x^2-1}\cos\varphi)z]^n \,d\varphi$$

$$= \frac{1}{\pi}\int_0^\pi \frac{d\varphi}{1-(x+\sqrt{x^2-1}\cos\varphi)z}$$

$$= \frac{1}{\pi z\sqrt{x^2-1}}\int_0^\pi \frac{d\varphi}{\dfrac{1-xz}{z\sqrt{x^2-1}} - \cos\varphi} . \tag{16}$$

Remembering that

$$\int_0^\pi \frac{d\varphi}{t-\cos\varphi} = \frac{\pi}{\sqrt{t^2-1}}$$

(where it is assumed that t does not lie in the interval $[-1, 1]$ and that the value of $\sqrt{(t^2-1)}$ is fixed in such a way that the inequality $|t - \sqrt{(t^2-1)}| < 1$ is satisfied), we see that the right side of eq. (16) equals $1/\sqrt{(1-2xz+z^2)}$.

We note that the series (15) converges uniformly for $-1 \leqslant x \leqslant +1$ and $|z| < 1$, because $|P_n(x)| \leqslant 1$ for $-1 \leqslant x \leqslant +1$ and, consequently, that $|P_n(x)z^n| \leqslant |z|^n$.

If $|z| > 1$, we define $z_1 = 1/z$. Then, $|z_1| < 1$ and we obtain

$$\frac{1}{\sqrt{1-2xz+z^2}} = \frac{z_1}{\sqrt{1-2xz_1+z_1^2}} = z_1 \sum_{n=0}^{\infty} P_n(x)\, z_1^n = \sum_{n=0}^{\infty} \frac{P_n(x)}{z^{n+1}} .$$

Thus,

$$\frac{1}{\sqrt{1-2xz+z^2}} = \begin{cases} \sum_{n=0}^{\infty} P_n(x) z^n & \text{for} \quad |z| < 1 \\ \sum_{n=0}^{\infty} \dfrac{P_n(x)}{z^{n+1}} & \text{for} \quad |z| > 1 \end{cases} \qquad (-1 \leqslant x \leqslant 1) .$$

6. *Recursion formulae relating the Legendre polynomials and their derivatives*

Beginning with the generating function, we can easily obtain recursion relations between the Legendre polynomials. Specifically, if we differentiate eq. (15) with respect to z and then multiply by $1-2xz+z^2$, we obtain

$$\frac{x-z}{\sqrt{1-2xz+z^2}} = (1-2xz+z^2) \sum_{n=1}^{\infty} nP_n(x)\, z^{n-1}$$

or

$$(x-z) \sum_{n=0}^{\infty} P_n(x)\, z^n = (1-2xz+z^2) \sum_{n=1}^{\infty} nP_n(x)\, z^{n-1} .$$

Therefore, if we equate the coefficients of like powers of z, we obtain

$$(n+1)\, P_{n+1}(x) - (2n+1)\, xP_n(x) + nP_{n-1}(x) = 0 \qquad (n = 1, 2, \ldots) , \tag{17}$$

$$P_1(x) - xP_0(x) = 0 . \tag{18}$$

Similarly, if we differentiate eq. (15) with respect to x and then multiply by $1-2xz+z^2$, we obtain

$$P_n(x) = \frac{\mathrm{d}P_{n+1}(x)}{\mathrm{d}x} + \frac{\mathrm{d}P_{n-1}(x)}{\mathrm{d}x} - 2x\frac{\mathrm{d}P_n(x)}{\mathrm{d}x} \tag{19}$$

or, by combining this equation with eq. (17), we obtain

$$nP_n(x) = x\frac{\mathrm{d}P_n(x)}{\mathrm{d}x} - \frac{\mathrm{d}P_{n-1}(x)}{\mathrm{d}x} . \tag{20}$$

By eliminating $x\mathrm{d}P_n(x)/\mathrm{d}x$ from eqs. (19) and (20), we obtain

$$(2n+1)\ P_n(x) = \frac{\mathrm{d}P_{n+1}(x)}{\mathrm{d}x} - \frac{\mathrm{d}P_{n-1}(x)}{\mathrm{d}x}\ . \tag{21}$$

This formula remains valid for $n = 0$ if we set $\mathrm{d}P_{-1}(x)/\mathrm{d}x = 0$. If, in formula (21), we set $n = 0, 1, 2, \dots, n$ and add, we obtain

$$\sum_{k=0}^{n} (2k+1)\ P_k(x) = \frac{\mathrm{d}P_{n+1}(x)}{\mathrm{d}x} + \frac{\mathrm{d}P_n(x)}{\mathrm{d}x} \tag{22}$$

7. *Legendre functions of the second kind*

To find the general solution to eq. (4), we need another solution that is linearly independent of the Legendre polynomials $P_n(x)$. We omit the proof, but point out that such a solution is of the form

$$Q_n(x) = \tfrac{1}{2} P_n(x) \ln \frac{x+1}{x-1} - \sum_{k=1}^{N} \frac{2n-4k+3}{(2k-1)(n-k+1)} P_{n-2k+1}(x)\ , \tag{23}$$

where $N = \frac{1}{2}n$ for even n and $N = \frac{1}{2}(n+1)$ for odd n. In particular, for $n = 0$, 1, 2, and 3, we have

$$Q_0(x) = \tfrac{1}{2} \ln \frac{x+1}{x-1}\ ,$$

$$Q_1(x) = \tfrac{1}{2} x \ln \frac{x+1}{x-1} - 1\ ,$$

$$Q_2(x) = \tfrac{1}{4}(3x^2 - 1) \ln \frac{x+1}{x-1} - \tfrac{3}{2} x\ ,$$

$$Q_3(x) = \tfrac{1}{4}(5x^3 - 3x) \ln \frac{x+1}{x-1} - \tfrac{5}{2} x^2 + \tfrac{2}{3}\ .$$

Since the functions $P_n(x)$ and $Q_n(x)$ are linearly independent, the general solution to eq. (4) can be written in the form

$$y = C_1 P_n(x) + C_2 Q_n(x)\ , \tag{24}$$

where C_1 and C_2 are arbitrary constants.

8. *Small-amplitude vibrations of a rotating string*

As a simple example of the application of Legendre polynomials, let us examine the problem of a vibrating homogeneous string of length l that is fixed at one end and that rotates freely around the point at which it is fixed. If we neglect gravity and air resistance, the equilibrium position of the string will take the form of a straight line that is rotating with angular velocity ω in a plane passing through the fixed point. The string may vibrate around this equilibrium position when displaced from it. In studying the

vibrations, we need not study the uniform motion of the line of equilibrium, but only the displacement of the string from that line. The displacement u is a function of the time t and the distance x from the fixed point; we shall assume that u is perpendicular to the plane of rotation of the string.

In the case of a rotating string, we need to find the acceleration of the point represented by the sum of two vectors: one vector being of constant length x, and the other (perpendicular to x) of variable length u. Both these vectors rotate with angular velocity ω.

Since u is parallel to the axis of rotation (perpendicular to the plane of rotation), the acceleration of this point will be $-\omega^2 x$ along the x-axis and $\partial^2 u/\partial t^2$ along the u-axis. The force acting on an element of length dx of the string at a distance x from the fixed point will be equal to

$$\rho\, dx \cdot \omega^2 x \ ,$$

where ρ is the density of the string.

The tension at the point x is determined by the sum of the forces acting on all elements of the string from the point x to its free end:

$$T(x) = \int_x^l \rho\omega^2 x\, dx = \tfrac{1}{2}\rho\omega^2(l^2 - x^2) \ .$$

From this, it is easy to obtain the equation for the free vibrations of a rotating string, namely,

$$\rho\, dx \frac{\partial^2 u}{\partial t^2} = \left[\tfrac{1}{2}\rho\omega^2(l^2 - x^2)\frac{\partial u}{\partial x}\right]_{x+dx} - \left[\tfrac{1}{2}\rho\omega^2(l^2 - x^2)\frac{\partial u}{\partial x}\right]_x$$

$$= \tfrac{1}{2}\rho\omega^2 \frac{\partial}{\partial x}\left[(l^2 - x^2)\frac{\partial u}{\partial x}\right] dx$$

or

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial}{\partial x}\left[(l^2 - x^2)\frac{\partial u}{\partial x}\right] \qquad (a^2 = \tfrac{1}{2}\omega^2) \ . \tag{25}$$

Obviously, we can solve the problem of the small-amplitude vibrations of a rotating string by finding the solution to eq. (25) that satisfies the boundary condition

$$u\,|_{x=0} = 0 \tag{26}$$

and the initial conditions

$$u|_{t=0} = f(x) \ , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x) \ . \tag{27}$$

Let us seek particular solutions to eq. (25) satisfying condition (26) and of the form

$$u = T(t)\ X(x) \ . \tag{28}$$

Substituting this equation into eq. (25), we obtain

$$\frac{T''(t)}{a^2 T(t)} = \frac{\frac{d}{dx}[(l^2 - x^2) X'(x)]}{X(x)} .$$

Denoting the common value of the two sides of this equation by $-\lambda$, we obtain the two equations

$$T''(t) + a^2 \lambda T(t) = 0 , \tag{29}$$

$$\frac{d}{dx}[(l^2 - x^2) X'(x)] + \lambda X(x) = 0 . \tag{30}$$

Setting $x = l\xi$, we transform eq. (30) into

$$\frac{d}{d\xi}\left[(1 - \xi^2) \frac{dX}{d\xi}\right] + \lambda X = 0 . \tag{31}$$

This is Legendre's equation.

The physical meaning of the problem implies that the displacement of the string $u(x, t)$ must remain bounded throughout the interval $[0, l]$. Therefore, we need to find solutions to eq. (30) that are bounded throughout this interval, including the end points. At the beginning of this chapter, it was shown that for $\lambda = n(n+1)$, where n is a positive integer, Legendre's equation (31) in the interval $[-1, 1]$ has a solution that is bounded at the points $\xi = \pm 1$. This solution is the Legendre polynomial $P_n(\xi)$. Consequently, returning to the variable x, we may assert that

$$X(x) = P_n(x/l) \tag{32}$$

is the solution to eq. (30) that is bounded at the points $x = \pm l$ for $\lambda = n(n+1)$.

Satisfying the boundary condition (26), we obtain

$$P_n(0) = 0 .$$

This is possible when $n = 2k - 1$, where k is a positive integer.

Thus, non-trivial solutions to eq. (30) with the boundary conditions

$$X(0) = 0 , \qquad X(l) \text{ is bounded} \tag{33}$$

are possible only at values

$$\lambda_k = 2k(2k - 1) \qquad (k = 1, 2, 3, \ldots) . \tag{34}$$

To these eigenvalues correspond the eigenfunctions

$$X_k(x) = P_{2k-1}(x/l) , \tag{35}$$

which form an orthogonal system of functions on the interval $[0, l]$.

For $\lambda = \lambda_k$, the general solution to eq. (29) is of the form

$$T_k(t) = a_k \cos\sqrt{2k(2k-1)}\, at + b_k \sin\sqrt{2k(2k-1)}\, at . \tag{36}$$

On the basis of eq. (28), we find that the functions

$$u_k(x, t) = [a_k \cos\sqrt{2k(2k-1)}\, at + b_k \sin\sqrt{2k(2k-1)}\, at]\, P_{2k-1}(x/l) \tag{37}$$

satisfy eq. (25) and the boundary condition (26) for arbitrary values of a_k and b_k. To solve this problem, we set up the series

$$u(x,t) = \sum_{k=1}^{\infty} [a_k \cos\sqrt{2k(2k-1)}\,at + b_k \sin\sqrt{2k(2k-1)}\,at]\, P_{2k-1}(x/l) \tag{38}$$

and, in order for the initial conditions (27) to be satisfied, we require that

$$u(x,0) = \sum_{k=1}^{\infty} a_k P_{2k-1}(x/l) = f(x)\ , \tag{39}$$

$$\frac{\partial u(x,0)}{\partial t} = \sum_{k=1}^{\infty} \sqrt{2k(2k-1)}\, ab_k P_{2k-1}(x/l) = F(x)\ . \tag{40}$$

Assuming that the series (39) converges uniformly, we can determine the coefficients a_k by multiplying both sides of eq. (39) by $P_{2k-1}(x/l)$ and integrating with respect to x from 0 to l. Then, remembering the orthogonality of the eigenfunctions, we obtain

$$\int_0^l f(x)\, P_{2k-1}(x/l)\, dx = a_k \int_0^l P_{2k-1}^2(x/l)\, dx = \tfrac{1}{2} l a_k \int_{-1}^{1} P_{2k-1}^2(\xi)\, d\xi = \frac{l}{4k-1} a_k\ .$$

Hence,

$$a_k = \frac{4k-1}{l} \int_0^l f(x)\, P_{2k-1}(x/l)\, dx\ . \tag{41}$$

In an analogous way, we obtain

$$b_k = \frac{4k-1}{al\sqrt{2k(2k-1)}} \int_0^l F(x)\, P_{2k-1}(x/l)\, dx\ . \tag{42}$$

Thus, the solution to the problem is given by the series (38), where a_k and b_k are determined by eqs. (41) and (42).

Rewriting eq. (38) in the form

$$u(x,t) = \sum_{k=1}^{\infty} A_k \sin(\sqrt{2k(2k-1)}\, at + \varphi_k)\, P_{2k-1}(x/l)\ , \tag{43}$$

we see that small-amplitude vibrations of a rotating string are a composite of harmonic vibrations. The frequency ω_k of the vibrations of the k-th harmonic is expressed by the formula

$$\omega_k = \sqrt{2k(2k-1)}\,a = \sqrt{k(2k-1)}\,\omega\ .$$

It follows from this that the frequencies of the vibrations depend on the angular velocity ω and not on the length of the string or its density (so long as the density is constant). When the length or the density is increased, the mass of the string is increased, which tends to decrease the frequency; the tension is also increased, which causes an increase in the frequency. These two factors counteract each other.

Problems

1. Show that

$$\int_0^1 P_n(x)\,dx = \begin{cases} 1 & \text{for} \quad n = 0\ , \\ 0 & \text{for} \quad n = 2k\ , \quad k > 0\ , \\ (-1)^k \dfrac{(2k)!}{2^{2k+1}\, k!\,(k+1)!} & \text{for} \quad n = 2k+1\ . \end{cases}$$

2. Show that

$$\int_0^1 xP_k(x)\,dx = \begin{cases} 0 & \text{for} \quad k = 2n+1 \quad (n > 0)\ , \\ \dfrac{(-1)^n\,(2n-2)!}{2^{2n}(n-1)!\,(n+1)!} & \text{for} \quad k = 2n \quad (n > 0)\ . \end{cases}$$

3. Expand the function $f(x)$ defined by

$$f(x) = \begin{cases} 0 & \text{for} \quad -1 \leqslant x < 0\ , \\ 1 & \text{for} \quad 0 < x \leqslant 1 \end{cases}$$

in a series of Legendre polynomials.
Answer:

$$f(x) = \frac{1}{2} + \frac{3}{2^2} P_1(x) - \frac{7\times 2!}{2^4\times 2!\times 1!} P_2(x) + \frac{11\times 4!}{2^6\times 3!\times 2!} P_3(x) - \dots$$

4. Show that the series

$$P_0(\cos\theta) + P_1(\cos\theta) + \dots + P_n(\cos\theta) + \dots$$

converges for $0 < \theta < \pi$.
Method of solution: Use Laplace's formula.

5. A homogeneous thread whose density varies according to the formula

$$\rho(x) = \frac{a}{\sqrt{b^2 - x^2}}$$

(where $a > 0$ and $b > l$ are constants) is fastened at the end $(x = 0)$ to a motionless axis and is fastened at the other end $(x = l)$ to a sphere whose mass is given by the formula

$$M = \frac{a}{l}\sqrt{b^2 - l^2}\ .$$

Show that when the thread rotates around the above axis at constant angular velocity ω, the equation for small-amplitude vibrations will be of the form

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial y^2}\ ,$$

where

$$y = \arcsin (x/b) .$$

Method of solution: The tension of the thread is given by the formula

$$T(x) = a\omega^2 \int_x^l \frac{x\,\mathrm{d}x}{\sqrt{b^2 - x^2}} + \omega^2 a\sqrt{b^2 - l^2} .$$

6. A homogeneous string is rotating as indicated in section 8 of this chapter. It is subjected to a force $\rho Y(x, t)$ that is continuously distributed along its entire length. Show that the forced vibrations of the string are given by the equation

$$u(x, t) = \sum_{k=1}^{\infty} T_k(t)\, P_{2k-1}(x/l) ,$$

where

$$T_k(t) = \frac{4k - 1}{al\sqrt{2k(2k - 1)}} \int_0^t \mathrm{d}\tau \int_0^l Y(\xi, \tau) \sin \omega_k(t - \tau)\, P_{2k-1}(\xi/l)\, \mathrm{d}\xi .$$

Chapter XVI

THE APPLICATION OF THE FOURIER METHOD TO THE STUDY OF SMALL-AMPLITUDE VIBRATIONS OF RECTANGULAR AND CIRCULAR MEMBRANES

1. *Free vibrations of a rectangular membrane*

Let us examine the small-amplitude vibrations of a homogeneous rectangular membrane which is fastened along the edges and has sides of length p and q (fig. 38).

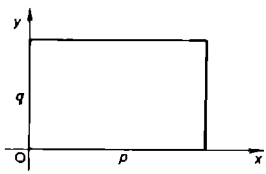

Fig. 38.

It was shown in Chapter VI that this problem amounts to solving the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

with the boundary conditions

$$u|_{x=0} = 0\ , \quad u|_{x=p} = 0\ , \quad u|_{y=0} = 0\ , \quad u|_{y=q} = 0 \tag{2}$$

and the initial conditions

$$u|_{t=0} = f(x, y)\ , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x, y)\ . \tag{3}$$

Let us seek particular solutions to eq. (1) of the form

$$u(x, y, t) = T(t)\ v(x, y)\ , \tag{4}$$

satisfying the boundary conditions (2).

Substituting eq. (4) into eq. (1), we obtain

$$\frac{T''(t)}{a^2 T(t)} = \frac{v_{xx} + v_{yy}}{v}\ .$$

Obviously, this equation can be satisfied only if both sides are equal to the same constant. We denote this constant by $-k^2$, and, when we consider the boundary conditions (2), we see that

$$T''(t) + (ak)^2 T(t) = 0 \tag{5}$$

$$v_{xx} + v_{yy} + k^2 v = 0 , \tag{6}$$

$$\begin{aligned} v|_{x=0} = 0 , \quad v|_{x=p} = 0 , \\ v|_{y=0} = 0 , \quad v|_{y=q} = 0 . \end{aligned} \tag{7}$$

Let us solve the boundary problem (6) - (7) by the Fourier method, setting

$$v(x, y) = X(x)\, Y(y) . \tag{8}$$

Substituting eq. (8) into eq. (6), we obtain

$$\frac{Y''(y)}{Y(y)} + k^2 = -\frac{X''(x)}{X(x)} ,$$

so that we obtain the two equations

$$X''(x) + k_1^2 X(x) = 0 , \quad Y''(y) + k_2^2 Y(y) = 0 , \tag{9}$$

where

$$k_2^2 = k^2 - k_1^2 \quad \text{or} \quad k^2 = k_1^2 + k_2^2 . \tag{10}$$

The general solutions to eqs. (9) are, as we know, of the following form:

$$X(x) = C_1 \cos k_1 x + C_2 \sin k_1 x , \quad Y(y) = C_3 \cos k_2 y + C_4 \sin k_2 y \tag{11}$$

From the boundary conditions (7), we obtain

$$X(0) = 0 , \quad X(p) = 0 ; \quad Y(0) = 0 , \quad Y(q) = 0 , \tag{12}$$

from which it is clear that $C_1 = C_3 = 0$ and if we set $C_2 = C_4 = 1$, we see that

$$X(x) = \sin k_1 x , \quad Y(y) = \sin k_2 y , \tag{13}$$

so that

$$\sin k_1 p = 0 , \quad \sin k_2 q = 0 . \tag{14}$$

It follows from eqs. (14) that k_1 and k_2 have an infinite set of values:

$$k_{1m} = m\pi/p , \quad k_{2m} = n\pi/q \quad (m, n = 1, 2, 3, \ldots) .$$

We then obtain from eq. (10) the corresponding values for the constant k^2:

$$k_{mn}^2 = k_{1m}^2 + k_{2n}^2 = \pi^2\left(\frac{m^2}{p^2} + \frac{n^2}{q^2}\right) . \tag{15}$$

Thus, the eigenvalues (15) correspond to the eigenfunctions

$$v_{mn}(x, y) = \sin\frac{m\pi x}{p} \sin\frac{n\pi y}{q} \tag{16}$$

of the boundary problem (6) - (7).

Turning now to eq. (5), we see that, for every eigenvalue $k^2 = k_{mn}^2$, its general solution is of the form

$$T_{mn}(t) = A_{mn} \cos ak_{mn}t + B_{mn} \sin ak_{mn}t\,. \tag{17}$$

Thus, on the basis of (4), (16), and (17), the particular solutions to eq. (1) that satisfy the boundary conditions (2) are of the form

$$u_{mn}(x, y, t) = (A_{mn} \cos ak_{mn}t + B_{mn} \sin ak_{mn}t) \sin \frac{m\pi x}{p} \sin \frac{n\pi y}{q}\,. \tag{18}$$

To satisfy the initial condítions (3), we set up the series

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (A_{mn} \cos ak_{mn}t + B_{mn} \sin ak_{mn}t) \sin \frac{m\pi x}{p} \sin \frac{n\pi y}{q}\,. \tag{19}$$

If this series and the series obtained from it by twice differentiating termwise with respect to x, y, and t all converge uniformly, its sum will obviously satisfy eq. (1) and the boundary conditions (2). To satisfy the initial conditions (3), it is necessary that

$$u|_{t=0} = f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{p} \sin \frac{n\pi y}{q}\,, \tag{20}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = F(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} ak_{mn}B_{mn} \sin \frac{m\pi x}{p} \sin \frac{n\pi y}{q}\,. \tag{21}$$

Assuming that the series (20) and (21) converge uniformly, we can determine the coefficients A_{mn} and B_{mn} by multiplying both sides of eqs. (20) and (21) by

$$\sin \frac{m_1\pi x}{p} \sin \frac{n_1\pi y}{q}$$

and integrating with respect to x from 0 to p and with respect to y from 0 to q. Remembering now that

$$\int_0^p \int_0^q \sin \frac{m\pi x}{p} \sin \frac{n\pi y}{q} \sin \frac{m_1\pi x}{p} \sin \frac{n_1\pi y}{q}\, dx\, dy = \begin{cases} 0, \text{ if } m \neq m_1 \text{ or } n \neq n_1, \\ \frac{1}{4}pq, \text{ if } m_1 = m,\ n_1 = n, \end{cases}$$

we obtain

$$\begin{aligned} A_{mn} &= \frac{4}{pq} \int_0^p \int_0^q f(x, y) \sin \frac{m\pi x}{p} \sin \frac{n\pi y}{q}\, dx\, dy \\ B_{mn} &= \frac{4}{apqk_{mn}} \int_0^p \int_0^q F(x, y) \sin \frac{m\pi x}{p} \sin \frac{n\pi y}{q}\, dx\, dy\,. \end{aligned} \tag{22}$$

The solution (19) can be rewritten in the form

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn} \sin \frac{m\pi x}{p} \sin \frac{n\pi y}{q} \sin (ak_{mn}t + \varphi_{mn})\,, \tag{23}$$

where

$$M_{mn} = A_{mn}^2 + B_{mn}^2\,, \qquad \varphi_{mn} = \arctan (A_{mn}/B_{mn})\,.$$

When we examine eq. (23), we see that its individual terms express a *harmonic* vibrational motion and that, consequently, the overall vibration of the membrane is a composite of an infinite set of natural harmonic vibrations of the type of standing waves.

The frequency of each natural vibration can be determined from the formula

$$\omega_{mn} = a\pi \sqrt{\frac{m^2}{p^2} + \frac{n^2}{q^2}}, \tag{24}$$

and the *period* of the vibrations can be determined from the formula

$$T_{mn} = \frac{2pq}{a\sqrt{m^2q^2 + n^2p^2}}. \tag{25}$$

We note the difference between a membrane and a string. In the case of the string, for every frequency of natural vibrations there is a corresponding shape of the string, and the string can easily be divided at the nodes into several equal portions. In the case of the membrane, however, each frequency may correspond to several shapes of the membrane, with different positions of the nodal lines along which the amplitudes of the natural harmonic vibrations are equal to zero. This is most easily seen in the example of a square membrane

$$p = q = \pi .$$

In this case, the frequency ω_{mn} is computed from the formula

$$\omega_{mn} = a\sqrt{m^2 + n^2} .$$

It is clear from this formula that the fundamental note, which is determined by the expression

$$u_{11} = M_{11} \sin (\omega_{11}t + \varphi_{11}) \sin x \sin y ,$$

has a frequency $\omega_{11} = a\sqrt{2}$; here, it is clear that for each frequency the nodal lines coincide with the sides of the square occupied by the membrane.

In these cases, when

$$m = 1 , \quad n = 2 \quad \text{or} \quad m = 2 , \quad n = 1 ,$$

we have two overtones:

$$u_{12} = M_{12} \sin (\omega_{12}t + \varphi_{12}) \sin x \sin 2y ,$$

$$u_{21} = M_{21} \sin (\omega_{21}t + \varphi_{21}) \sin 2x \sin y$$

with the same frequency

$$\omega = \omega_{12} = \omega_{21} = a\sqrt{5} .$$

Clearly, for this frequency, the nodal lines are determined from the equation

$$\alpha \sin x \sin 2y + \beta \sin 2x \sin y = 0 ,$$

or

$$\alpha \cos y + \beta \cos x = 0 .$$

The simplest of the nodal lines are indicated in fig. 39 by the dashed lines. More complicated nodal lines are obtained with the same frequency when $\alpha \neq \pm\beta$ and $\alpha, \beta \neq 0$ but we do not show them.

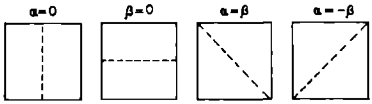

Fig. 39.

We can study these nodal lines and the resultant overtones in a manner analogous to that above.

Forced vibrations of a rectangular membrane are studied in the same way as were the forced vibrations of a string, except that the external force $\Phi(x, y, t)$ is expanded, not in a single, but in a double Fourier series.

2. *Free vibrations of a circular membrane*

Let us examine the problem of the vibrations of a circular membrane of radius l that is fastened at its edge. This problem is reduced to solving the wave equation in polar coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \tag{26}$$

with boundary conditions

$$u|_{r=l} = 0 \tag{27}$$

and initial conditions

$$u|_{t=0} = f(r, \varphi) , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(r, \varphi) . \tag{28}$$

It is clear from the physical nature of the problem that the solution $u(r, \varphi, t)$ must be a single-valued periodic function of φ with period 2π, and must remain bounded at all points of the membrane, including the center, where $r = 0$.

Applying the Fourier method, we set

$$u(r, \varphi, t) = T(t)\, v(r, \varphi) . \tag{29}$$

We obtain the equation for $T(t)$:

$$T''(t) + a^2\lambda^2 T(t) = 0 ,$$

its general solution

$$T(t) = C_1 \cos a\lambda t + C_2 \sin a\lambda t, \tag{30}$$

and the following boundary problem for the function $v(r, \varphi)$:

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \varphi^2} + \lambda^2 v = 0 , \tag{31}$$

$$v|_{r=l} = 0 , \tag{32}$$

$$v|_{r=0} = \text{finite value}, \qquad v(r, \varphi) = v(r, \varphi + 2\pi) . \tag{33}$$

Let us seek a solution to eq. (31) in the form

$$v(r, \varphi) = R(r)\ \Phi(\varphi) . \tag{34}$$

Substituting this equation into eq. (31) and separating the variables, we obtain

$$\frac{\Phi''(\varphi)}{\Phi(\varphi)} = -\frac{r^2 R''(r) + rR'(r) + \lambda^2 r^2 R(r)}{R(r)} = -p^2 ,$$

from which, by considering (32), (33), and (34), we obtain the two boundary problems:

$$\Phi''(\varphi) + p^2\Phi(\varphi) = 0 , \tag{35}$$

$$\Phi(\varphi) = \Phi(\varphi + 2\pi) , \qquad \Phi'(\varphi) = \Phi'(\varphi + 2\pi) ; \tag{36}$$

$$R''(r) + \frac{1}{r} R'(r) + \left(\lambda^2 - \frac{p^2}{r^2}\right) R(r) = 0 , \tag{37}$$

$$R(l) = 0 , \qquad R(0) = \text{finite value} . \tag{38}$$

It is easy to see that non-trivial periodic solutions to problem (35)-(36) exist only if $p = n$ (where n is an integer) and that they are of the form

$$\Phi_n(\varphi) = A_n \cos n\varphi + B_n \sin n\varphi \qquad (n = 0, 1, 2, \ldots) .$$

Let us return to eq. (37). Its general solution for $p = n$ is of the form

$$R_n(r) = D_n J_n(\lambda r) + \epsilon_n Y_n(\lambda r) .$$

It follows from the second of the conditions (38) that $\epsilon_n = 0$. The first condition gives

$$J_n(\lambda l) = 0 .$$

Setting $\lambda l = \mu$, we obtain the transcendental equation for determining μ

$$J_n(\mu) = 0 , \tag{39}$$

which, as we know, has an infinite number of positive roots

$$\mu_1^{(n)}, \mu_2^{(n)}, \mu_3^{(n)}, \ldots ,$$

to which correspond the values

$$\lambda_{nm} = \mu_m^{(n)}/l \qquad (m = 1, 2, \ldots , n = 0, 1, 2, \ldots)$$

and the corresponding solutions to the problem (37) - (38)

$$R_{nm}(r) = J_n(\mu_m^{(n)} r/l) .$$

Turning to the boundary problem (31) - (33), we see that two linearly independent eigenfunctions correspond to the eigenvalue $\lambda_{nm}^2 = (\mu_m^{(n)}/l)^2$, that is,

$$J_n(\mu_m^{(n)} r/l) \cos n\varphi , \quad J_n(\mu_m^{(n)} r/l) \sin n\varphi \quad (m = 1, 2, \dots , \ n = 0, 1, 2, \dots).$$

From the above, it follows that we may set up an infinite number of particular solutions to eq. (26) that satisfy the boundary condition (27) and that are of the form

$$u_{nm}(r, \varphi, t) = \left[\left(A_{nm} \cos \frac{a\mu_m^{(n)} t}{l} + B_{nm} \sin \frac{a\mu_m^{(n)} t}{l}\right) \cos n\varphi \right.$$
$$\left. + \left(C_{nm} \cos \frac{a\mu_m^{(n)} t}{l} + D_{nm} \sin \frac{a\mu_m^{(n)} t}{l}\right) \sin n\varphi \right] J_n\left(\frac{\mu_m^{(n)} r}{l}\right)$$

To satisfy the initial conditions (28), let us set up the series

$$u(r, \varphi, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left[\left(A_{nm} \cos \frac{a\mu_m^{(n)} t}{l} + B_{nm} \sin \frac{a\mu_m^{(n)} t}{l}\right) \cos n\varphi \right.$$
$$\left. + \left(C_{nm} \cos \frac{a\mu_m^{(n)} t}{l} + D_{nm} \sin \frac{a\mu_m^{(n)} t}{l}\right) \sin n\varphi \right] J_n\left(\frac{\mu_m^{(n)} r}{l}\right) . \tag{40}$$

The coefficients A_{nm}, B_{nm}, C_{nm}, and D_{nm} are determined from the boundary conditions (28). For if we set $t = 0$ in the series (40), we obtain

$$f(r, \varphi) = \sum_{m=1}^{\infty} A_{0m} J_0\left(\frac{\mu_m^{(0)} r}{l}\right) + \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} A_{nm} J_n\left(\frac{\mu_m^{(n)} r}{l}\right)\right) \cos n\varphi$$
$$+ \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} C_{nm} J_n\left(\frac{\mu_m^{(n)} r}{l}\right)\right) \sin n\varphi . \tag{41}$$

This series is the expansion of a periodic function $f(r,\varphi)$ in a Fourier series in the interval $(0, 2\pi)$; consequently, the coefficients of $\cos n\varphi$ and $\sin n\varphi$ in this series must be the Fourier coefficients. In other words,

$$\frac{1}{2\pi} \int_0^{2\pi} f(r, \varphi)\, d\varphi = \sum_{m=1}^{\infty} A_{0m} J_0\left(\frac{\mu_m^{(0)} r}{l}\right) , \tag{42}$$

$$\frac{1}{\pi} \int_0^{2\pi} f(r, \varphi) \cos n\varphi\, d\varphi = \sum_{m=1}^{\infty} A_{nm} J_n\left(\frac{\mu_m^{(n)} r}{l}\right) , \tag{43}$$

$$\frac{1}{\pi} \int_0^{2\pi} f(r, \varphi) \sin n\varphi\, d\varphi = \sum_{m=1}^{\infty} C_{nm} J_n\left(\frac{\mu_m^{(n)} r}{l}\right) . \tag{44}$$

When we examine these equations, we see that they are expansions of an arbitrary function $\Phi(r)$ in a series of Bessel functions:

$$\Phi(r) = \sum_{m=1}^{\infty} a_m J_n\left(\frac{\mu_m^{(n)} r}{l}\right).$$

It was shown in Chapter XII that the coefficients a_m are determined by the formula

$$a_m = \frac{2}{l^2 J_{n+1}^2(\mu_m^{(n)})} \int_0^l r\Phi(r)\, J_n\left(\frac{\mu_m^{(n)} r}{l}\right) dr\,.$$

We then obtain without difficulty

$$A_{0m} = \frac{2}{\pi l^2 J_1^2(\mu_m^{(0)})} \int_0^l \int_0^{2\pi} f(r,\varphi)\, J_0\left(\frac{\mu_m^{(0)} r}{l}\right) r\, dr\, d\varphi \qquad , \quad (45)$$

$$A_{nm} = \frac{2}{\pi l^2 J_{n+1}^2(\mu_m^{(n)})} \int_0^l \int_0^{2\pi} f(r,\varphi)\, J_n\left(\frac{\mu_m^{(n)} r}{l}\right) \cos n\varphi\, r\, dr\, d\varphi \;, \quad (46)$$

$$C_{nm} = \frac{2}{\pi l^2 J_{n+1}^2(\mu_m^{(n)})} \int_0^l \int_0^{2\pi} f(r,\varphi)\, J_n\left(\frac{\mu_m^{(n)} r}{l}\right) \sin n\varphi\, r\, dr\, d\varphi\,. \quad (47)$$

By a similar process, we can determine the coefficients B_{0m}, B_{nm}, and D_{nm}; we need only replace $f(r,\varphi)$ by $F(r,\varphi)$ in the formulae (45), (46), and (47), and divide the corresponding expressions by $a\mu_m^{(n)}/l$. Thus, all the coefficients in the expansion (40) are determined, and we can rewrite the solution obtained for the problem (26) - (28) in the form

$$u(r,\varphi,t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} M_{nm} J_n\left(\frac{\mu_m^{(n)} r}{l}\right) \sin(n\varphi + \psi_{nm}) \sin\left(\frac{\mu_m^{(n)} a t}{l} + \nu_{nm}\right), \quad (48)$$

where the constants M_{nm}, ψ_{nm}, and ν_{nm} are related in an obvious way to the constants A_{nm}, B_{nm}, C_{nm}, and D_{nm}.

It is clear from eq. (48) that the overall vibration of a circular membrane is a composite of an infinite set of natural harmonic vibrations of frequency

$$\omega_{nm} = \frac{\mu_m^{(n)}}{l} \sqrt{T_0/\sigma}\;,$$

where T_0 is the tension and σ is the surface density of the membrane.

For $n = 0$ and $m = 1$, we have the *fundamental note* with lowest frequency

$$\omega_{01} = \frac{\mu_1^{(0)}}{l} \sqrt{T_0/\sigma}\,.$$

Formula (48) also shows that, in the case of a circular membrane, the standing waves of varying frequency have nodal lines. The simplest of these lines are determined by the equations

$$J_n\left(\frac{\mu_m^{(n)} r}{l}\right) = 0\;, \qquad \sin(n\varphi + \psi_{nm}) = 0\,. \quad (49)$$

The first of these equations determines $m - 1$ circles, which are concentric with the edge of the membrane, and which have the following equations

$$r_1 = \frac{\mu_1^{(n)}}{\mu_m^{(n)}} l, \quad r_2 = \frac{\mu_2^{(n)}}{\mu_m^{(n)}} l, \quad \ldots, \quad r_{m-1} = \frac{\mu_{m-1}^{(n)}}{\mu_m^{(n)}} l .$$

The second of eqs. (49) determines n diameters of the membrane with equations

$$\varphi_1 = -\frac{\psi_{nm}}{n}, \quad \varphi_2 = \frac{\pi}{n} - \frac{\psi_{nm}}{n}, \quad \ldots, \quad \varphi_n = \frac{(n-1)\pi}{n} - \frac{\psi_{nm}}{n} .$$

Fig. 40 shows several simple cases of the positions of the nodal lines.

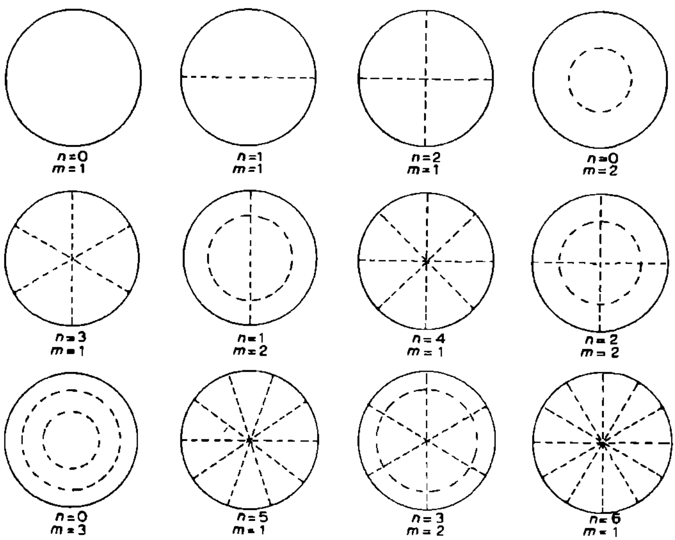

Fig. 40.

In the case of radial vibrations of a circular membrane, the initial functions depend only on r:

$$u|_{t=0} = f(r), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = F(r) . \tag{50}$$

It then follows from formulae (45), (46), and (47) that we have the analogous formulae

$$A_{0m} = \frac{2}{l^2 J_1{}^2(\mu_m{}^{(0)})} \int_0^l rf(r)\, J_0\left(\frac{\mu_m{}^{(0)} r}{l}\right) dr \qquad ,$$

$$B_{0m} = \frac{2}{al\mu_m{}^{(0)} J_1{}^2(\mu_m{}^{(0)})} \int_0^l rF(r)\, J_0\left(\frac{\mu_m{}^{(0)} r}{l}\right) dr ,$$

and for positive n, the coefficients A_{nm}, B_{nm}, C_{nm}, and D_{nm} are equal to zero. The series (40) then becomes the series

$$u(r, t) = \sum_{m=1}^{\infty} \left(A_{0m} \cos \frac{a\mu_m{}^{(0)} t}{l} + B_{0m} \sin \frac{a\mu_m{}^{(0)} t}{l}\right) J_0\left(\frac{\mu_m{}^{(0)} r}{l}\right), \quad (51)$$

where the $\mu_m{}^{(0)}$ are the positive roots of the equation $J_0(\mu) = 0$.

Problems

1. A homogeneous square membrane that, at the initial instant $t = 0$, has a shape represented by $Axy(b - x)(b - y)$ (where A is a small positive number) begins to vibrate with no initial velocity. Investigate the free vibrations of the membrane, which is fastened at its edges.
Answer:

$$u(x, y, t) = \frac{64Ab^4}{\pi^6} \sum_{n,m=0}^{\infty} \frac{\sin \frac{(2n+1)\pi x}{b} \sin \frac{(2m+1)\pi y}{b}}{(2n+1)^3 (2m+1)^3}$$

$$\times \cos \sqrt{(2n+1)^2 + (2m+1)^2}\, \frac{a\pi t}{b} .$$

2. A homogeneous rectangular membrane $0 \leqslant x \leqslant l$, $0 \leqslant y \leqslant m$ is fastened at its edges and, at the initial instant $t = 0$, receives a blow in the neighbourhood of the center, so that

$$\lim_{\epsilon \to 0} \iint_{\sigma_\epsilon} v_0 \, dx\, dy = A ,$$

where v_0 is the initial velocity and A is a constant. Determine the free vibrations of the membrane.
Answer:

$$u(x, y, t) = \frac{4A}{a\pi ml} \sum_{k,\nu=1}^{\infty} \frac{\psi_{k\nu}(\frac{1}{2}l, \frac{1}{2}m)}{\mu_{k\nu}} \psi_{k\nu}(x, y) \sin \mu_{k\nu}\pi at ,$$

where

$$\psi_{k\nu}(x, y) = \sin \frac{k\pi x}{l} \sin \frac{\nu\pi y}{m} , \qquad \mu_{k\nu} = \sqrt{(k/l)^2 + (\nu/m)^2} .$$

3. A liquid with density q is poured into a vessel that has the shape of a circular cylinder of height h. The bottom of the vessel is a thin film of

surface density ρ that is subjected to a uniform tension T. Show that the equation of the radial vibrations of such a film has the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + b^2(u+h) = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}, \qquad a = \sqrt{T_0/\rho}, \qquad b = \sqrt{q/T_0}.$$

4. A homogeneous membrane that has the shape of a circle of radius R lies on the surface of a liquid with density q. Show that the period of the fundamental note of the radial vibrations of such a membrane is given by the formula

$$T = \frac{2\pi l\sqrt{T_0\rho}}{\sqrt{\mu^2 T_0{}^2 + q^2R^2}},$$

where μ is the smallest positive root of the equation $J_0(\mu) = 0$.
Method of solution: In the derivation of the differential equation for the small-amplitude vibrations of the membrane, remember that an element $d\sigma$ of its surface is subjected to a hydrostatic pressure $-qu\,d\sigma$. Neglect the (apparent) additional mass of the water at the surface of the membrane.

5. A homogeneous membrane that is fastened at the edges has the shape of a ring formed by concentric circles of radii R_1 and R_2. Show that the fundamental note of such a membrane is determined by

$$u = A\{J_0(\mu_1 r)\,Y_0(\mu_1R_1) + Y_0(\mu_1 r)\,J_0(\mu_1R_1)\}\cos a\mu_1 t,$$

where μ_1 is the smallest positive root of the equation

$$J_0(\mu R_1)\;Y_0(\mu R_2) - J_0(\mu R_2)\;Y_0(\mu R_1) = 0.$$

6. Find the natural vibrations of a homogeneous circular membrane of radius R that is fastened along the edge, if its shape at an initial instant is that of the surface of a paraboloid of revolution, and the initial velocities are equal to zero.
Answer:

$$u(r, t) = 8A\sum_{n=1}^{\infty}\frac{J_0(\mu_n r/R)}{\mu_n{}^3 J_1(\mu_n)}\cos\frac{a\mu_n t}{R} \qquad (A = \text{constant}),$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the equation $J_0(\mu) = 0$.

7. A circular homogeneous membrane of radius R that is fastened along the edge is in a state of equilibrium with tension T_0. At the instant $t = 0$, a uniformly distributed harmonic force $\rho A \sin\omega t$ is applied to the surface of the membrane. Find the radial vibrations of the membrane.
Answer:

$$u(r, t) = \frac{A}{\omega^2}\left[\frac{J_0(\omega r/a)}{J_0(\omega R/a)} - 1\right]\sin\omega t - \frac{2A\omega R^3}{a}\sum_{n=1}^{\infty}\frac{\sin(\mu_n at/R)\;J_0(\mu_n r/R)}{\mu_n{}^2(\omega^2R - a^2\mu_n{}^2)\;J_0'(\mu_n)},$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the equation $J_0(\mu) = 0$.

8. A homogeneous membrane fastened at the edges has the shape of a ring

formed by circles of radii l_1 and l_2. Show that, when the initial conditions of vibration are

$$u|_{t=0} = f(r, \varphi) , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 ,$$

the displacements of the membrane from the equilibrium position are expressed by the formulae

$$u(r, \varphi, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} W_{nm}(r) \left(A_{nm} \cos n\varphi + B_{nm} \sin n\varphi\right) \cos k_{nm}at ,$$

$$A_{nm} = \frac{1}{\pi L} \int_0^{2\pi} \int_{l_1}^{l_2} f(r, \varphi) \; W_{nm}(r) \cos n\varphi r \, dr \, d\varphi ,$$

$$B_{nm} = \frac{1}{\pi L} \int_0^{2\pi} \int_{l_1}^{l_2} f(r, \varphi) \; W_{nm}(r) \; \sin n\varphi r \, dr \, d\varphi ,$$

where

$$W_{nm}(r) = \frac{J_n(k_{nm}r)}{J_n(k_{nm}l_1)} - \frac{Y_n(k_{nm}r)}{Y_n(k_{nm}l_1)} , \qquad L = \left[\frac{r^2}{2} \; \frac{J'_n(k_{nm}r)}{J_n(k_{nm}l_1)} - \frac{Y'_n(k_{nm}r)}{Y_n(k_{nm}l)}\right]_{l_1}^{l_2} ,$$

and k_{nm} are roots of the transcendental equation

$$J_n(kl_1) \; Y_n(kl_2) - J_n(kl_2) \; Y_n(kl_1) = 0 .$$

PART II

DIFFERENTIAL EQUATIONS OF THE ELLIPTIC TYPE

Chapter XVII

INTEGRAL FORMULAE THAT ARE APPLICABLE TO THE THEORY OF DIFFERENTIAL EQUATIONS OF THE ELLIPTIC TYPE

1. *Definitions and notations*

Integral relationships, to which the present chapter is devoted, have a wide application in mathematical physics, especially in the theory of elliptic equations. The study of these equations will be begun in the next chapter.

We begin by introducing a new system of notation, which will be more convenient for what we are about to do. We have been denoting the Cartesian coordinates of a point in space by x, y, and z and the point itself by a capital Roman letter. In what follows, we shall frequently use the following system of notation: points in space will be denoted by lower-case letters, for example, x, ξ, and so on, and their coordinates by the same letters with subscripts 1, 2, 3. For example, we denote by x_1, x_2, x_3 the coordinates of the point x; we denote by ξ_1, ξ_2, ξ_3 the coordinates of the point ξ, etc. We shall call the respective coordinate axes 1, 2, 3. We shall denote by $|x|$, $|\xi|$, etc., the distance of the points x, ξ, etc., from the coordinate origin, and we shall denote by $|x-\xi|$, $|y-\eta|$, etc., the distance between the points x and ξ, y and η, etc. If it is clear from the context what points we are speaking about, the distance between them will be denoted by r.

The integral and other relationships that we shall examine will be taken over certain volumes, surfaces, or curves. Completely rigorous definitions of the concepts of surface and curve are given in topology and are rather difficult. Just as above, we shall rely basically on an intuitive approach to these concepts. A curve, for example, can be thought of as the locus of a moving point; a surface can be thought of as the figure swept out by the motion of a curve or as the boundary of a solid.

A *region* (which may be either three- or two-dimensional) is that portion of space or of a surface that satisfies the following conditions:

(a) Two arbitrary points of the region can be connected by a curve, every point of which belongs to the region (the property of connectedness) *;

(b) To every point x of the region corresponds a number $\eta = \eta(x)$ such

* Regions in space or in a plane can be defined also in terms of the concept of a broken line, that is, without resorting to the general concept of a curve, which is not sufficiently rigorously defined above. When we confine ourselves to an examination of two-dimensional regions in a plane, we do not have to use the general concept of a surface.

that all points of space (or of the surface) whose distance from x is less than η also belong to the region.

We shall call a region in a plane a *plane region*.

The set of points in space (or on the surface) that are at a distance less than any arbitrary number both from points belonging to the region and from points not belonging to the region is called the *boundary* of the region. We note that points of the boundary of a region do not belong to the region.

The set of points of a region and its boundary is called a *closed region*. The points of a closed region that do not belong to the boundary are called *interior* points.

We shall always assume the boundaries of regions to be surfaces (or curves). The boundary may contain several closed surfaces (or curves). For example, the boundary of a sphere from whose interior portion a sphere of smaller radius has been deleted consists of two spherical surfaces.

In connection with the properties of boundaries, we shall make certain assumptions that will give a precise meaning to all the relationships that we use. Unless the contrary is stated, the boundaries of regions will be assumed to be continuously smooth. This means, for example, that everywhere on the boundary of a three-dimensional region, except possibly for a finite number of curves of finite length, there exists a unique normal (a unique tangent plane) with direction cosines that are continuous functions of the points of the boundary. The boundaries of two-dimensional regions will interest us only in the case in which these regions are plane. In this case, to characterize the local properties of the boundary it will also be sufficient to examine only the normal to the boundary and its two direction cosines. On a continuously smooth boundary of a plane region, the direction cosines of the normal are continuous and the normal is unique, except possibly at a finite number of points.

A portion of a curve that is connected and that consists only of interior points (for example, a straight line segment without its end points) is analogous to a region (one-dimensional region). The two end points of such a one-dimensional region constitute its boundary. These end points (such as the end points of a segment) are not considered as belonging to the region.

When we consider the boundaries of three-dimensional regions, we shall sometimes also assume that, at each point x of the boundary, we may introduce a local Cartesian coordinate system with origin at the point x such that the portion of the boundary that lies within some sphere with center at point x can be represented by the equation

$$\xi_3 = f(\xi_1, \xi_2) ,$$

where the function $f(\xi_1, \xi_2)$ and its first-order derivatives are continuous and vanish at the point x. We shall see in section 6 of Chapter XIX that, under this condition, the boundary will be smooth.

Any region containing a point is a *neighbourhood* of that point. Depending on the type of problem that we are studying, we may consider regions of a different number of dimensions that constitute a portion of space, of a surface, or of a curve.

If all the points of a region belong to a bounded portion of space (for example, if they can be included in a sphere of finite radius), then this region is said to be *bounded* or *finite*. In the opposite case, the region is said to be *infinite*. Every closed surface S partitions that portion of space containing points not belonging to S into two regions, one finite and the other infinite. The infinite region is said to be situated outside S and the finite inside S. The surface S is the common boundary of these regions. A normal to the surface S that is directed toward the infinite region (outside S) is called the *outer normal*. A normal to S with the opposite direction is correspondingly called the *inner normal*. In what follows, we shall use only the outer normal to a region.

A plane partitions space into two infinite regions, each of which is called a *half-space*. Analogously, a straight line divides a plane into two infinite plane regions (half-planes).

We shall use the following notations:

V, S, L are three-, two- and one-dimensional regions, respectively;
$\mathrm{d}V$, $\mathrm{d}S$, $\mathrm{d}L$ are the dimensions (volume, area, length) of infinitesimal elements of the corresponding regions *;
$\mathcal{F}V$, $\mathcal{F}S$, $\mathcal{F}L$ are the boundaries of the corresponding regions.

When it is not likely to cause confusion, we shall, for brevity, speak simply of the "region V" instead of the "closed region V", and so on. To show that a point belongs to a particular region or boundary, we shall use the symbol $\in$. For example, the expressions

$$x \in V\,, \qquad x \in V - \mathcal{F}V\,, \qquad x \in \mathcal{F}V$$

denote respectively that x is a point of the region V, that x is an interior point of the region V, and that x is a point of the boundary of the region V.

2. *The Ostrogradskii-Gauss formula and the Green theorem*

Suppose that $A_i(x)$, where $i = 1, 2, 3$, are functions with continuous first derivatives in the region V. Let us represent the integral over the region V of the derivative $\partial A_1/\partial x$, in the form of an iterated integral:

$$\iiint_V \frac{\partial A_1}{\partial x_1}\,\mathrm{d}V = \iint_\sigma \mathrm{d}x_2\,\mathrm{d}x_3 \int_{l(x_2,x_3)} \frac{\partial A_1}{\partial x_1}\,\mathrm{d}x_1\,,$$

where σ is the region in the plane of the axes 2 and 3 that is formed by the projections of the points of the boundary $\mathcal{F}V$ of the region V; and $l(x_2,x_3)$ is the set of segments of the straight line passing parallel to the axis 1 through the point with coordinates x_2, x_3 in σ that are in the region V. The line integral of $\partial A_1/\partial x_1$ along any segment of the set $l(x_2,x_3)$ is equal to the difference in the values of A_1 at the ends of the segment (corresponding to the upper and lower limits of integration).

We now note that the surface $\mathcal{F}V$ can be partitioned into three parts –

* For measure concepts multiple integrals. and so on. see V.I.Smirnov [1]). Volume 2. p. 88.

S_1, S_2, and S_3. Here, S_1 and S_2 are formed by the ends of the segments of the set $l(x_2,x_3)$ that correspond, respectively, to the lower and upper limits of integration (the points at which straight lines parallel to the axis 1 enter or leave the region V); S_3 is made up of points which lie on the tangent (to $\mathcal{F}V$) parallel to the axis 1 and which are not ends of the segments examined above. We denote by (n_x, x_1) the angle between the axis 1 and the outer normal n_x to $\mathcal{F}V$ at the point $x \in \mathcal{F}V$. We also introduce the function sgn q *, which is equal to 1 if q is positive, to -1 if q is negative, and to zero if q is zero. It is easy to see that

$$\operatorname{sgn}\cos(n_x, x_1) = \begin{cases} -1 & \text{if} \quad x \in S_1 , \\ 1 & \text{if} \quad x \in S_2 , \\ 0 & \text{if} \quad x \in S_3 . \end{cases}$$

Let us now examine the integrand in the integral over σ. After integrating over $l(x_2,x_3)$, the integrand takes the form of a sum, each of whose terms is computed from the value of the function A_1 at the boundary $\mathcal{F}V$; specifically, each term takes the form $A_1(x)$ sgn cos (n_x, x_1). The total number of terms is equal to the number of intersections of the corresponding straight line (parallel to the axis 1) with boundary $\mathcal{F}V$. Thus, to every point x on S_1 and S_2 there corresponds the quantity $A_1(x)$ sgn cos (n_x, x_1); using the set of values of this function on S_1 and S_2, we can completely determine the integrand being considered. This makes it possible to transform the integral over σ into an integral over the surface $\mathcal{F}V$. To do this, we note that

$$\mathrm{d}x_2\,\mathrm{d}x_3 = \mathrm{d}S(x)\ |\cos(n_x, x_1)| = \mathrm{d}S(x)\cos(n_x, x_1)\operatorname{sgn}\cos(n_x, x_1) ,$$

where $\mathrm{d}S(x)$ is an infinitesimal neighbourhood of the point $x \in \mathcal{F}V$ on the boundary $\mathcal{F}V$ (an element of $\mathcal{F}V$). Carrying out the corresponding substitution and extending the integration over all points $x \in S_1 + S_2$, we obtain

$$\iint_{\sigma} \mathrm{d}x_2\,\mathrm{d}x_3 \int_{l(x_2,x_3)} \frac{\partial A_1}{\partial x_1}\,\mathrm{d}x_1$$

$$= \iint_{S_1+S_2} A_1 \operatorname{sgn}\cos(n_x, x_1)\ |\cos(n_x, x_1)|\ \mathrm{d}S(x) = \iint_{S_1+S_2} A_1 \cos(n, x_1)\,\mathrm{d}S .$$

For brevity, the argument x is omitted in the integrand. But the integration can also be taken over S_3, since there $\cos(n, x_1) = 0$. Consequently,

$$\iiint_V \frac{\partial A_1}{\partial x_1}\,\mathrm{d}x_1 = \iint_{\mathcal{F}V} A_1 \cos(n, x_1)\,\mathrm{d}S .$$

By replacing the subscript 1 by the subscripts 2 and 3, we obtain analogous relationships for the functions A_2 and A_3. By adding all these relationships, we arrive at the Ostrogradskii-Gauss formula:

$$\iiint_V \sum_{\alpha=1}^{3} \frac{\partial A_\alpha}{\partial x_\alpha}\,\mathrm{d}V = \iint_{\mathcal{F}V} \sum_{\alpha=1}^{3} A_\alpha \cos(n, x_\alpha)\,\mathrm{d}S . \tag{1}$$

* This is an abbreviation of the Latin word *signum*, meaning "sign". The abbreviation is read "signum".

In the derivation of the Ostrogradskii-Gauss formula, we did not take into consideration the fact that the normal may not exist on isolated curves of the surface $\mathcal{F}V$. This is justifiable since the measure (area) of the set of points belonging to these curves is obviously equal to zero, as a consequence of which the exclusion of these points does not affect the value of the limit to which the integral sums tend. If we wished, we could derive eq. (1) by partitioning the region V into subregions, such that in each of these subregions the integration would be carried out only over the smooth portion of the boundaries. By adding the results, we would again arrive at the formula for the entire region V.

Our first application of the Ostrogradskii-Gauss formula will be to derive Green's theorem, which play an important role in mathematical physics.

Let us examine the second-order linear differential expression

$$\mathfrak{M}u = \sum_{\alpha,\beta=1}^{3} a_{\alpha\beta} \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \sum_{\alpha=1}^{3} b_\alpha \frac{\partial u}{\partial x_\alpha} + cu \, , \tag{2}$$

where $a_{\alpha\beta}$, b_α, and c are functions of the point x. If the functions $a_{\alpha\beta}$ and the functions

$$e_\alpha \equiv b_\alpha - \sum_{\beta=1}^{3} \frac{\partial a_{\alpha\beta}}{\partial x_\beta} \, , \tag{3}$$

have continuous first derivatives, we may put the differential expression $\mathfrak{M}u$ in the form

$$\mathfrak{M}u = \sum_{\alpha,\beta=1}^{3} \frac{\partial}{\partial x_\alpha} a_{\alpha\beta} \frac{\partial u}{\partial x_\beta} + \sum_{\alpha=1}^{3} e_\alpha \frac{\partial u}{\partial x_\alpha} + cu \, . \tag{4}$$

The differential expression

$$\mathfrak{N}u \equiv \sum_{\alpha,\beta=1}^{3} \frac{\partial}{\partial x_\alpha} a_{\alpha\beta} \frac{\partial u}{\partial x_\beta} - \sum_{\alpha=1}^{3} \frac{\partial (e_\alpha u)}{\partial x_\alpha} + cu \tag{5}$$

is called the *conjugate* of the differential expression $\mathfrak{M}u$. Putting the expression $\mathfrak{N}u$ in the form

$$\mathfrak{N}u = \sum_{\alpha,\beta=1}^{3} \frac{\partial}{\partial x_\alpha} a_{\alpha\beta} \frac{\partial u}{\partial x_\beta} - \sum_{\alpha=1}^{3} e_\alpha \frac{\partial u}{\partial x_\alpha} + \left(c - \sum_{\alpha=1}^{3} \frac{\partial e_\alpha}{\partial x_\alpha}\right) u$$

we easily see that this property is mutual; that is, that the expression $\mathfrak{M}u$ is the conjugate of $\mathfrak{N}u$.

If $\mathfrak{M}u = \mathfrak{N}u$, the differential expression $\mathfrak{M}u$ is said to be *self-conjugate*. Necessary and sufficient conditions for the differential expression $\mathfrak{M}u$ to be self-conjugate are that

$$e_\alpha \equiv b_\alpha - \sum_{\beta=1}^{3} \frac{\partial a_{\alpha\beta}}{\partial x_\beta} = 0 \qquad (\alpha = 1, 2, 3) \, .$$

Let us set up the differential expression

$$v\mathfrak{M}u - u\mathfrak{N}v = \sum_{\alpha,\beta=1}^{3} \frac{\partial}{\partial x_\beta} a_{\alpha\beta}\left(v\frac{\partial u}{\partial x_\alpha} - u\frac{\partial v}{\partial x_\alpha}\right) + \sum_{\alpha=1}^{3} \frac{\partial}{\partial x_\alpha} e_\alpha uv \, .$$

If the functions u and v and their first and second derivatives are continuous in the region V, then, by integrating this expression over V and applying the Ostrogradskii-Gauss formula, we obtain Green's theorem:

$$\iiint_V (v\mathfrak{M}u - u\mathfrak{N}v)\,\mathrm{d}V = \iint_{\mathcal{F}V}\left[\sum_{\alpha,\beta=1}^{3} n_\beta a_{\alpha\beta}\left(v\frac{\partial u}{\partial x_\alpha} - u\frac{\partial v}{\partial x_\alpha}\right) + \sum_{\alpha=1}^{3} e_\alpha n_\alpha uv\right]\mathrm{d}S \, . \tag{6}$$

Green's theorem is also valid when the functions u and v have integrable second-order derivatives that are continuous *only inside* the region V *. To show this, let us examine the region V' that is contained within the region V together with its boundary. Since the expression $v\mathfrak{M}u - u\mathfrak{N}v$ is integrable, as $V' \to V$ the limit of the integral over V' does not depend on the path by which V' approaches V and, by definition, is the integral over the region V. The integrand on the right side of eq. (6) is continuous in the region V, right up to its boundary. Therefore, as V' approaches V, the integral of this expression over the boundary $\mathcal{F}V'$ of the region V' changes continuously and converges to a limit that must be the integral over $\mathcal{F}V$. But since $V' \neq V$, eq. (6) is valid; consequently, as V' approaches V, its left and right sides tend to the same limit.

An important special case of Green's theorem is the case in which

$$\mathfrak{M}u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} + cu \, .$$

This differential expression is self-conjugate, so that

$$\mathfrak{N}v = \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 v}{\partial x_3^2} + cv \, .$$

The expression

$$\sum_{\alpha,\beta=1}^{3} n_\beta a_{\alpha\beta}\frac{\partial}{\partial x_\alpha} = \sum_{\beta=1}^{3} n_\beta \frac{\partial}{\partial x_\beta} \equiv \frac{\mathrm{d}}{\mathrm{d}n}$$

is, in the present case, a differential operator ** in the direction of the

* For example, the second derivatives of the functions u and v may increase without bound as they approach the boundary of the region V, thus having an infinite discontinuity at points of the boundary.

** An operator is a rule of correspondence such that to any function belonging to some specified class of functions there corresponds some new function belonging either to the same or to a different class. In the present case, the rule of correspondence is clear from the expression for the operator, and the specified class of functions over which the operator in question is defined is indicated in the derivation of Green's theorem, namely, the class of functions that are defined in the region V, that have continuous second derivatives within the region V, and that are integrable throughout the entire region up to its boundary.

outer normal n to $\mathcal{F}V$, as a consequence of which Green's theorem takes the form

$$\iiint_V (v\,\Delta u - u\,\Delta v)\,\mathrm{d}V = \iint_{\mathcal{F}V} \left(v\frac{\mathrm{d}u}{\mathrm{d}n} - u\frac{\mathrm{d}v}{\mathrm{d}n}\right)\mathrm{d}S\,, \tag{7}$$

where Δ denotes the *Laplacian operator:*

$$\frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial x_3{}^2}\,.$$

If S is a plane region, we have

$$\iint_S (v\mathcal{M}u - u\mathcal{N}v)\,\mathrm{d}S$$
$$= \iint_{\mathcal{F}S}\left[\sum_{\alpha,\beta=1}^{2} n_\beta\, a_{\alpha\beta}\left(v\frac{\partial u}{\partial x_\alpha} - u\frac{\partial v}{\partial x_\alpha}\right) + \sum_{\alpha=1}^{2} e_\alpha\, n_\alpha\, uv\right]\mathrm{d}L\,. \tag{8}$$

This formula is analogous to eq. (6) and is also called Green's theorem. When

$$\mathcal{M} = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2}\,,$$

it takes the form

$$\iint_S (v\,\Delta u - u\,\Delta v)\,\mathrm{d}S = \int_{\mathcal{F}S}\left(v\frac{\mathrm{d}u}{\mathrm{d}n} - u\frac{\mathrm{d}v}{\mathrm{d}n}\right)\mathrm{d}L\,, \tag{9}$$

where $\mathrm{d}/\mathrm{d}n$ is the differential operator in the direction of the outer normal to the boundary $\mathcal{F}S$ of the region S.

Problem

Derive Green's theorem (9) for a plane region.

3*. *Transformation of Green's theorem*

Green's theorem (6) can be transformed into a simpler form. To do this, we set up a correspondence between every point of the boundary $\mathcal{F}V$ and the straight line passing through that point with direction cosines

$$\nu_i = \frac{1}{a}\sum_{\beta=1}^{3} a_{i\beta}\, n_\beta\,, \tag{10}$$

+ The material in sections 3 - 6 of this chapter is used only in Chapter XXVII. Hence. these sections can be omitted until one begins Chapter XXVII.

where

$$a = \left[\sum_{\alpha=1}^{3} \left(\sum_{\beta=1}^{3} a_{\alpha\beta}\, n_\beta\right)^2\right]^{\frac{1}{2}} \tag{11}$$

We shall call this straight line the *conormal*. Noting that

$$\sum_{\alpha,\beta=1}^{3} n_\beta\, a_{\alpha\beta} \frac{\partial}{\partial x_\alpha} = a \sum_{\alpha=1}^{3} \nu_\alpha \frac{\partial}{\partial x_\alpha} \equiv a \frac{\mathrm{d}}{\mathrm{d}\nu}\,, \tag{12}$$

where $\mathrm{d}/\mathrm{d}\nu$ denotes differentiation in the direction of the conormal, and defining

$$b \equiv \sum_{\alpha=1}^{3} e_\alpha n_\alpha\,,$$

we reduce Green's theorem (6) to the form

$$\iiint\limits_V (vMu - uNv)\,\mathrm{d}V = \iint\limits_{\mathcal{F}V} \left[a\left(v\frac{\mathrm{d}u}{\mathrm{d}\nu} - u\frac{\mathrm{d}v}{\mathrm{d}\nu}\right) + buv\right] \mathrm{d}S\,. \tag{13}$$

By introducing the notations

$$\mathcal{P}u \equiv a\frac{\mathrm{d}u}{\mathrm{d}\nu} + \beta u\,, \qquad Qv \equiv a\frac{\mathrm{d}v}{\mathrm{d}\nu} + (\beta - b)v\,, \tag{14}$$

where β is an arbitrary continuous function, we may likewise reduce Green's theorem to the form

$$\iiint\limits_V (v\mathfrak{M}u - u\mathfrak{N}v)\,\mathrm{d}V = \iint\limits_{\mathcal{F}V} (v\mathcal{P}u - uQv)\,\mathrm{d}S\,. \tag{15}$$

In the case of a plane region, formulae (13) and (15) take the form

$$\iint\limits_S (v\mathfrak{M}u - u\mathfrak{N}v)\,\mathrm{d}S = \int\limits_{\mathcal{F}S} \left[a\left(v\frac{\mathrm{d}u}{\mathrm{d}\nu} - u\frac{\mathrm{d}v}{\mathrm{d}\nu}\right) + buv\right] \mathrm{d}L\,, \tag{16}$$

$$\iint\limits_S (v\mathfrak{M}u - u\mathfrak{N}v)\,\mathrm{d}S = \int\limits_{\mathcal{F}S} (v\mathcal{P}u - uQv)\,\mathrm{d}L\,, \tag{17}$$

where the differentiation in the direction of the conormal is defined by formulae analogous to formulae (10) - (12).

4. *Lévy's functions*

In this section, we shall study functions that play an important role in the theory of elliptic differential equations of general form.

Let us suppose that the differential expression (2) is of the elliptic type, that is, that there exists a positive number ϵ such that the condition

$$\sum_{\alpha=1}^{3} \lambda_\alpha^2 = 1$$

implies the inequality

$$\sum_{\alpha,\beta=1}^{3} a_{\alpha\beta}\,\lambda_\alpha\,\lambda_\beta \geqslant \epsilon\ . \tag{18}$$

On the basis of this inequality, the determinant

$$A(x) \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} ,$$

of the coefficients $a_{ij} = a_{ji}$ does not vanish. We denote by $\tilde{a}_{ij}$ the cofactor of the element a_{ij} divided by the value of the determinant A. By a well-known property of determinants,

$$\sum_{\alpha=1}^{3} a_{i\alpha}\,\tilde{a}_{\alpha j} = \delta_{ij} \equiv \begin{cases} 1 & \text{for} \quad i = j\ , \\ 0 & \text{for} \quad i \neq j\ . \end{cases} \tag{19}$$

Furthermore, it is easy to see that the inequality (18) implies the inequality

$$\sum_{\alpha,\beta=1}^{3} \tilde{a}_{\alpha\beta}\,\lambda_\alpha\,\lambda_\beta > 0\ . \tag{20}$$

To see this, it is sufficient to reduce the quadratic form in the inequality (18) by means of an orthogonal transformation to the form

$$c_{11}\,\mu_1^{\,2} + c_{22}\,\mu_2^{\,2} + c_{33}\,\mu_3^{\,2}\ ;$$

then, the form appearing in the inequality (20) becomes

$$c_{22}\,c_{33}\,\mu_1^{\,2} + c_{11}\,c_{33}\,\mu_2^{\,2} + c_{11}\,c_{22}\,\mu_3^{\,2}\ .$$

Since, on the basis of the inequality (18) c_{11}, c_{22}, and c_{33} are all positive, the assertion is obvious.

We denote by x and ξ two points with coordinates x_1, x_2, x_3 and ξ_1, ξ_2 ξ_3, respectively, and we consider the function

$$H(\xi, x) = \frac{1}{4\pi}\left[A(x) \sum_{\alpha,\beta=1}^{3} \tilde{a}_{\alpha\beta}(x)\ (\xi_\alpha - x_\alpha)(\xi_\beta - x_\beta)\right]^{-\frac{1}{2}} \tag{21}$$

For $x \neq \xi$,

$$\sum_{\alpha,\beta=1}^{3} a_{\alpha\beta}(x)\,\frac{\partial^2 H}{\partial \xi_\alpha\,\partial \xi_\beta} = 0\ . \tag{22}$$

This is proved as follows: first,

$$\frac{\partial H}{\partial \xi_i} = -\frac{1}{4\pi}\,[A(x)]^{-\frac{1}{2}}\,\frac{\displaystyle\sum_{\alpha=1}^{3} \tilde{a}_{i\alpha}\,r_\alpha}{\left(\displaystyle\sum_{\alpha,\beta=1}^{3} \tilde{a}_{\alpha\beta}\,r_\alpha\,r_\beta\right)^{\frac{3}{2}}}\ , \tag{23}$$

where, for brevity, we use the notaion

$$\xi_i - x_i \equiv r_i \quad (i = 1, 2, 3) .$$

Furthermore,

$$\frac{\partial^2 H}{\partial \xi_i \, \partial \xi_j} = - \frac{1}{4\pi} [A(x)]^{-\frac{1}{2}} \left\{ \frac{\tilde{a}_{ij}}{\left(\sum\limits_{\alpha,\beta=1}^{3} \tilde{a}_{\alpha\beta} \, r_\alpha \, r_\beta \right)^{\frac{3}{2}}} - 3 \frac{\sum\limits_{\alpha,\beta=1}^{3} \tilde{a}_{i\alpha} \, \tilde{a}_{j\beta} \, r_\alpha \, r_\beta}{\left(\sum\limits_{\alpha,\beta=1}^{3} \tilde{a}_{\alpha\beta} \, r_\alpha \, r_\beta \right)^{\frac{5}{2}}} \right\},$$

$$\sum_{\gamma,\delta=1}^{3} a_{\gamma\delta} \frac{\partial^2 H}{\partial \xi_\gamma \, \partial \xi_\delta} = - \frac{1}{4\pi} [A(x)]^{-\frac{1}{2}} \left[\sum_{\alpha,\beta=1}^{3} a_{\alpha\beta} \, r_\alpha \, r_\beta \right]^{-\frac{3}{2}}$$

$$\times \left\{ \sum_{\gamma,\delta=1}^{3} a_{\gamma\delta} \, \tilde{a}_{\gamma\delta} - 3 \frac{\sum\limits_{\alpha,\beta,\gamma,\delta=1}^{3} \tilde{a}_{\gamma\delta} \, \tilde{a}_{\gamma\alpha} \, a_{\delta\beta} \, r_\alpha \, r_\beta}{\sum\limits_{\alpha,\beta=1}^{3} \tilde{a}_{\alpha\beta} \, r_\alpha \, r_\beta} \right\}. \tag{24}$$

Remembering that $a_{ij} = a_{ji}$ and $\tilde{a}_{ij} = \tilde{a}_{ji}$, we obtain, on the basis of the identity (19), the following equations:

$$\sum_{\gamma,\delta=1}^{3} a_{\gamma\delta} \, \tilde{a}_{\gamma\delta} = 3 ,$$

$$\sum_{\alpha,\beta,\gamma,\delta=1}^{3} a_{\gamma\delta} \, \tilde{a}_{\gamma\alpha} \, \tilde{a}_{\delta\beta} \, r_\alpha \, r_\beta = \sum_{\alpha,\beta,\gamma=1}^{3} a_{\gamma\alpha} \, \tilde{a}_{\gamma\beta} \, r_\alpha \, r_\beta = \sum_{\alpha,\beta=1}^{3} \tilde{a}_{\alpha\beta} \, r_\alpha \, r_\beta .$$

It follows from these relationships that the right side of eq. (24) vanishes, and this proves eq. (22). If the coefficients $a_{ij} = 0$ for $i \neq j$ and $a_{ij} = \frac{1}{2}$ for $i = j$, then

$$H(\xi, x) = \frac{1}{4\pi} \, \frac{1}{r} ,$$

where $r \equiv |\xi - x|$ is the distance between the points x and ξ. In the general case, we have the following relations in every bounded closed region contained in the region V:

$$|H| < \frac{B}{r} , \qquad \left| \frac{\partial H}{\partial \xi_i} \right| < \frac{B_1}{r^2} , \qquad \left| \frac{\partial^2 H}{\partial \xi_i \, \partial \xi_j} \right| < \frac{B_2}{r^3} , \tag{25}$$

where $i, j = 1, 2, 3$ and B, B_1, and B_2 are positive numbers. These relations are proved in an analogous fashion. Let us prove, for example, the second of them. We have

$$r^2 \frac{\partial H}{\partial \xi_i} = -\frac{1}{4\pi}[A(x)]^{-\frac{1}{2}} \frac{r^1 \sum_{\alpha=1}^{3} \tilde{a}_{i\alpha}(\xi_\alpha - x_\alpha)}{\left[\sum_{\alpha,\beta=1}^{3} \tilde{a}_{\alpha\beta}(\xi_\alpha - x_\alpha)(\xi_\beta - x_\beta)\right]^{\frac{3}{2}}}$$

$$= -\frac{1}{4\pi}[A(x)]^{-\frac{1}{2}} \frac{\sum_{\alpha=1}^{3} \tilde{a}_{i\alpha}\mu_\alpha}{\left[\sum_{\alpha,\beta=1}^{3} \tilde{a}_{\alpha\beta}\mu_\alpha \mu_\beta\right]^{\frac{3}{2}}},$$

where

$$\mu_i = \frac{\xi_i - x_i}{r}$$

are the direction cosines of the straight line passing through the points x and ξ. Since the form $\Sigma_{\alpha,\beta=1}^{3} \tilde{a}_{\alpha\beta} r_\alpha r_\beta$ is positive definite, there exists a positive number C^* such that

$$\sum_{\alpha,\beta=1}^{3} \tilde{a}_{\alpha\beta} r_\alpha r_\beta > C^*,$$

so that

$$r^2 \left|\frac{\partial H}{\partial \xi_i}\right| < \frac{1}{4\pi}[A(x)]^{-\frac{1}{2}} \left|\sum_{\alpha=1}^{3} \tilde{a}_{i\alpha} r_\alpha\right| (C^*)^{-\frac{3}{2}}.$$

The right side of this inequality is bounded. Let B_1 be its greatest value in the region that we are considering. Dividing both sides of the inequality by r, we obtain the second of the inequalities (25).

Suppose that a function $\varphi(\xi, x)$ and its first and second derivatives are continuous for $\xi \neq x$ in the given region V (with respect to the coordinates of the point ξ). Suppose, in addition, that in every closed region contained in V, $\varphi(\xi, x)$ uniformly satisfies the inequalities

$$|\varphi| < \frac{C_1}{r^{1-\lambda}}, \quad \left|\frac{\partial\varphi}{\partial\xi_i}\right| < \frac{C_2}{r^{2-\lambda}}, \quad \left|\frac{\partial^2\varphi}{\partial\xi_i\,\partial\xi_j}\right| < \frac{C_3}{r^{3-\lambda}}, \tag{26}$$

where $i, j = 1, 2, 3$ and C_1, C_2, C_3, and λ are positive numbers not depending on the choice of the point x. Then the expression

$$L(\xi, x) \equiv H(\xi, x) + \varphi(\xi, x)$$

is called Lévy's function. The function $H(\xi, x)$ is the principal part of Lévy's function.

Problems

1. Show that if the coefficients a_{ij} are constants, the function $H(\xi, x)$ is a solution to the equation

$$\sum_{\alpha,\beta=1}^{3} a_{\alpha\beta} \frac{\partial^2 u}{\partial \xi_\alpha \partial \xi_\beta} = 0 .$$

2. Show that for $r > \delta$,

$$\left| \sum_{\alpha,\beta=1}^{3} [a_{\alpha\beta}(x) - a_{\alpha\beta}(\xi)] \frac{\partial^2 H}{\partial \xi_\alpha \partial \xi_\beta} \right| < \frac{B(\delta)}{\delta} ,$$

where $B(\delta)$ and δ are positive numbers.

5. *The Green-Stokes theorem*

We denote by $J(x,\rho)$ the neighbourhood of the point x defined by the inequality

$$\sum_{\alpha,\beta=1}^{3} \tilde{a}_{\alpha\beta}(x) \, (\xi_\alpha - x_\alpha)(\xi_\beta - x_\beta) \leqslant \rho^2 \tag{27}$$

As we know from analytic geometry, such a neighbourhood is an ellipsoid whose volume is equal to

$$V_J = \tfrac{4}{3}\pi [A(x)]^{\frac{1}{2}} \rho^3 . \tag{28}$$

Suppose that V is a closed region and that x is an interior point of it. We choose ρ small enough so that the neighbourhood $J(x,\rho)$ lies entirely inside V. In the region $V - J - \mathcal{F}V$, Lévy's function $L(\xi, x)$ and its first two derivatives are continuous. Consequently, in the region $V - J$ we may apply Green's theorem (15), by setting $v(\xi) = L(\xi, x)$ in this region. We then obtain

$$\iiint\limits_{V-J} (L\mathcal{M}_\xi u - u\mathcal{N}_\xi L) \, \mathrm{d}V_\xi = \iint\limits_{\mathcal{F}V + \mathcal{F}J} (L\mathcal{P}_\xi u - uQ_\xi L) \, \mathrm{d}S_\xi , \tag{29}$$

where the subscripts indicate that the differentiation and integration are with respect to the coordinates of the point ξ.

Our intention is to take the limit of this expression as ρ approaches zero. We note that the integrands increase without bound in the neighbourhood of the point x.

Setting

$$L(\xi, x) = H(\xi, x) + \varphi(\xi, x)$$

and keeping the relationship (14) in mind, we may write

$$L\mathcal{P}_\xi u - uQ_\xi L = - ua(\xi) \frac{\mathrm{d}H}{\mathrm{d}\nu} + \psi(\xi, x) , \tag{30}$$

where

$$\psi(\xi, x) = - ua \frac{\mathrm{d}\varphi}{\mathrm{d}\nu} + [(c - b) + \mathcal{P}_\xi u] \, (H + \varphi) .$$

Since the functions u and $\mathcal{P}u$ are bounded, the inequalities (25) and (26) imply the inequality

$$|\psi(\xi, x)| < \frac{B^*}{r^{2-\lambda}} + \frac{B_1^*}{r^{1-\lambda}} + \frac{B_2^*}{r},$$

where λ, B^*, B_1^*, and B_2^* are positive constants. For a sufficiently small neighbourhood of the point x, the first term on the right side of this inequality becomes overwhelmingly large in comparison with the remaining terms. Therefore, there exists a positive number δ such that

$$|\psi(\xi, x)| < \frac{B^*}{r^{2-\lambda}} \quad \text{for} \quad r < \delta \ . \tag{31}$$

Let us now consider the expression $L\mathfrak{M}u - u\mathfrak{N}L$. Recalling (21), we reduce this expression to the form

$$L\mathfrak{M}_\xi u - u\mathfrak{N}_\xi L = -u \sum_{\alpha,\beta=1}^{3} a_{\alpha\beta}(\xi) \frac{\partial^2 H}{\partial\xi_\alpha \, \partial\xi_\beta} + \psi_1(\xi, x) \ , \tag{32}$$

where $\psi_1(\xi, x)$ is a function not containing derivatives of the function $\psi(\xi, x)$ with respect to ξ_i higher than the second or derivatives of the function $H(\xi, x)$ higher than the first. Therefore, in a sufficiently small neighbourhood of the point x,

$$|\psi_1(\xi, x)| < \frac{C_1^*}{r^{3-\lambda}} \quad \text{for} \quad r < \delta_1 \ ,$$

where C_1, λ, and δ_1 are positive constants independent of the choice of the point x. Furthermore, on the basis of the identity (22), we have

$$\sum_{\alpha,\beta=1}^{3} a_{\alpha\beta}(\xi) \frac{\partial^2 H}{\partial\xi_\alpha \, \partial\xi_\beta} = \sum_{\alpha,\beta=1}^{3} [a_{\alpha\beta}(\xi) - a_{\alpha\beta}(x)] \frac{\partial^2 H}{\partial\xi_\alpha \, \partial x_\beta}$$

With an accuracy up to higher-order terms in r,

$$a_{ij}(\xi) - a_{ij}(x) = \left.\frac{\partial a_{ij}}{\partial r}\right|_{\xi=x} r \ , \tag{33}$$

where $\partial/\partial r$ denotes differentiation in the direction of the straight line connecting the points x and ξ. Since the derivatives of the functions $a_{ij}(\xi)$ are continuous by hypothesis and, consequently, are bounded, the last of the inequalities (25) implies that, for sufficiently small values of r,

$$\left| \sum_{\alpha,\beta=1}^{3} a_{\alpha\beta}(\xi) \frac{\partial^2 H}{\partial\xi_\alpha \, \partial\xi_\beta} \right| < \frac{C_2^*}{r^2} \ , \qquad C_2^* > 0 \ .$$

When we consider the estimates that we have found for the terms on the right side of eq. (32), we conclude that, for sufficiently small positive δ^*,

$$|L\mathfrak{M}u - u\mathfrak{N}L| < \frac{C^*}{r^{3-\lambda}} \quad \text{if} \quad r < \delta^* \ , \tag{34}$$

where C^* and $\lambda < 1$ are positive constants, independent of the choice of the point x.

In the integral relationships (29), let us pass to the limit as ρ ap-

proaches zero. On the basis of the inequality (34), the integral on the left side of eq. (29) converges to a finite limit

$$\iiint\limits_V (L\mathcal{M}_\xi u - u\mathcal{N}_\xi L)\,\mathrm{d}V_\xi = \lim_{\rho\to 0} \iiint\limits_{V-J} (L\mathcal{M}_\xi u - u\mathcal{N}_\xi L)\,\mathrm{d}V_\xi\ . \qquad (35)$$

Furthermore, on the basis of eq. (30), we obtain

$$\iint\limits_{\mathcal{F}J} (L\mathcal{P}_\xi u - uQ_\xi L)\,\mathrm{d}S_\xi = -\iint\limits_{\mathcal{F}J} ua(\xi)\frac{\mathrm{d}H}{\mathrm{d}\nu}\,\mathrm{d}S_\xi + \iint\limits_{\mathcal{F}J}\psi(\xi,x)\,\mathrm{d}S_\xi\ . \qquad (36)$$

From the inequality (31), the second of the integrals on the right side approaches zero as ρ approaches zero. For

$$\left|\iint\limits_{\mathcal{F}J}\psi(\xi,x)\,\mathrm{d}S_\xi\right| \leqslant \iint\limits_{\mathcal{F}J}|\psi(\xi,x)|\,\mathrm{d}S_\xi < B^* \iint\limits_{\mathcal{F}J}\frac{\mathrm{d}S_\xi}{r^{2-\lambda}} < B^*\frac{S_J}{r_m{}^{2-\lambda}}\,,$$

where S_J is the area of the surface $\mathcal{F}J$ and r_m is the shortest distance between the point x and points on $\mathcal{F}J$. As ρ approaches zero, the ellipsoid (27) becomes smaller but keeps the same shape, since the coefficients $\tilde{a}_{ij}(x)$ do not depend on ρ. Therefore, the area of its surface

$$S_J = \bar{B}r_m{}^2\,,$$

where the number $\bar{B}$ does not depend on ρ. Thus,

$$\left|\iint\limits_{\mathcal{F}J}\psi(\xi,x)\,\mathrm{d}S_\xi\right| < B^*\bar{B}r_m{}^\lambda\,.$$

As ρ approaches zero, the right side of this inequality will also approach zero because ρ and r_m vanish simultaneously. This proves the assertion made above.

Let us now examine the first integral on the right side of the inequality (36). From eq. (12),

$$\frac{\mathrm{d}H}{\mathrm{d}\nu} = \frac{1}{a(\xi)}\sum_{\alpha,\beta=1}^{3} n_\beta\, a_{\alpha\beta}(\xi)\,\frac{\partial H}{\partial\xi_\alpha}\,,$$

where the n_i (with $i = 1, 2, 3$) are the direction cosines of the outer normal to the boundary $\mathcal{F}J$. When we write the expressions for the derivatives $\partial H/\partial\xi_i$ in explicit form, and when we recall that, on the surface $\mathcal{F}J$,

$$\sum_{\alpha,\beta=1}^{3}\tilde{a}_{\alpha\beta}(x)\,(\xi_\alpha - x_\alpha)(\xi_\beta - x_\beta) = \rho^2\,,$$

we obtain

$$-u(\xi)\,a(\xi)\,\frac{\partial H}{\partial\nu} = \frac{1}{4\pi}\,\frac{u(\xi)}{\sqrt{A(x)}}\,\frac{1}{\rho^3}\sum_{\alpha,\beta,\gamma=1}^{3} n_\beta\, a_{\alpha\beta}(\xi)\,\tilde{a}_{\alpha\gamma}(x)\,(\xi_\gamma - x_\gamma)\,. \qquad (37)$$

For small values of r,

$$u(\xi) = u(x) + \left.\frac{\partial u}{\partial r}\right|_{\xi=x} r = u(x) + O(r)\,,$$

where $O(r)$ denotes the set of terms that become infinitesimally small simultaneously with r. Furthermore, from (33) and (19), for small values of r,

$$\sum_{\alpha=1}^{3} a_{\alpha i}(\xi)\,\tilde{a}_{\alpha j}(x) = \sum_{\alpha=1}^{3} \left[a_{\alpha i}(x)\,\tilde{a}_{\alpha j}(x) + \left.\frac{\partial a_{\alpha i}}{\partial r}\right|_{\xi=x} r\,\tilde{a}_{\alpha j}(x)\right] = \delta_{ij} + O(r)\ .$$

Summing over α in eq. (37), and substituting the expression for $u(\xi)$, we obtain

$$\iint_{\mathcal{F}J} u(\xi)\,a(\xi)\,\frac{\mathrm{d}H}{\mathrm{d}\nu}\,\mathrm{d}S_\xi = \frac{1}{4\pi}\,\frac{u(x)}{\sqrt{A(x)}}\,\frac{1+O(r)}{\rho^3}\iint_{\mathcal{F}J}\sum_{\beta=1}^{3} n_\beta(\xi_\beta - x_\beta)\,\mathrm{d}S_\xi\ . \tag{38}$$

The sum $\Sigma_{\beta=1}^{3}\, n_\beta(\xi_\beta - x_\beta)$ in the integrand is equal to the projection $r\cos(r,n)$ of the segment r drawn from the point x to the point ξ along the outer normal n to the surface $\mathcal{F}J$ at the point ξ. This projection is negative, since the outer normal to $\mathcal{F}J$, which is the boundary of the region $V-J$, is directed out from $V-J$, that is, inside the ellipsoid J (fig. 41). We note also that the volume of the cone constructed with the element $\mathrm{d}S$ as its base and with vertex at the point x, is, with an accuracy up to higher-order terms, equal to $\frac{1}{3}r|\cos(n,r)|\,\mathrm{d}S$. Since the union of all such cones is the region $J(x,\rho)$, we conclude that the integral on the right side of eq. (38) is equal to three times the volume of the ellipsoid $J(x,\rho)$. Therefore, from eq. (28) and from a consideration of the sign of $\cos(r,n)$, we obtain

$$\iint_{\mathcal{F}J}\sum_{\beta=1}^{3} n_\beta(\xi_\beta - x_\beta)\,\mathrm{d}S_\xi = -\,4\pi\sqrt{A(x)}\,\rho^3$$

$\sqrt{A(x)}$

from which, after substituting in eq. (38), we obtain

$$\iint_{\mathcal{F}J} u(\xi)\,a(\xi)\,\frac{\mathrm{d}H}{\mathrm{d}\nu}\,\mathrm{d}S_\xi = u(x)\,[1+O(r)]\ .$$

On the basis of the calculations that we have made, we conclude that, as ρ approaches zero, the expression in (36) approaches the limit

$$\lim_{\rho\to 0}\iint_{\mathcal{F}J}(L\,\mathcal{P}_\xi u - uQ_\xi L)\,\mathrm{d}S_\xi = -\,u(x)\ .$$

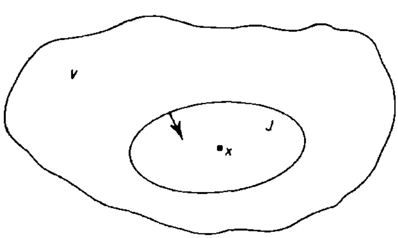

Fig. 41.

In the light of the limit relationships that we have obtained, when we pass to the limit as ρ approaches zero in the integral relationship (29), we obtain the Green-Stokes theorem

$$u(x) = \iint\limits_{\mathcal{F}V} (L\mathcal{P}_\xi u - uQ_\xi L)\, dS_\xi - \iiint\limits_V (L\mathcal{M}_\xi u - u\mathcal{N}_\xi L)\, dV_\xi , \tag{39}$$

which plays an extremely important role in the theory of differential equations of the elliptic type.

In the frequently encountered case in which

$$\mathcal{M}_\xi u = \Delta u \equiv \sum_{\alpha=1}^{3} \frac{\partial^2 u}{\partial \xi_\alpha^2}$$

(that is, when $a_{ij} = 0$ for $i \neq j$ and $a_{ij} = \frac{1}{2}$ for $i = j$), the Green-Stokes theorem takes the form

$$u(x) = \iint\limits_{\mathcal{F}V} \left(L\frac{du}{dn} - u\frac{dL}{dn}\right) dS - \iiint\limits_V (L\,\Delta u - u\,\Delta L)\, dV , \tag{40}$$

where

$$L = \frac{1}{4\pi}\,\frac{1}{r} + \varphi(\xi, x) . \tag{41}$$

Problem

Assume that the function u satisfies the equation $\Delta u = 0$ in a finite region with boundary S. Then derive the formula

$$u(x) = \frac{1}{4\pi}\iint\limits_S \left[\frac{1}{r}\frac{du}{dn} - u\frac{d}{dn}\left(\frac{1}{r}\right)\right] dS .$$

6*. *The Green-Stokes theorem for two dimensions*

The construction of the Lévy functions and the derivation of the Green-Stokes theorem for a plane is almost the same as for the three-dimensional case. Let us consider the function

$$H(\xi, x) = \frac{1}{2\pi}[A(x)]^{-\frac{1}{2}} \ln\left[\sum_{\alpha,\beta=1}^{2} \tilde{a}_{\alpha\beta}(x)\,(\xi_\alpha - x_\alpha)\,(\xi_\beta - x_\beta)\right]^{-\frac{1}{2}} ,$$

defined in a closed bounded plane region S. The notation on the right side is analogous to the notation used in section 4. The identity,

$$\sum_{\alpha,\beta=1}^{2} a_{\alpha\beta}(x)\,\frac{\partial^2 H}{\partial \xi_\alpha\, \partial \xi_\beta} = 0 ,$$

is valid and, in an arbitrary closed region contained in S, we have the inequalities

$$|H| < B\left|\ln \frac{1}{r}\right|, \qquad \left|\frac{\partial H}{\partial \xi_i}\right| < \frac{B_1}{r}, \qquad \left|\frac{\partial^2 H}{\partial \xi_i\, \partial \xi_j}\right| < \frac{B_2}{r^2},$$

where $i, j = 1, 2$ and B, B_1, and B_2 are positive numbers that are independent of the choice of the point x.

We shall call a function of the form

$$L(\xi, x) = H(\xi, x) + \varphi(\xi, x)$$

Lévy's function, if the function $\varphi(\xi, x)$ is bounded in the interval in question, if for $\xi \neq x$ it and its first and second derivatives with respect to the coordinates of the point ξ are continuous, and if, in an arbitrary closed region contained in S, it satisfies the inequalities

$$\left|\frac{\partial \varphi}{\partial \xi_i}\right| < \frac{C_1}{r^{1-\lambda}}, \qquad \left|\frac{\partial^2 \varphi}{\partial \xi_i\, \partial \xi_j}\right| < \frac{C_2}{r^{2-\lambda}}$$

where $i, j = 1, 2$ and λ, C_1, and C_2 are positive numbers that are independent of the choice of the point x.

With this definition of Lévy's functions for a plane, the *Green-Stokes theorem*,

$$u(x) = \int\limits_{\mathcal{F}S} (L\mathcal{P}_\xi u - u\,Q_\xi L)\, dS \;-\; \iint\limits_{S} (L\mathcal{M}_\xi u - u\mathcal{N}_\xi L)\, dS_\xi, \tag{42}$$

where S is a plane region, is valid.

Problem

Assume that the function u satisfies the equation $\Delta u = 0$ in a bounded plane region with boundary L. Then derive the formula

$$u(x) = \frac{1}{2\pi} \int\limits_{L} \left(\frac{du}{dn} \ln \frac{1}{r} - u \frac{d}{dn} \ln \frac{1}{r}\right) dL_\xi .$$

7. *Representation of certain differential expressions in orthogonal coordinate systems*

A number of integral formulae used in mathematical physics contain differential expressions. We have thus far been expressing these in orthogonal Cartesian coordinates. For example, in the Ostrogradskii-Gauss formula (1) and in Green's theorem (7), we found the expressions

$$\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}, \tag{43}$$

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}. \tag{44}$$

One also comes across Stokes' theorem (which we give here without proof *)

$$\iint_S B_n \, dS = \int_{\mathcal{F}S} A_\tau \, dL ,$$

where S is a continuously smooth two-sided surface (in space) with a continuously smooth boundary $\mathcal{F}S$ and the functions B_n and A_τ are expressed in terms of the given functions A_1, A_2, and A_3 by the formulae

$$B_n = \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) \cos(n, x_1) + \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) \cos(n, x_2) + \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) \cos(n, x_3) , \tag{45}$$

$$A_\tau = A_1 \cos(\tau, x_1) + A_2 \cos(\tau, x_2) + A_3 \cos(\tau, x_3) .$$

Here, $\cos(n, x_i)$ and $\cos(\tau, n_i)$ for $i = 1, 2, 3$ are the direction cosines of the normal n to the surface S and the tangent τ to its boundary $\mathcal{F}S$. The positive direction of the normals to S can be chosen arbitrarily; then, the positive direction for the line integral $\int_{\mathcal{F}S} A_\tau \, dL$ around the boundary $\mathcal{F}S$ must be counterclockwise, looking from the end of the vector of any normal to S. The functions A_1, A_2, and A_3 are assumed to be defined in the region V containing the surface S; these functions, together with their first derivatives, are assumed to be continuous. These functions can be chosen as the components of some vector $\boldsymbol{A}$. Then, the function B_n can be regarded as the projection onto the normal n of the vector whose components are

$$B_i = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}, \qquad i, j, k = \begin{cases} 1, 2, 3, \\ 2, 3, 1, \\ 3, 1, 2. \end{cases}$$

This vector is known as the *curl* of the vector A (that is, $\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}$).

Let us determine the form of the differential expressions (43) - (45) in arbitrary orthogonal coordinate systems.

Let us recall the definition of orthogonal coordinates. Here, we shall examine only coordinates in space, leaving the matter of coordinates in two-dimensional regions to the reader.

Let us suppose that the point x is defined in terms of the three parameters τ_1, τ_2, and τ_3; that is,

$$x = x(\tau_1, \tau_2, \tau_3)$$

or

$$x_1 = x_1(\tau_1, \tau_2, \tau_3) , \qquad x_2 = x_2(\tau_1, \tau_2, \tau_3) , \qquad x_3 = x_3(\tau_1, \tau_2, \tau_3) . \tag{46}$$

If these three functions, which define the coordinates of the point x in terms of the parameters τ_i, are single-valued, then, to every set of values τ_1, τ_2, and τ_3, there corresponds one definite point x. Let us suppose not only

* See V. I. Smirnov [1)], Vol. 2, pp. 64 and 70.

that the functions (46) are single-valued, but that they also have continuous partial derivatives. Let us examine the system of equations

$$dx_i = \frac{\partial x_i}{\partial \tau_1} d\tau_1 + \frac{\partial x_i}{\partial \tau_2} d\tau_2 + \frac{\partial x_i}{\partial \tau_3} d\tau_3 \tag{47}$$

with respect to the differentials $d\tau_1$, $d\tau_2$, $d\tau_3$. The determinant D of this system, which is composed of the partial derivatives $\partial x_i/\partial \tau_i$, is called the *Jacobian* or *functional determinant* of the system of functions (46). Obviously, the Jacobian of the system (46) is a function of the parameters τ_1, τ_2, and τ_3.

The following proposition is proved in the theory of differential equations *: If the Jacobian of the system (46) does not vanish in some neighbourhood T of the values of the parameters $\tau_1 = \tau^0{}_1$, $\tau_2 = \tau^0{}_2$, $\tau_3 = \tau^0{}_3$, to which the point x^0 with coordinates $x_1 = x^0{}_1$, $x_2 = x^0{}_2$, $x_3 = x^0{}_3$ corresponds, then, in some neighbourhood X of the point x^0, the system (46) admits a set of single-valued inverse functions

$$\tau_1 = \tau_1(x_1, x_2, x_3)\ , \qquad \tau_2 = \tau_2(x_1, x_2, x_3)\ , \qquad \tau_3 = \tau_3(x_1, x_2, x_3)\ ,$$

and these functions $\tau_i = \tau_i(x_1, x_2, x_3)$ have continuous first derivatives with respect to x_1, x_2, and x_3 in the neighbourhood X and they assume the values $\tau^0{}_i$ at the point x^0.

Thus, under the above conditions, to every point $x \in X$ there corresponds a definite set of parameters τ_1, τ_2, τ_3, which, when substituted into eqs. (46), gives the Cartesian coordinates of the point x. In other words, there exists a one-to-one correspondence between the points x and the triples of the parameters τ_1, τ_2, and τ_3; because of this correspondence we may regard these parameters as coordinates of the point x. If at least one of eqs. (46) is non-linear with respect to x_1, x_2, and x_3, these coordinates are said to be curvilinear, since they correspond to a curvilinear coordinate grid.

It is general practice to choose curvilinear coordinates which are in one-to-one correspondence with the points of the region to be studied, except possibly at certain points or lines where the Jacobian of the system of functions (46) vanishes. These points (or lines) are called the *singular* points (or lines) of the corresponding coordinates.

Surfaces on which one of the curvilinear coordinates takes a constant value are called coordinate surfaces. A surface on which the coordinate τ_i is constant will be referred to as the surface τ_i. The set of surfaces τ_i forms a system of surfaces τ_i. There are three systems of coordinate surfaces, corresponding to τ_1, τ_2, and τ_3. An intersection of coordinate surfaces forms a coordinate curve; the set of coordinate curves constitutes a coordinate grid. Only one coordinate varies along a coordinate curve. There are also three systems of coordinate lines. Along a curve of the system τ_i only the coordinate τ_i varies. Three pairs of coordinate surfaces, one pair from each system, form a *curvilinear coordinate parallelepiped*, whose edges are segments of coordinate curves.

* See V.I.Smirnov [1]. Vol. 3. Part 1. p. 19.

If every pair of coordinate surfaces of different systems intersects at a right angle, the curvilinear coordinates are said to be orthogonal. Obviously, in this case, the coordinate curves also intersect at right angles.

Let us examine the displacement of some point x along the coordinate curve τ_j passing through it, over a distance corresponding to an increment in the curvilinear coordinate of $d\tau_j$. It follows from the system (47) that the Cartesian coordinates of the point x then receive the increments

$$dx_i = \frac{\partial x_i}{\partial \tau_j} d\tau_j \qquad (i = 1, 2, 3) .$$

Consequently, the direction cosines of the tangent to the curve τ_j at the point x are proportional to the partial derivatives $\partial x_1/\partial \tau_j$, $\partial x_2/\partial \tau_j$, $\partial x_3/\partial \tau_j$. We then arrive at the following *orthogonality condition:*

$$\sum_{\alpha=1}^{3} \frac{\partial x_\alpha}{\partial \tau_j} \frac{\partial x_\alpha}{\partial \tau_k} = 0 \qquad \text{if} \quad j \neq k .$$

The displacement corresponding to the increment in the curvilinear coordinate $d\tau_j$ is equal to

$$ds_j = \sqrt{dx_1{}^2 + dx_2{}^2 + dx_3{}^2} = d\tau_j \sqrt{\left(\frac{\partial x_1}{\partial \tau_j}\right)^2 + \left(\frac{\partial x_2}{\partial \tau_j}\right)^2 + \left(\frac{\partial x_3}{\partial \tau_j}\right)^2} = h_j \, d\tau_j ,$$

where

$$h_j \equiv \sqrt{\left(\frac{\partial x_1}{\partial \tau_j}\right)^2 + \left(\frac{\partial x_2}{\partial \tau_j}\right)^2 + \left(\frac{\partial x_3}{\partial \tau_j}\right)^2}$$

The quantities h_j (where $j = 1, 2, 3$) are called the Lamé coordinate parameters.

The volume of an infinitesimal curvilinear coordinate parallelepiped is obviously equal to

$$ds_1 \, ds_2 \, ds_3 = h_1 h_2 h_3 \, d\tau_1 \, d\tau_2 \, d\tau_3 .$$

The product $h_1 h_2 h_3$ of the Lamé parameters is (except for sign) equal to the Jacobian of the transformation. To show this, it is sufficient to take the square of the Jacobian, using the rule for "column by column" multiplication. As a result of the orthogonality, we obtain

$$D^2 = \begin{vmatrix} h_1{}^2 & 0 & 0 \\ 0 & h_2{}^2 & 0 \\ 0 & 0 & h_3{}^2 \end{vmatrix} = (h_1 h_2 h_3)^2 .$$

Thus, at the singular points of the coordinates, at least one of the Lamé parameters will vanish.

In what follows, we shall use only two types of curvilinear coordinates, *cylindrical* and *spherical*.

The cylindrical coordinates r, φ, z of a point x are defined by the system of equations

$$x_1 = r \cos \varphi , \qquad x_2 = r \sin \varphi , \qquad x_3 = z \qquad (0 \leqslant \varphi \leqslant 2\pi) .$$

The Lamé coordinate parameters have the values

$$h_1 \equiv h_r = 1\,, \quad h_2 \equiv h_\varphi = r\,, \quad h_3 \equiv h_z = 1\,. \tag{48}$$

The coordinate surfaces r form a system of circular cylindrical surfaces of radius r with a common axis (coinciding with axis 3 of the Cartesian coordinate system). This axis is called the *cylindrical coordinate axis*. The coordinate surfaces φ form a system of half-planes; the boundary of each half-plane is the cylindrical coordinate axis. The coordinate surfaces z form a system of planes perpendicular to the cylindrical coordinate axis. The coordinate r represents the distance of the point x from axis 3 of the Cartesian coordinate system (or, equivalently, from the cylindrical coordinate axis); φ represents the angle between the coordinate half-plane passing through the point x and the coordinate half-plane in which axis 1 of the Cartesian coordinate system lies; z coincides with the Cartesian coordinate x_3.

Through every point x that does not lie on the cylindrical coordinate axis there pass one coordinate surface r, one coordinate surface φ, and one coordinate surface z. On the cylindrical coordinate axis, the parameter $h_\varphi = 0$, and consequently, it is a singular line. On this axis, the coordinate φ does not have a definite value.

The spherical coordinates r, θ, and φ of a point x are defined by the system of equations

$$x_1 = r \sin\theta \cos\varphi\,, \quad x_2 = r\sin\theta\sin\varphi\,, \quad x_3 = r\cos\theta\,;$$
$$0 \leqslant \theta \leqslant \pi\,, \quad 0 \leqslant \varphi \leqslant 2\pi\,.$$

The Lamé coordinate parameters have the values

$$h_1 \equiv h_r = 1\,, \quad h_2 \equiv h_\theta = r\,, \quad h_3 \equiv h_\varphi = r\sin\theta \tag{49}$$

The coordinate surfaces r form a system of spherical surfaces with a common center at the point $x_1 = x_2 = x_3 = 0$, called the *spherical coordinate origin*. The θ surfaces form a system of circular cones with a common axis coinciding with axis 3 of the Cartesian coordinates. It is called the *polar* axis. The φ surfaces form a system of half-planes passing through the polar axis. The coordinate r represents the length of the radius vector of the point x; the coordinate θ represents the angle between the radius vector and the polar axis; the coordinate φ represents the angle between the coordinate half-plane passing through the point x and the coordinate half-plane in which axis 1 of the Cartesian coordinates lies.

Through every point x not on the polar axis there pass one r-surface, one θ-surface, and one φ-surface. On the polar axis, the parameter $h_\varphi = 0$; consequently, it is a singular line. On this axis, the coordinate φ does not have a definite value. At the singular point $r = 0$, the coordinate θ is also undefined.

Let us now calculate the differential expressions in these orthogonal coordinates.

We shall begin with the Ostrogradskii-Gauss formula (1), taking for the functions A_1, A_2, and A_3 the components of some vector A and for the region V the curvilinear coordinate parallelepiped formed by the six coordi-

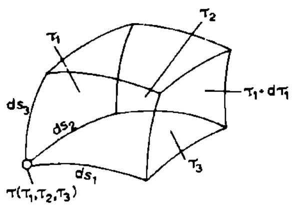

Fig. 42.

nate surfaces: τ_1, $\tau_1 + d\tau_1$, τ_2, $\tau_2 + d\tau_2$, τ_3, $\tau_3 + d\tau_3$ (fig. 42). The lengths of the edges of this parallelepiped are equal to

$$ds_1 = h_1 \, d\tau_1 , \qquad ds_2 = h_2 \, d\tau_2 , \qquad ds_3 = h_3 \, d\tau_3 . \tag{50}$$

Breaking up the integral over the surface of the parallelepiped into the sum of integrals over its faces, we obtain

$$\iiint_V \left(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}\right) dV = \sum_{\alpha=1}^{3} \left(\iint_{S_{\tau_\alpha}} A_n \, dS + \iint_{S_{\tau_\alpha + d\tau_\alpha}} A_n \, dS_\alpha \right) ,$$

where S_{τ_α} and $S_{\tau_\alpha + d\tau_\alpha}$ are the faces formed by the coordinate surfaces τ_α and $\tau_\alpha + d\tau_\alpha$, and $A_n = \Sigma_{\beta=1}^{3} A_\beta \cos(n, x_\beta)$ are the projections of the vector A onto the normal to the corresponding faces. Note that, on three of the faces of the parallelepiped, the direction of the outer normal coincides with the direction of the coordinate axis normal to the face. However, on the faces opposite these, the outer normal is directed oppositely. Therefore, by the mean-value theorem,

$$\iiint_V \left(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}\right) dV = \left(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}\right) \bar{V} ,$$

$$\iint_{S_{\tau_\alpha}} A_n \, dS_\alpha = - \iint_{S_{\tau_\alpha}} A_{\tau_\alpha} \, dS_\alpha = - A_{\tau_\alpha} \bar{S}_\alpha , \qquad \alpha = 1, 2, 3 ,$$

$$\iint_{S_{\tau_\alpha + d\tau_\alpha}} A_n \, dS_\alpha = \iint_{S_{\tau_\alpha + d\tau_\alpha}} A_{\tau_\alpha} \, dS_\alpha = A_{\tau_\alpha + d\tau_\alpha} \bar{S}_{\tau_\alpha + d\tau_\alpha} , \qquad \alpha = 1, 2, 3 ,$$

where $\bar{V}$ is the volume of the parallelepiped, $\bar{S}_{\tau_\alpha}$ and $\bar{S}_{\tau_\alpha + d\tau_\alpha}$ are the areas of the faces of the parallelepiped, and the values of the functions on the right sides of these equations are taken at certain internal points of the corresponding regions of integration.

With an accuracy up to second-order terms, we may set

$$A_{\tau_\alpha + d\tau_\alpha} \bar{S}_{\tau_\alpha + d\tau_\alpha} = A_{\tau_\alpha} \bar{S}_{\tau_\alpha} + \frac{\partial A_{\tau_\alpha} \bar{S}_{\tau_\alpha}}{\partial \tau_\alpha} \, d\tau_\alpha .$$

Substituting these expressions into the Ostrogradskii-Gauss formula, we obtain

$$\left(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}\right) V = \sum_{\alpha=1}^{3} \left[\left(A_\alpha \bar{S}_\alpha + \frac{\partial}{\partial \tau_\alpha} A_\alpha \bar{S}_\alpha \, \mathrm{d}\tau_\alpha\right) - A_\alpha \bar{S}_\alpha\right]$$
$$= \sum_{\alpha=1}^{3} \frac{\partial A_\alpha \bar{S}_\alpha}{\partial \tau_\alpha} \mathrm{d}\tau_\alpha \, .$$

Substituting the values

$$\bar{V} = \mathrm{d}s_1 \, \mathrm{d}s_2 \, \mathrm{d}s_3 = h_1 h_2 h_3 \, \mathrm{d}\tau_1 \, \mathrm{d}\tau_2 \, \mathrm{d}\tau_3 \ , \tag{51a}$$

$$\bar{S}_\alpha = \mathrm{d}s_\beta \, \mathrm{d}s_\gamma = h_\beta h_\gamma \, \mathrm{d}\tau_\beta \, \mathrm{d}\tau_\gamma \ , \qquad \alpha, \beta, \gamma = \begin{cases} 1, 2, 3 \ , \\ 2, 3, 1 \ , \\ 3, 1, 2 \ , \end{cases} \tag{51b}$$

and taking the limit as V approaches zero, we obtain the desired formula:

$$\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial \tau_1} h_2 h_3 A_1 + \frac{\partial}{\partial \tau_2} h_3 h_1 A_2 + \frac{\partial}{\partial \tau_3} h_1 h_2 A_3\right] . \tag{52}$$

Setting

$$A_i \equiv \partial u / \partial x_i \qquad (i = 1, 2, 3)$$

and noticing that

$$\frac{\partial u}{\partial s_\alpha} = \frac{\partial u}{h_\alpha \, \partial \tau_\alpha} \qquad (\alpha = 1, 2, 3) \ , \tag{53}$$

we also obtain the formula

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}$$
$$= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial \tau_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial u}{\partial \tau_1}\right) + \frac{\partial}{\partial \tau_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial u}{\partial \tau_2}\right) + \frac{\partial}{\partial \tau_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial u}{\partial \tau_3}\right)\right] . \tag{54}$$

We now apply Stokes' integral theorem to one of the faces of the parallelepiped that we are considering; let us take the face formed by the τ_1-surface (fig. 42). Breaking up the integral on the boundary into the sum of the integrals over its edges, applying the mean-value theorem, and keeping relationship (50) in mind, we obtain

$$B_1 \bar{S}_1 = A_2 \, \mathrm{d}s_2 - A_3 \, \mathrm{d}s_3 + A_3 \, \mathrm{d}s_3 + \frac{\partial}{\partial \tau_2} A_3 \, \mathrm{d}s_3 \, \mathrm{d}\tau_2 - A_2 \, \mathrm{d}s_2 - \frac{\partial}{\partial \tau_3} A_2 \mathrm{d}s_2 \mathrm{d}\tau_3$$
$$= \frac{\partial}{\partial \tau_2} h_3 \, A_3 \, \mathrm{d}\tau_3 \, \mathrm{d}\tau_2 - \frac{\partial}{\partial \tau_3} h_2 \, A_2 \, \mathrm{d}\tau_2 \, \mathrm{d}\tau_3 \ ,$$

where B_1 is the projection of the vector $\boldsymbol{B}$ in the direction of τ_1. In this relationship, it is assumed that the normal to the contour is directed in the direction of increase of the coordinate τ_1. The corresponding path around the contour is shown in fig. 43.

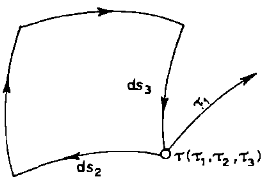

Fig. 43.

Let us substitute the value of the area S_1 given by eq. (51b) into the above equation, so that

$$B_1 = \frac{1}{h_2 h_3}\left(\frac{\partial h_3 A_3}{\partial \tau_2} - \frac{\partial h_2 A_2}{\partial \tau_3}\right).$$

Hence, by means of a cyclic permutation of subscripts, we see that, in general,

$$B_\alpha = \frac{1}{h_\beta h_\gamma}\left(\frac{\partial h_\gamma A_\gamma}{\partial \tau_\beta} - \frac{\partial h_\beta A_\beta}{\partial \tau_\gamma}\right), \qquad \alpha, \beta, \gamma = \begin{cases} 1, 2, 3, \\ 2, 3, 1, \\ 3, 1, 2. \end{cases} \tag{55}$$

When we project the vector B (with coordinates determines by these formulae) in an arbitrary direction n and equate the expression obtained with the expression given in eq. (45), we obtain the formula

$$\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right)\cos(n, x_1) + \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right)\cos(n, x_2) + \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)\cos(n, x_3)$$
$$= \frac{1}{h_2 h_3}\left(\frac{\partial h_3 A_3}{\partial \tau_2} - \frac{\partial h_2 A_2}{\partial \tau_3}\right)\cos(n, \tau_1) + \frac{1}{h_3 h_1}\left(\frac{\partial h_1 A_1}{\partial \tau_3} - \frac{\partial h_3 A_3}{\partial \tau_1}\right)\cos(n, \tau_2)$$
$$+ \frac{1}{h_1 h_2}\left(\frac{\partial h_2 A_2}{\partial \tau_1} - \frac{\partial h_1 A_1}{\partial \tau_2}\right)\cos(n, \tau_3). \tag{56}$$

Problems

1. Derive formula (54) by using Green's theorem (7).
   Method: Set $v = -1$ in Green's theorem.

2. Show that in orthogonal curvilinear coordinates in a plane the relationship (54) takes the form

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial \tau_1}\left(\frac{h_2}{h_1}\frac{\partial u}{\partial \tau_1}\right) + \frac{\partial}{\partial \tau_2}\left(\frac{h_1}{h_2}\frac{\partial u}{\partial \tau_2}\right)\right],$$

   where the parameters h_1 and h_2 have the same meaning as in the three-dimensional case.

3. Show that when we make the change of variables

$x_1 = (c + r\cos\theta)\cos\varphi\ , \qquad x_2 = (c + r\cos\theta)\sin\varphi\ , \qquad x_3 = r\sin\theta\ ,$

the differential equation

$$\frac{\partial^2 u}{\partial x_1{}^2} + \frac{\partial^2 u}{\partial x_2{}^2} + \frac{\partial^2 u}{\partial x_3{}^2} = 0$$

takes the form

$$\frac{\partial}{\partial r}\left[r(c + r\cos\theta)\frac{\partial u}{\partial r}\right] + \frac{1}{r}\frac{\partial}{\partial\theta}\left[(c + r\cos\theta)\frac{\partial u}{\partial\theta}\right] + \frac{r}{c + r\cos\theta}\frac{\partial^2 u}{\partial\varphi^2} = 0\ .$$

4. Show that if the coordinate surfaces are given by the equations

$$(x_1{}^2 + x_2{}^2)^{\frac{1}{2}} = \frac{c\sinh\tau_1}{\cosh\tau_1 - \cos\tau_2}\ , \quad x_2 = x_1\tanh\tau_3\ , \quad x_3 = \frac{c\sin\tau_2}{\cosh\tau_1 - \cos\tau_2}\ ,$$

we obtain an orthogonal curvilinear coordinate system with parameters

$$h_1 = h_2 = \frac{c}{\cosh\tau_1 - \cos\tau_2}\ , \qquad h_3 = \frac{c\sinh\tau_1}{\cosh\tau_1 - \cos\tau_2}\ .$$

This coordinate system is called a *toroidal* system. Its coordinate surfaces represent the surfaces of tori, spheres, and planes.

5. Show that if the coordinate surfaces are given by the equations

$$(x_1{}^2 + x_2{}^2)^{\frac{1}{2}} = \frac{c\sin\tau_2}{\cosh\tau_1 - \cos\tau_2}\ , \quad x_2 = x_1\tan\tau_3\ , \quad x_3 = \frac{c\sinh\tau_1}{\cosh\tau_1 - \cos\tau_2}\ ,$$

we obtain an orthogonal curvilinear coordinate system with parameters

$$h_1 = h_2 = \frac{c}{\cosh\tau_1 - \cos\tau_2}\ , \qquad h_3 = \frac{c\sin\tau_2}{\cosh\tau_1 - \cos\tau_2}\ .$$

This coordinate system is called a bipolar system.

Chapter XVIII

LAPLACE AND POISSON EQUATIONS

1. *Laplace and Poisson equations. Examples of problems leading to the Laplace equation*

The equation

$$\frac{\partial^2 u}{\partial x_1} + \frac{\partial^2 u}{\partial x_2} + \frac{\partial^2 u}{\partial x_3} = 0 , \tag{1}$$

where x_1, x_2, and x_3 are orthogonal Cartesian coordinates, is called the *Laplace equation*. The expression on the left side is called the Laplacian function and the rule for forming this expression is called the *Laplacian operator*. The Laplacian operator is commonly denoted by the symbol Δ, so that eq. (1) may be written in the form

$$\Delta u = 0 .$$

The non-homogeneous equation

$$\frac{\partial^2 u}{\partial x_1} + \frac{\partial^2 u}{\partial x_2} + \frac{\partial^2 u}{\partial x_3} = f \tag{2}$$

or

$$\Delta u = f ,$$

where f is a known function, is called the Poisson equation.

The form of the differential expressions on the left sides of the Laplace and Poisson equations are the same in all orthogonal Cartesian coordinates. When we change to curvilinear coordinates, this form may change, and, for orthogonal curvilinear coordinates, it may be defined in terms of the relationships of section 7 of the preceding chapter. In particular, by using eqs. (54), (48), and (49) of Chapter XVII, we see that in cylindrical coordinates r, φ, z,

$$\Delta \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} , \tag{3}$$

and in spherical coordinates r, θ, φ,

$$\Delta \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2} . \tag{4}$$

Numerous problems in the theory of heat flow, electrostatics, hydrodynamics, and so on lead to Laplace and Poisson equations. For example, let us examine certain problems which lead to the Laplace equations.

1. *The problem of a steady thermal state of a homogeneous body.* Sup-

pose that we have a homogeneous isotropic body that is insulated from the surrounding space and whose thermal state does not vary with time. Let us denote the portion of space occupied by the body by V, its surface by $\mathcal{F}V$, and the temperature at any point $x \in V$ by $u(x)$.

Let us show that at every internal point x of this body the function $u(x)$ satisfies the Laplace equation.

Let us take some region V_1 inside the body that is bounded by an arbitrary surface $\mathcal{F}V_1$ and let us examine the amount of heat that flows through an element $\mathrm{d}S_1$ of its surface per unit of time. From the Fourier principle, it is proportional to the area of the element and the normal derivative $\mathrm{d}u/\mathrm{d}n$, where n denotes the direction of the *outer* normal to the surface. In other words, this quantity of heat is equal to the product

$$k \frac{\mathrm{d}u}{\mathrm{d}n} \mathrm{d}S_1 .$$

The proportionality constant k is called the coefficient of internal heat conductivity of the body.

Let us examine the flow of heat in the body. We know from thermodynamics that heat flows from points at high temperatures to points at low temperatures. Consequently, when the derivative $\partial u/\partial n$ is negative, heat will flow from the internal portion of the body bounded by the surface $\mathcal{F}V_1$ into the region surrounding this surface. If this derivative is positive, the reverse situation will take place.

From this it follows that the double integral

$$k \iint_{\mathcal{F}V_1} \frac{\mathrm{d}u}{\mathrm{d}n} \mathrm{d}S_1 \tag{5}$$

yields the algebraic sum of the amount of heat that flows through the surface $\mathcal{F}V_1$ per unit of time. A negative sign represents a heat loss and a positive sign a heat gain.

If we assume that there are no sources of heat within the body and that no heat is converted into other forms of energy, the integral (5) must be equal to zero. Otherwise, heat would accumulate or be lost within the body and, consequently, the temperature of the body would change with time, which contradicts the hypothesis of a steady state.

Thus, in the case in question, we have

$$\iint_{\mathcal{F}V_1} \frac{\mathrm{d}u}{\mathrm{d}n} \mathrm{d}S_1 = 0 . \tag{6}$$

Let us apply Green's theorem, eq. (7) of Chapter XVII, to the region V_1:

$$\iiint_{V_1} (u \Delta v - v \Delta u) \,\mathrm{d}V = \iint_{\mathcal{F}V_1} \left(u \frac{\mathrm{d}v}{\mathrm{d}n} - v \frac{\mathrm{d}u}{\mathrm{d}n}\right) \mathrm{d}S_1$$

and let us set $v = 1$.

Remembering now that the integral (5) is equal to zero, we obtain

$$\iiint_{V_1} \Delta u \, dV_1 = 0 .$$

Since the region V_1 is arbitrary, it follows that

$$\Delta u = 0 ;$$

that is, the function $u(x)$ satisfies the Laplace equation.

Let us now suppose that we know the temperature distribution over the surface $\mathcal{F}V$ of the body and that we wish to determine the temperature of an arbitrary point *within* the body.

Obviously, we shall solve this problem if we find the solution to the Laplace equation that satisfies the boundary condition

$$u = f(x) \quad \text{when} \quad x \in \mathcal{F}V , \tag{7}$$

where $f(x)$ denotes the temperature at the point x on the surface $\mathcal{F}V$.

2. *The problem of equilibrium for electric charges on the surface of a conductor.* Let us examine a constant electric field that is created in space by some system of electric charges. If a discrete number of charges q_1, $q_2, \ldots, q_n$ are located at the points $\xi_1, \xi_2, \ldots, \xi_n$, the potential of the field at a point x will be equal to

$$u = \sum_{\alpha=1}^{n} \frac{q_\alpha}{r_\alpha} , \tag{8}$$

where $r_\alpha = |\xi_\alpha - x|$ is the distance from the charge q_α to the point x. However, if the charges are continuously distributed along a curve L, over a surface S, or throughout a volume V, the potential of the field at a point will be expressed by one of the integrals

$$u = \int_\alpha \frac{\rho_2}{r} \, dL , \quad u = \iint_S \frac{\rho_1}{r} \, dS , \quad u = \iiint_V \frac{\rho}{r} \, dV , \tag{9}$$

where r is the distance from the element of the curve (surface, volume) to the point in question. In these formulae, the quantities ρ_2, ρ_1, and ρ denote, respectively, the linear, surface, and volume charge densities:

$$\rho_2 = \lim_{\Delta L \to 0} \frac{\Delta q}{\Delta L} = \frac{dq}{dL} , \quad \rho_1 = \lim_{\Delta S \to 0} \frac{\Delta q}{\Delta S} = \frac{dq}{dS} , \quad \rho = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V} = \frac{dq}{dV} , \tag{10}$$

where Δq is the charge on the element of curve L (or of surface S or of volume V). In the general case, the potential of the field is equal to the sum of the potentials caused separately by each of these forms of charge distribution.

Let us suppose that a finite volume V of space is occupied by a conducting medium, that is, one in which charges can move freely, and that the remaining portion of space is a dielectric, that is, a medium in which the charges cannot move.

In a steady state, the potential of the field will be the same at all points of the region V including its boundary. Otherwise the electric charges would move so as to equalize the potential, and the field would

vary. From this, it is immediately obvious that in the region V the potential u of the field satisfies the Laplace equation

$$\Delta u = 0 . \tag{11}$$

Within the conductor, the charges of opposite sign must neutralize each other. This is true because any excess charges (of either sign) within the conductor are displaced outward as a result of the repulsion between charges of like sign until all of them are uniformly distributed on the boundary of the conductor. Consequently, if a steady state is attained, the excess charges are located on the boundary $\mathcal{F}V$ of the conductor in the form of an infinitesimally thin shell.

The potential of this shell at the point x is given by the integral

$$u = \iint\limits_{\mathcal{F}V} \frac{\rho_1}{r} \, dS , \tag{12}$$

where r is the distance from a variable point ξ on the surface of the conductor to the point x.

If the point x is outside the conductor, the function $1/r$ satisfies the Laplace equation. For

$$\frac{\partial}{\partial x_i} \frac{1}{r} = \frac{x_i - \xi_i}{r^3} , \qquad \frac{\partial^2}{\partial x_i^2} \frac{1}{r} = - 3 \, \frac{(x_i - \xi_i)^2}{r^4} + \frac{1}{r^3} ,$$

and hence,

$$\Delta \frac{1}{r} = \sum_{\alpha=1}^{3} \frac{\partial^2}{\partial x_\alpha^2} \frac{1}{r} = \frac{3}{r^3} - \frac{3}{r^4} \sum_{\alpha=1}^{3} (x_\alpha - \xi_\alpha)^2 = 0 .$$

Consequently, the potential u determined by formula (12) satisfies the Laplace equation. To prove this assertion, it is sufficient to apply to the integral (12) the rule for differentiating with respect to a parameter (we may do this since, by hypothesis, the point x lies outside the surface S and, consequently, the integrand in eq. (12) is nowhere infinite).

Thus, the potential u also satisfies the Laplace equation at every point *outside* the conductor.

Let us now discuss the phenomena that take place at infinitely distant points of a space filled by a dielectric, and on the surface of the conductor itself.

As we shall show below, the integral (12) (together with its first-order partial derivatives) vanishes at infinitely distant points. Thus, the products ru, $r^2(\partial u/\partial x_i)$ $(i = 1, 2, 3)$ remain bounded when the distance r from a point x to the coordinate origin increases without bound. With regard to the phenomena that take place on the surface of the conductor, it will be shown that the potential u remains bounded and continuous when the point x moves through the surface of the conductor. On the other hand, the normal derivatives of the potential u have a finite discontinuity in this case. This discontinuity is characterized by the equation

$$\frac{du}{dn_i} - \frac{du}{dn_e} = 4\pi\rho_1 , \tag{13}$$

where du/dn_i and du/dn_e are the limiting values of the expression

$$\sum_{\alpha=1}^{3} \frac{\partial u}{\partial x_\alpha} \cos (n, x_\alpha)$$

as x approaches the point $\xi \in \mathcal{F}V$ along the inner and outer normals to $\mathcal{F}V$ at the point ξ, respectively.

Let us use eq. (13) for stating the so-called electrostatic problem of finding the charge density of a layer that is continuously distributed on the surface of a given conductor (provided the conductor is in a state of electric equilibrium).

Let us suppose that for a given conductor this condition is fulfilled. Then, from the above, we see that the potential inside the conductor will have a constant value and, consequently,

$$du/dn_i = 0 .$$

It follows from this equation and from formula (13) that

$$\rho_1 = -\frac{1}{4\pi} \frac{du}{dn_e} ; \tag{14}$$

that is, the desired charge density of the layer will be found if we determine the potential u of this layer at points lying within the conductor.

Thus, the problem that we have stated amounts to finding the function u which, at all points of the space surrounding the conductor, (1) satisfies the Laplace equation, (2) approaches zero at infinity, and (3) satisfies the condition

$$u(x) = \text{constant} \quad \text{when} \quad x \in \mathcal{F}V .$$

3. *The problem of the motion of an incompressible liquid.* Let us investigate the steady-state motion of an incompressible liquid. Let us denote by $\mathbf{v}$ the velocity vector of the liquid and let v_1, v_2, and v_3 be its projections onto fixed coordinate axes. We shall henceforth assume that these projections do not depend directly on the time t. We shall call such a motion of a liquid steady-state.

Let us now suppose that the motion of the liquid takes place with a velocity potential u; in other words, we shall assume that

$$v_i = \partial u/\partial x_i \quad (i = 1, 2, 3) . \tag{15}$$

Let us show that this potential satisfies the Laplace equation

$$\Delta u = 0 . \tag{16}$$

It was shown in Chapter VI of Part I that the projections v_1, v_2, and v_3 of the vector $\mathbf{v}$, and the density ρ of a liquid are related by the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \sum_{\alpha=1}^{3} \frac{\partial \rho v_\alpha}{\partial x_\alpha} = 0 .$$

Remembering that the density ρ of an incompressible liquid is constant, we may rewrite this equation in the form

$$\sum_{\alpha=1}^{3} \frac{\partial v_\alpha}{\partial x_\alpha} = 0 .$$

Instead of v_α (where $\alpha = 1, 2, 3$), let us substitute their values as given by eq. (15). This gives us the Laplace equation (16).

The boundary conditions will depend on the nature of the hydrodynamic problem in question. For example, for rigid walls of a basin, we have

$$\mathrm{d}u/\mathrm{d}n = 0 ,$$

where n is the normal to the wall. If a rigid body is moving in the liquid according to some given function, then, on the surface of the body,

$$\mathrm{d}u/\mathrm{d}n = f(x) , \tag{17}$$

where $f(x)$ is the given function. More complicated boundary conditions are obtained for a free surface of a liquid (see Chapter XXIV).

Furthermore, we need to consider the conditions that must be satisfied by the potential at infinity. In many hydrodynamic problems, it is assumed that a disturbance-causing motion in the liquid, if it acts in a bounded region of space, does not change the state of rest of the liquid at an infinite distance from that region. Then, the partial derivatives $\partial u/\partial x_i$ vanish at infinity. It can then be shown that the quantities

$$|ru| , \qquad \left| r^2 \frac{\partial u}{\partial x_i} \right| \qquad (i = 1, 2, 3)$$

will be bounded when r increases without bound.

The assumption that the liquid is at rest at infinity is made, for example, in the extremely important hydrodynamic problem of the motion of a rigid body in an incompressible liquid that fills all space.

Problem

Show that the problem of steady-state temperature distribution in a bounded homogeneous body in which the sources of heat are distributed continuously, reduces to integration to the equation

$$\Delta u = - 4\pi f_1 \quad \text{when} \quad x \in V - \mathcal{F}V$$

under the condition that

$$\frac{\mathrm{d}u}{\mathrm{d}n} + ku + f = 0 \quad \text{when} \quad x \in \mathcal{F}V ,$$

where V is the region occupied by the body and k, f_1, and f are functions such that f_1 is given within the body and f and k are given on its surface.

2. *Boundary-value problems*

We have examined a number of physical problems. Each of them was reduced to the mathematical problem of finding a function u that satisfies the Laplace equation $\Delta u = 0$ at all *internal* points of a given region V and that satisfies some given condition *on the boundary* $\mathcal{F}V$ of the region V. This condition is called a *boundary condition*, and the mathematical problem connected with it is called a boundary-value problem.

There may be boundary-value problems not only for the Laplace equation but for any equations of the elliptic type.

There are three basic types of boundary-value problems, depending on the form of the boundary condition *:

1. $u(x) = \psi(x)$, when $x \in \mathcal{F}V$ is the first boundary-value problem or the *Dirichlet problem*.

2. $du/dn = \psi(x)$, when $x \in \mathcal{F}V$ is the second boundary-value problem or the *Neumann problem*.

3. $du/dn + \beta u = \psi(x)$, when $x \in \mathcal{F}V$ is the third or *mixed* boundary-value problem.

Here, ψ and β are continuous functions defined on the boundary surface $\mathcal{F}V$ and du/dn denotes the derivative at a point of the surface $\mathcal{F}V$ in the direction of the outer normal to it.

The study of a wide class of *steady-state* physical processes and phenomena leads to boundary-value problems of this kind. In particular, the examples examined in the preceding section led to the Dirichlet and Neumann problems. However, problems with other boundary conditions are also encountered. Among these are, for example, hydrodynamic problems in which free surfaces of liquid media are examined. If the physical medium in question is non-homogeneous, but consists of several homogeneous portions, certain coupling conditions, etc., must be satisfies on the boundaries.

If the region in which a solution to the equation is sought is bounded, the boundary-value problem is said to be *internal*. On the other hand, if this region is the portion of space lying outside some bounded region, the boundary-value problem is said to be *external*. If the boundary of the region is a plane, the boundary-value problem is said to be posed for a half-space. The problem of the thermal state of a homogeneous body that was formulated in the preceding section is an example of an internal Dirichlet problem and the electrostatic problem is an example of an external problem.

Let us now make precise the mathematical formulation of a boundary-value problem. As we stated in the introduction, a mathematical physical problem is said to be correctly stated if a solution exists that is unique and that is a continious function of the given conditions of the problem.

The requirements contained in the formulation of the concept of correctness reflect our general view that there are a wide class of physical phenomena: (1) which must arise when certain necessary conditions are met (a solution exists); (2) which are completely determined by these con-

* See also Chapter XXVII, section 2.

ditions (the solution is unique), and (3) which are changed only slightly if the conditions are changed only slightly (the continuous dependence of the solution on the given conditions of the problem) *. A correct statement of the problem usually assures a physical meaning to the solution.

The conditions assuring correctness of the statement of a boundary-value problem vary somewhat for the different types of problem. However, there is a basic group of conditions that enter into all these formulations. It amounts to the following: a function constituting a solution to a boundary-value problem (stated for a second-order partial differential equation) must

(1) be continuous in the region for which the problem is posed, up to the boundary of the region,
(2) have continuous second derivatives within the region and satisfy the given equation (for example, Laplace's, Poisson's, etc.),
(3) satisfy the given boundary condition on the boundary of the region, and
(4) (if the region is three-dimensional and infinite) approach zero as we displace a given point an infinite distance along an arbitrary ray contained in the region.

Solutions to boundary-value problems in three-dimensional regions satisfying the conditions enumerated will be called *regular* solutions.

As we shall show, regular solutions to the fundamental boundary-value problems are unique (sometimes under certain additional conditions) and they depend continuously on the boundary conditions. We shall not go into the existence of solutions, which would require the application of a special mathematical apparatus. We note only that regular solutions exist only when the given boundary condition is sufficiently smooth. In practice, this point is not especially important, because any boundary condition with a physical meaning can be approximated to any desired degree of accuracy by smooth functions. Within the framework of an idealized view of physical objects as being continuous, this approximation can have the same physical meaning as the original condition. A more theoretical solution to the problem of the existence of solutions is given in the theory of *generalized solutions*, which we shall touch on in Chapter XXXIX.

We note, in conclusion, that the solutions to correctly stated boundary-value problems for an arbitrary equation of the elliptic type are never less smooth (in the sense of having fewer continuous derivatives) than the functions that determine them (the coefficients in an equation and the given conditions of a problem). Usually, they are differentiable infinitely many times at all internal points of the region being studied. This property of the solutions of boundary-value problems is closely related to the fact that the study of steady-state physical processes – of equilibria that are the end

* Of course, we may not assert that an incorrectly stated problem in the sense defined does not have a physical meaning. For example, we might wish that the last requirement (the continuous dependence on the given conditions of the problem) might not be satisfied in order to investigate the conditions of stability of the process, etc. However, in the overwhelming majority of physical problems (if not in all) that are studied by the methods of mathematical physics, the requirement for correct stating (as defined above) is a necessary part of the rigorous formulation of the mathematical statement of the problem.

results of a preceding equalization process – leads to boundary-value problems. It is obvious from physical considerations that not only the solutions to a problem but also the boundary conditions (which rather precisely determine the nature of the phenomenon) will in this case be extremely smooth.

3. *Harmonic functions*

A function $u(x)$ is said to be *harmonic at a point* x if it has continuous second derivatives and satisfies the Laplace equation at that point. A function $u(x)$ is said to be *harmonic in a closed region* V if it

(1) is continuous throughout that region,
(2) is harmonic at all interior points of the region, and
(3) (when the region V is infinite) approaches zero as we displace a point x infinitely far along an arbitrary ray belonging to the region.

We note that on the basis of this definition, regular solutions to boundary-value problems for the Laplace equation are harmonic functions in the region in question.

Let us establish some important properties of harmonic functions.

AN EXTREME-VALUE THEOREM. *If a function $u(x)$ is harmonic in a region V, it does not have maxima or minima within that region but attains its largest and smallest values on the boundary.*

Proof: Let us suppose that the function u attains a maximum at a point $x \in V - \mathcal{F}V$. Consider a spherical surface σ, with center at point x, lying entirely within the region V. The radius of the sphere can be chosen sufficiently small so that

$$u(x) > u_{\max} + \epsilon \ , \tag{18}$$

where $u_{\max}$ is the maximum value of u on σ, and ϵ is a positive number. Also, we can find a sufficiently small positive number η such that at an arbitrary point ξ lying either on or within the surface σ

$$\eta|x - \xi|^2 < \tfrac{1}{2}\epsilon \ ,$$

where $|x - \xi|$ is the distance between the points x and ξ. Then, on the basis of the inequality (18), the function

$$v(\xi) = u(\xi) + \eta|x - \xi|^2$$

will exceed its greatest value on σ at the point $\xi = x$. This means that its maximum must be attained within the surface σ. But at the point where the maximum is attained, the second derivatives with respect to the coordinates of the point ξ cannot exceed zero. However,

$$\Delta_\xi v = \frac{\partial^2 v}{\partial \xi_1^2} + \frac{\partial^2 v}{\partial \xi_2^2} + \frac{\partial^2 v}{\partial \xi_3^2} = \eta\Delta_\xi|x - \xi|^2 = 6\eta > 0 \ .$$

The contradiction shows the impossibility of inequality (18), from which it follows that the function u cannot have a maximum value within the region V. In a similar manner, one can show that the function u cannot have a

minimum within V. But like every continuous function, it has a maximum and a minimum in the closed region (Weierstrass' theorem). Since they cannot be within the region V, they must be on its boundary.

Let us note a useful corollary.

COROLLARY. *If two functions u and v are harmonic in the region V, satisfaction on the boundary of the region of one of the inequalities*

$$u \leq v \quad \text{or} \quad |u| \leq v$$

implies the satisfaction of the same inequality within the region also.

Proof: If the function $u - v$, which is harmonic in the region V, is non-positive on the boundary of the region, it must be non-positive everywhere in the region, since within the region it cannot exceed its maximum value on the boundary. The assertion thus follows in the case of $u \leq v$. The inequality $|u| \leq v$ is equivalent to the two inequalities $u \leq v$ and $-v \leq u$. From what has been shown, the satisfaction of each of these on the boundary implies their satisfaction within the region as well. Hence, the assertion is also true in the case of the inequality $|u| \leq v$.

By using the extreme-value theorem, let us prove the following lemma.

REMOVABLE-SINGULARITY LEMMA. *Suppose that a point $\xi = x$ is an isolated singular point of a function $u(\xi)$ and that, at all points of some neighbourhood Ω of the point x, the function $u(\xi)$ is harmonic. Then, either the function $u(\xi)$ increases no more slowly as ξ approaches x than does $1/r$ (where $r = |x - \xi|$ is the distance between the points x and ξ), or the function $u(\xi)$ has a removable singularity at the point x and can be redefined at that point in such a way that it will be harmonic there.*

Proof: Let us choose a positive number a sufficiently small that the sphere consisting of points r such that $r \leq a$ belongs entirely to the region Ω. We shall show in section 6 without using the present lemma, that it is possible to define a function that is harmonic in a sphere and that coincides on its surface with a given continuous function. We denote by $v(\xi)$ a function that is harmonic in the sphere $r \leq a$, having the same values on its surface as does the function $u(\xi)$. Let us consider the function

$$\frac{1}{r} - \frac{1}{a}.$$

It is non-negative within the sphere $r \leq a$ and harmonic in the region V_ϵ that is obtained by removing from the sphere $r \leq a$ an arbitrarily small neighbourhood $r \leq \epsilon$ of the point x. As ξ approaches x, the function increases in proportion to $1/r$. Therefore, if the function $u(\xi)$ increases more slowly than $1/r$ as ξ approaches x (that is, if the product ru approaches 0 as ξ approaches x), then there exists a number η, which approaches 0 as ϵ approaches 0 such that

$$|u - v| \leq \eta\left(\frac{1}{r} - \frac{1}{a}\right) \quad \text{for} \quad r = \epsilon \quad \text{and} \quad r = a\,. \tag{19}$$

For η, we can take the smallest value of the expression

$$|u - v| \frac{ra}{a - r}$$

with $r = \epsilon$. Since the functions $u - v$ and $\eta(1/r - 1/a)$ are both harmonic in the region V_ϵ, then from the corollary to the extreme-value theorem, the inequality (19) remains valid for $\epsilon \leqslant r \leqslant a$. Let us fix the point ξ by giving the left side of the inequality (19) and also the function $(1/r - 1/a)$ certain fixed values; then let the radius ϵ approach zero. The right side of the inequality (19) will approach zero and, since its left side is independent of ϵ, we have $u = v$ for all $\xi \neq x$ and $r \leqslant a$.

Thus, if the function $u(\xi)$ increases more slowly than $1/r$ as ξ approaches x, then, for $\xi \neq x$, it coincides with the bounded function v and, consequently, it is bounded for $\xi \neq x$. Then, since $u = v$ for all $\xi \neq x$, we may set $u(x)$ identically equal to $v(x)$ at the singular point $\xi = x$; that is, the point x is a removable singular point for the function $u(\xi)$.

Thus, a function that is harmonic at all points of a region except for a finite number of isolated points x^i (where $i = 1, 2, 3, \ldots$), at which it has a non-removable singularity, increases at least as rapidly as $1/|\xi - x^i|$ as these points are approached. The function has no other kind of singular point. An example of a function with a non-removable singularity at a point x^i that is harmonic at all remaining points in space is the function $1/|\xi - x^i|$.

To examine functions that are harmonic in infinite regions, let us place every point x in space in correspondence with a point ξ with coordinates

$$\xi_i = x_i \frac{a^2}{|x|^2} \quad (i = 1, 2, 3, \ |x|^2 \equiv x_1{}^2 + x_2{}^2 + x_3{}^2, \ a = \text{constant}) . \tag{20}$$

The transformation given by eq. (20) is called an *inversion* with respect to the spherical surface of radius a with center at the point $x = 0$. The points x and ξ are said to be *harmonically conjugate* with respect to the spherical surface referred to.

Since the relations

$$\frac{\xi_i}{x_i} = \frac{a^2}{|x|^2} \quad (i = 1, 2, 3)$$

have the same values for all i, the two harmonically conjugate points x and ξ lie on a single ray drawn through the point $|x| = 0$. Furthermore, if we calculate the distance $|\xi| = \sqrt{(\xi_1{}^2 + \xi_2{}^2 + \xi_3{}^2)}$ of the point ξ from the origin of the ray by use of eq. (20), we see that $|\xi|\,|x| = a^2$.

From this it follows that the geometry of the transformation in question is the same as if the space were reflected in the surface Σ of the sphere of radius a with center at the point $|x| = 0$. Points lying on Σ are mapped into themselves and points lying outside (inside) Σ are mapped into points lying inside (outside) Σ. In particular, an infinitely distant point is mapped into the point $|x| = 0$ and the point $|x| = 0$ is mapped into an infinitely distant point. It is easy to show that, with this inversion, curves are mapped into curves, surfaces into surfaces, and regions into regions. Infinite regions are mapped into regions containing the coordinate origin and regions containing the coordinate origin are mapped into infinite regions.

Since the property of conjugacy of two points is mutual, that is, since

each is mapped into the other upon inversion, arbitrary sets of points also have this property. In particular, if a region V is mapped into a region V', then the region V' will be mapped into the region V. The regions V and V' are said to be *conjugate* to each other.

Suppose that V' is conjugate to the region V under inversion with respect to the surface of a sphere of unit radius. Let us prove the following theorem.

THEOREM (Kelvin). *If a function $u(x)$ is harmonic in the region V, the* function

$$v(\xi) \equiv \frac{1}{|\xi|}\, u\left(\frac{\xi_1}{|\xi|^2}, \frac{\xi_2}{|\xi|^2}, \frac{\xi_3}{|\xi|^2}\right) \tag{21}$$

will be harmonic in the region V'.

Proof: Let us introduce the spherical coordinates r, θ, and φ, with origin at the point $|x| = 0$. Then, the point $\xi(r', \theta, \varphi) \in V'$, where $r' \equiv 1/r$, will be harmonically conjugate to the point $x(r, \theta, \varphi) \in V$. Therefore, eq. (21) takes the form

$$v(r', \theta, \varphi) = ru(r, \theta, \varphi) = \frac{1}{r'}\, u\left(\frac{1}{r'}, \theta, \varphi\right), \qquad r' \equiv \frac{1}{r}. \tag{22}$$

Let us first assume that the region V' does not contain the point $r' = 0$. Substituting the function v in the Laplace equation in spherical coordinates (see formula (4)), we obtain

$$\Delta_\xi v = \frac{\partial}{\partial r'}\left(r'^2 \frac{\partial v}{\partial r'}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial v}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial\varphi^2} = 0, \tag{23}$$

and since

$$\frac{\partial}{\partial r'} = \frac{\partial r}{\partial r'}\frac{\partial}{\partial r} = -\frac{1}{r'^2}\frac{\partial}{\partial r} = -r^2\frac{\partial}{\partial r},$$

we obtain

$$\frac{\partial}{\partial r'}\left\{r'^2 \frac{\partial}{\partial r'}\left[\frac{1}{r'}\, u\left(\frac{1}{r'}, \theta, \varphi\right)\right]\right\} = r^2 \frac{\partial^2}{\partial r^2}[ru(r, \theta, \varphi)] = r\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right),$$

so that

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta \frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\theta^2} = 0.$$

Since the function u is harmonic in the region V, this equation is identically satisfied when $x(r, \theta, \varphi) \in V$, that is, when $\xi(r', \theta, \varphi) \in V'$. Consequently, the function $v(\xi)$ satisfies the Laplace equation (23) when $\xi \in V'$. Then, as can easily be seen by direct differentiation, the existence and continuity of the derivatives of $u(x)$ in the region V imply the existence and continuity of the derivatives of the same order of the function $v(\xi)$ in the region V'. Thus, the theorem is proven for the assumption that the point $r' = 0$ does not belong to the region V'.

Let us now suppose that the point $r' = 0$ does belong to the region V'. This point is singular for the function

$$v = \frac{1}{r'}\, u\left(\frac{1}{r'}, \theta, \varphi\right) .$$

Let us show that this is a removable singularity.

Suppose that ξ' is an arbitrary point of the region V' that does not coincide with the point $r' = 0$, and that ω is a sphere with center at the point $r' = 0$ and radius so small that the point ξ' lies outside the sphere. Then, the region $V' - \omega$ does not contain the point $r' = 0$ and, from what we have shown above, the function v is harmonic within this region and, in particular, harmonic at the point ξ'. Consequently, the function v is harmonic at all points of some neighbourhood of the point $r' = 0$, except for this point itself (where it is not defined). By the preceding lemma, as r' approaches zero the function v either remains bounded or increases at least as rapidly as $1/r'$. However, the latter is impossible. For it follows from eq. (22) that

$$r'v(r', \theta, \varphi) = u\left(\frac{1}{r'}, \theta, \varphi\right) .$$

As r' approaches zero, the function $u(1/r', \theta, \varphi)$ approaches a limit that is equal to its value at an infinitely distant point. But since the function u is harmonic by hypothesis, this limit is equal to zero, and, consequently,

$$\lim_{r' \to 0} r'v = 0 .$$

Thus, the function v is bounded in a neighbourhood of its singular point and hence, it may be redefined so as to be harmonic throughout the entire region V'. This completes the proof of Kelvin's theorem.

A lemma on the behaviour of a harmonic function at infinity follows from Kelvin's theorem:

A function u that is harmonic in an infinite region satisfies the inequalities

$$|u(x)| < \frac{A}{|x|}, \quad \left|\frac{\partial u}{\partial x_i}\right| < \frac{A}{|x|^2} \quad (i = 1, 2, 3, \ |x| = \sqrt{x_1^2 + x_2^2 + x_3^2} > r_0) , \tag{24}$$

where A and r_0 are properly chosen constants.

Proof: Suppose that ξ is a point that is a harmonic conjugate of the point x. The function $v(\xi) = |x|u(x)$ is harmonic at the point $\xi = 0$ and in some neighbourhood $|\xi| < \epsilon$ (on the basis of Kelvin's theorem); it is therefore bounded there. This implies the first of the inequalities (24) for $r_0 = 1/\epsilon$ and for some value $A > A_0$, where A_0 is the maximum of the function $|v(\xi)|$ for $|\xi| < \epsilon$. Furthermore, noting that, for $a = 1$, the formula

$$\frac{\partial}{\partial x_i} = \sum_{\alpha=1}^{3} \frac{\partial \xi_\alpha}{\partial x_i} \frac{\partial}{\partial \xi_\alpha} = \frac{1}{|x|^2} \frac{\partial}{\partial \xi_i} - \frac{2x_i}{|x|^3} \sum_{\alpha=1}^{3} \frac{x_\alpha}{|x|} \frac{\partial}{\partial \xi_\alpha}$$

follows from eq. (20), we obtain by direct differentiation

$$\frac{\partial u}{\partial x_i} = \frac{1}{|x|^2}\frac{\partial}{\partial \xi_i}|\xi|v(\xi) - \frac{2x_i}{|x|^3}\sum_{\alpha=1}^{3}\frac{x_\alpha}{|x|}\frac{\partial}{\partial \xi_\alpha}|\xi|v(\xi)$$

$$= \frac{1}{|x|^3}\frac{\partial v}{\partial \xi_i} + \frac{1}{|x|^2}\frac{\xi_i}{|\xi|}v - \frac{2x_i}{|x|^3}\sum_{\alpha=1}^{3}\frac{x_\alpha}{|x|}\left[\frac{1}{|x|}\frac{\partial v}{\partial \xi_\alpha} + \frac{\xi_\alpha}{|\xi|}v\right].$$

Since the ratios $x_j/|x|$ and $\xi_j/|\xi|$ (where $j = 1,2,3$) and also (in the neighbourhood $|\xi| < \epsilon$) the functions v and $\partial v/\partial \xi_j$ are bounded, there exists a positive number A_i such that

$$\left|\frac{\partial u}{\partial x_i}\right| < \frac{A_i}{|x|^2}.$$

If, for A, we choose the greatest of the numbers A_0, A_j (where $j = 1,2,3$), we obtain all the inequalities (24).

Problems

1. Show that if a point x is a harmonic conjugate of a point ξ, then the point ξ is a harmonic conjugate of the point x, that is, that the property of conjugacy is mutual.
2. Show that Kelvin's theorem remains valid upon inversion of the general form
$$\xi_i = y_i + \frac{a^2(x_i - y_i)}{|x-y|^2} \qquad (i = 1,2,3),$$
where y is an arbitrary fixed point.
3. Show that inversion represents a conformal mapping of space, that is, that angles between curves are preserved upon inversion.
Method: Examine the mapping of elements of arc length.
4. Generalize the extreme-value theorem to infinite regions.

4. *Uniqueness of the solutions to boundary-value problems*

Let us prove the uniqueness of the solution to Dirichlet's problem for the Laplace and Poisson equations. Let us suppose that Dirichlet's problem

$$\begin{aligned} \Delta u &= f \quad \text{when} \quad x \in V - \mathcal{F}V, \\ u &= \psi \quad \text{when} \quad x \in \mathcal{F}V \end{aligned} \tag{25}$$

has two distinct solutions u_1 and u_2. Then, the difference $w = u_1 - u_2$ is harmonic in the region V and vanishes on its boundary.

If the region V is bounded, we may immediately apply the extreme-value theorem. Inside the region V, the harmonic function w cannot have values either greater or less than its boundary value, which is zero. There-

fore, it is equal to zero throughout the interior of the region; that is, the functions u_1 and u_2 coincide within the region in question. If the region V is infinite, we use Kelvin's theorem, setting

$$w^*(\xi) \equiv |x|\, w(x) \ ,$$

where ξ is the point with coordinates

$$\xi_i \equiv x_i/|x| \ .$$

The function $w^*(\xi)$ is harmonic in the bounded region V' (conjugate to the region V) and vanishes on its boundary because of the boundary condition for the function w. Consequently, on the basis of the above, $w^*(\xi)$ is equal to zero, and hence the function

$$w(x) = |\xi|\, w^*(\xi) \ .$$

is also equal to zero. This completes the proof.

It is also easy to show that the solution to the Dirichlet problem in question depends continuously on the boundary condition. Suppose that u_1 and u_2 are solutions to two Dirichlet problems for a single region, and that the boundary values of these solutions differ by not more than an amount ϵ. Then, the function w, identically equal to $u_1 - u_2$, is harmonic and differs from zero by no more than ϵ at points of the boundary of the region. If the region V is bounded, then, on the basis of the extreme-value theorem, the function $u_1 - u_2$ cannot differ from zero by more than ϵ at an arbitrary point within the region. Consequently, throughout the entire region,

$$|u_1 - u_2| \leq \epsilon \ ,$$

from which the above assertion follows. If the region V is infinite but the point $|x| = 0$ does not belong to the region, then, by using Kelvin's theorem, we obtain the function

$$w^*(\xi) \equiv |x|\, w(x) \ ,$$

which is harmonic in the bounded region V' conjugate to the region V. The boundary values of the function w^* do not exceed $A\epsilon$, where A is the maximum value of the quantity $|x|$ on the boundary $\mathcal{F}V$. Consequently, from what we have shown, $w(x) < (A/B)\epsilon$ when ξ is an element of V'. Hence, $w^*(\xi) < A\epsilon$, where B is the smallest value of the quantity $|x|$ on the boundary $\mathcal{F}V$. Thus, the assertion is proven. When the point $|x| = 0$ belongs to the region V, we translate the coordinate origin before applying Kelvin's theorem, after which we again obtain the required result.

To examine Neumann's problem and the mixed problem, let us turn to Green's theorem (7) of Chapter XVII. By using the identity

$$v\left(\frac{du}{dn} + \beta u\right) - u\left(\frac{dv}{dn} + \beta v\right) \equiv v\,\frac{du}{dn} - u\,\frac{dv}{dn} \ , \tag{26}$$

where β is an arbitrary continuous function, and by using the notation

$$\mathcal{P} \equiv \frac{d}{dn} + \beta \tag{27}$$

for brevity in writing, we obtain Green's theorem in the form

$$\iint\limits_{\mathcal{F}V} (v\mathcal{P}u - u\mathcal{P}v)\, dS = \iiint\limits_{V} (v\,\Delta u - u\,\Delta v)\, dV\,, \tag{28}$$

where V is a bounded region. Setting one of the functions appearing in this formula equal to unity and the other equal to the square of the harmonic function w, we arrive at *Dirichlet's formula*

$$\iiint\limits_{V} \left[\left(\frac{\partial w}{\partial x_1}\right)^2 + \left(\frac{\partial w}{\partial x_2}\right)^2 + \left(\frac{\partial w}{\partial x_3}\right)^2\right] dV = \iint\limits_{\mathcal{F}V} w\mathcal{P}w\, dS - \tfrac{1}{2} \iint\limits_{\mathcal{F}V} \beta w^2\, dS\,. \tag{29}$$

Let us use this formula to establish the conditions for uniqueness of the solutions to the interior mixed problem and the interior Neumann problem for the Laplace and Poisson equations.

With the notation (27), both these problems can be written in the same form:

$$\begin{aligned} \Delta u &= f \quad \text{when} \quad x \in V - \mathcal{F}V\,, \\ \mathcal{P}u &= \psi \quad \text{when} \quad x \in \mathcal{F}V \end{aligned} \tag{30}$$

If β is not identically equal to zero, eq. (30) corresponds to the mixed problem, and for β identically equal to zero, it refers to the Neumann problem.

Let us suppose that the problem (30) has two distinct solutions u_1 and u_2, that are continuous and have continuous first derivatives throughout the region V. Then, their difference $w = u_1 - u_2$ is a solution to the homogeneous boundary problem for the Laplace equation:

$$\Delta w = 0 \quad \text{when} \quad x \in V - \mathcal{F}V\,, \qquad \mathcal{P}w = 0 \quad \text{when} \quad x \in \mathcal{F}V\,,$$

which satisfies the same continuity conditions. Then, for non-negative β, it follows from Dirichlet's formula that

$$\iiint\limits_{V} \left[\left(\frac{\partial w}{\partial x_1}\right)^2 + \left(\frac{\partial w}{\partial x_2}\right)^2 + \left(\frac{\partial w}{\partial x_3}\right)^2\right] dV \leq 0\,.$$

Since all the terms in the integrand are non-negative, and since the expression itself is, by hypothesis, continuous, it follows that $\partial w/\partial x_i = 0$ (for $i = 1, 2, 3$); that is,

$$w = u_1 - u_2 = \text{constant}\,.$$

To determine the admissible values of the constant on the right side of this equation, let us turn to the boundary condition for the homogeneous problem that we are examining. If β is identically equal to zero (the Neumann problem), any constant satisfies the boundary condition. Consequently, any constant is a solution to the homogeneous Neumann problem, and therefore, *the solution to the homogeneous Neumann problem is determined except for an arbitrary constant*. However, if β is different from zero for at least a portion of the boundary $\mathcal{F}V$, then this constant will be equal to zero; that is, *the solution to the mixed problem is unique*.

Physical problems in which the appearance of a constant in the solution

is unimportant (if the choice of the zero value of the function u may be arbitrary) or in which this constant term can be determined from supplementary requirements on the behaviour of the function u at the boundary, are the ones that usually lead to Neumann's problem. For example, we are often interested in a solution whose average value on the boundary of the region is equal to zero. This leads to the condition

$$\iint_{\mathcal{F}V} u \, dS = 0 \,. \tag{31}$$

Such a solution is obviously unique.

Thus, supplementary conditions that yield a correct statement of Neumann's problem can be established, depending on the content of the physical problem that is being studied.

Let us turn to the exterior problems.

Suppose that V is an infinite region with a finite boundary $\mathcal{F}V$. Let us take a finite portion V^* of the region V that lies within a spherical surface Σ containing the boundary $\mathcal{F}V$. By applying Dirichlet's formula (29) to the region V^*, we obtain

$$\iiint_{V^*} \left[\left(\frac{\partial w}{\partial x_1}\right)^2 + \left(\frac{\partial w}{\partial x_2}\right)^2 + \left(\frac{\partial w}{\partial x_3}\right)^2\right] dV$$
$$= \iint_{\mathcal{F}V} w\mathcal{P}w \, dS - \tfrac{1}{2} \iint_{\mathcal{F}V} \beta w^2 \, dS + \iint_{\Sigma} \left(w \frac{dw}{dn} + \tfrac{1}{2}\beta w^2\right) dS \,.$$

Let us increase the radius of the surface Σ without bound. On the basis of the lemma on the behaviour of a harmonic function at infinity, the terms $(\partial w/\partial x_i)^2$ decrease (in the neighbourhood of an infinitely distant point) at least as rapidly as $1/r^4$, whereas, with increase in the surface Σ, the volume of the region V^* increases only as r^3. Consequently, the integral over V^* is then reduced to an improper integral over the volume V. The integral

$$\iint_{\Sigma} w \frac{\partial w}{\partial n} \, dS$$

approaches zero with increase in the radius of the surface Σ since, on the basis of the same lemma, the expression $w\partial w/\partial n$ then decreases on Σ as $1/r^3$, whereas the area of the surface Σ increases only as r^2. Since we are interested in the values of the function β only on the boundary $\mathcal{F}V$, we choose β so that, with increasing Σ, the integral

$$\iint_{\Sigma} \beta w^2 \, dS$$

will tend to zero. When we take the limit, we arrive at *Dirichlet's formula for an infinite region:*

$$\iiint_{V} \left[\left(\frac{\partial w}{\partial x_1}\right)^2 + \left(\frac{\partial w}{\partial x_2}\right)^2 + \left(\frac{\partial w}{\partial x_3}\right)^2\right] dV = \iint_{\mathcal{F}V} w\mathcal{P}w \, dS - \tfrac{1}{2} \iint_{\mathcal{F}V} \beta w^2 \, dS \,. \tag{32}$$

This formula coincides exactly with formula (29). Therefore, from the same considerations as were made above, we conclude that for non-negative β the difference $w = u_1 - u_2$ of the two solutions u_1 and u_2 of the exterior problem

$$\begin{aligned} \Delta u &= f \quad \text{when} \quad x \in V - \mathcal{F}V , \\ \mathcal{P}u \equiv \frac{du}{dn} + \beta u &= \psi \quad \text{when} \quad x \in \mathcal{F}V , \end{aligned} \tag{33}$$

satisfies the relations $\partial w/\partial x_i = 0$ (for $i = 1, 2, 3$); it then follows that $w =$ constant. The constant on the right side of the last relationship must be equal to zero both for the exterior mixed problem and for the exterior Neumann problem since, at an infinitely distant point, all harmonic functions have the same value, namely zero. Thus, for non-negative β, the regular solution to the exterior problem (33) is unique.

Let us turn now to the question of the conditions for the existence of solutions to Neumann's problem for the Laplace equation. Setting $v = 1$ and $\Delta u = 0$ in Green's theorem (7) of Chapter XVII, we obtain

$$\iint_S \frac{du}{dn} \, dS = 0 , \tag{34}$$

where S is an arbitrary surface forming the boundary of a finite region in which the function u is harmonic. It then follows that the boundary condition

$$du/dn = \psi \quad \text{when} \quad x \in \mathcal{F}V ,$$

for the interior Neumann problem for the Laplace equation cannot be given arbitrarily, but must satisfy the condition

$$\iint_{\mathcal{F}V} \psi \, dS = 0 . \tag{35}$$

This result admits a simple interpretation. For example, let us examine a temperature field. According to the Fourier principle, the amount of heat flowing through an element of surface dS is proportional to the product $(du/dn)dS$, where du/dn is the derivative of the temperature u in the direction of the normal to the element dS. If the temperature field does not change with time, then the total amount of heat flowing through an arbitrary closed surface contained within the limits of the body in question is equal to zero. Thus, eq. (34) or eq. (35) represents the condition of a steady-state field.

Let us note that the property expressed by eq. (34) is peculiar only to harmonic functions (see problem 1).

However, condition (35) does not extend to the external Neumann problem. To show this, we again introduce the region V^* which we examined in the derivation of Dirichlet's formula (32). Applying formula (35) to the region V^*, we obtain

$$\iint_{\mathcal{F}V} \frac{du}{dn} \, dS = - \iint_{\Sigma} \frac{du}{dn} \, dS .$$

When the radius of the surface Σ increases without bound, the integral over Σ cannot approach zero because the proof of the lemma on the behaviour of harmonic function at infinity implies that the integrand can decrease only as $1/r^2$; that is, the integral over Σ cannot vanish with increase in the surface Σ. Consequently, formula (34) and, with it, formula (35), do not become harmonic functions in an infinite region.

Recalling the interpretation of formula (34), we see that within the framework of this interpretation an interaction between a medium in an *infinite* region and the exterior space must be considered as taking place not only on the boundary $\mathcal{F}V$ of the region but also at an *infinitely distant point*. As a consequence, equilibrium on the boundary $\mathcal{F}V$ may not be attained.

Problems

1. Suppose that u is a function with continuous derivatives of the first two orders throughout a region V and that

$$\iint_S \frac{\mathrm{d}u}{\mathrm{d}n}\,\mathrm{d}S = 0$$

for *any* choice of closed surface S that does not extend beyond the boundaries of V. Show that the function u is harmonic in the region V.
Method: Use Green's theorem, setting $v = 1$.

2. Prove the extreme-value theorem by means of formula (34).
Method: Use the fact that on a spherical surface of sufficiently small radius surrounding the point of maximum or minimum value of the function u, the derivative $\mathrm{d}u/\mathrm{d}n$ does not change sign.

3. By using Dirichlet's integral formula, show the uniqueness of the solution to the Dirichlet problem.

4. By applying the extreme-value theorem to a sphere of infinite radius, prove Liouville's theorem (that is, that a function that is harmonic and bounded throughout all space is identically equal to a constant).

5. By using Kelvin's theorem, show that the exterior Dirichlet problem can be reduced to the interior problem.

5. *The fundamental solutions to Laplace's equation. The basic formula in the theory of harmonic functions*

As we saw in section 1, the function

$$\frac{1}{r} = \frac{1}{\sqrt{\sum_{\alpha=1}^{3} (\xi_\alpha - x_\alpha)^2}}, \tag{36}$$

where ξ_j and x_j (for $j = 1, 2, 3$) are the coordinates of two points ξ and x,

satisfies Laplace's equation for $\xi \neq x$. Since the expression $1/r$ is symmetric with respect to the coordinates of the points ξ and x, this expression is valid under differentiation with respect to the coordinates both of the point ξ and of the point x. At $\xi = x$, the function $1/r$ has an infinite discontinuity.

If a function $\varphi(\xi, x)$ is harmonic in a region V with respect to the coordinates of the point ξ and if it and its first derivatives are continuous, we shall call the function

$$L(\xi, x) = \frac{1}{4\pi}\left[\frac{1}{r} + \varphi(\xi, x)\right] \tag{37}$$

the *fundamental* solution to Laplace's equation in the region V.

By using the properties of fundamental solutions, we can derive some important integral formulae relating the value of an arbitrary sufficiently smooth function at an arbitrary point (within or on the boundary of the domain of its definition) with the set of values of this function and its normal derivatives on the boundary of the region being considered.

Let us first examine bounded regions. Suppose that V is such a region. When a point x lies outside the region V, the fundamental solution $L(\xi, x)$ is harmonic in this region. Therefore, by setting

$$v(\xi) = L(\xi, x)$$

in Green's theorem (7), Chapter XVII, we obtain

$$\iint\limits_{\mathcal{F}V} \left(L\frac{\mathrm{d}u}{\mathrm{d}n} - u\frac{\mathrm{d}L}{\mathrm{d}n}\right) \mathrm{d}S_\xi = \iiint\limits_V L\,\Delta u\,\mathrm{d}V_\xi\,, \qquad x \in R_E - V\,, \tag{38}$$

where R_E denotes *all space* and the point x is treated as a parameter. When the point x lies within the region V, we may apply Green's theorem to the region $V - \Omega_\epsilon$, where Ω_ϵ is a sphere within the region V of arbitrarily small radius ϵ with center at the point x. Here, instead of the relationship (38), we obtain

$$\iint\limits_{\mathcal{F}V} \left(L\frac{\mathrm{d}u}{\mathrm{d}n} - u\frac{\mathrm{d}L}{\mathrm{d}n}\right) \mathrm{d}S_\xi = \iiint\limits_{V-\Omega_\epsilon} L\,\Delta u\,\mathrm{d}V_\xi - \iint\limits_{\mathcal{F}\Omega_\epsilon} L\frac{\mathrm{d}u}{\mathrm{d}n}\,\mathrm{d}S_\xi + \iint\limits_{\mathcal{F}\Omega_\epsilon} u\frac{\mathrm{d}L}{\mathrm{d}n}\,\mathrm{d}S_\xi\,.$$

As ϵ approaches zero, the integral

$$\iiint\limits_{V-\Omega_\epsilon} L\,\Delta u\,\mathrm{d}V_\xi$$

approaches the improper integral

$$\iiint\limits_V L\,\Delta u\,\mathrm{d}V_\xi\,,$$

if this improper integral exists. The integral

$$\iint\limits_{\mathcal{F}\Omega_\epsilon} L\frac{\mathrm{d}u}{\mathrm{d}n}\,\mathrm{d}S_\xi$$

approaches zero because the derivative du/dn is continuous (by the assumption made in the derivation of Green's theorem) and therefore bounded, and the function $L(\xi, x)$ increases on $\mathcal{F}\Omega_\epsilon$ as $1/\epsilon$, whereas the area of the surface $\mathcal{F}\Omega_\epsilon$ decreases as ϵ^2.

Let us examine the behaviour of the integrals of $u\,dL/dn$. From eq. (37),

$$\iint_{\mathcal{F}\Omega_\epsilon} u \frac{dL}{dn}\, dS_\xi = \frac{1}{4\pi} \iint_{\mathcal{F}\Omega_\epsilon} u \frac{d\varphi}{dn}\, dS_\xi + \frac{1}{4\pi} \iint_{\mathcal{F}\Omega_\epsilon} u \frac{d}{dn}\left(\frac{1}{r}\right) dS_\xi \,.$$

The first of the integrals on the right side vanishes as ϵ approaches zero because the integrand is bounded. By using the fact that

$$\mathcal{F}\Omega_\epsilon \frac{d}{dn} = -\frac{d}{dr}$$

on the surface of the sphere, we can transform the second integrand, since the outward normal to the boundary of the region $V - \Omega_\epsilon$ is directed along the radius r within the sphere Ω_ϵ. This yields

$$\frac{1}{4\pi} \iint_{\mathcal{F}\Omega_\epsilon} u \frac{d}{dn}\left(\frac{1}{r}\right) dS_\xi = \frac{1}{4\pi} \iint_{\mathcal{F}\Omega_\epsilon} \frac{u}{r^2}\, dS_\xi = \frac{1}{4\pi\epsilon^2} \iint_{\mathcal{F}\Omega_\epsilon} u\, dS_\xi \,.$$

By the mean-value theorem,

$$\iint_{\mathcal{F}\Omega_\epsilon} u\, dS_\xi = u_{\mathrm{av}} \iint_{\mathcal{F}\Omega_\epsilon} dS_\xi \,,$$

where u_{av} is the value of the function u at some point belonging to the sphere Ω_ϵ. Noting that the integral

$$\iint_{\mathcal{F}\Omega_\epsilon} dS_\xi$$

is equal to the area $4\pi\epsilon^2$ of the surface $\mathcal{F}\Omega_\epsilon$ and that, as ϵ approaches zero, the value of u_{av} approaches $u(x)$ since the function u is continuous, we obtain

$$\lim_{\epsilon\to 0} \iint_{\mathcal{F}\Omega_\epsilon} u \frac{dL}{dn}\, dS_\xi = \lim_{\epsilon\to 0} \frac{u_{\mathrm{av}}}{4\pi\epsilon^2} \iint_{\mathcal{F}\Omega_\epsilon} dS_\xi = \lim_{\epsilon\to 0} u_{\mathrm{av}} = u(x) \,. \tag{39}$$

Using the values of these limits, we finally obtain

$$\iint_{\mathcal{F}V} \left(L \frac{du}{dn} - u \frac{dL}{dn}\right) dS_\xi = \iiint_V L\, \Delta u\, dV_\xi + u(x) \qquad (x \in V - \mathcal{F}V) \,. \tag{40}$$

Let us assume, finally, that the point x is located on the boundary of the surface $\mathcal{F}V$. By applying Green's theorem to the region $V - \Omega'_\epsilon$, where Ω'_ϵ is that portion of the sphere Ω_ϵ (of small radius ϵ with center at x) that lies in the region V, we obtain

$$\iint_{\mathcal{F}V - \omega_\epsilon} \left(L \frac{du}{dn} - u \frac{dL}{dn}\right) dS_\xi = \iiint_{V - \Omega'_\epsilon} L\, \Delta u\, dV - \iint_{\omega'_\epsilon} L \frac{du}{dn}\, dS_\epsilon + \iint_{\omega'_\epsilon} u \frac{dL}{dn}\, dS_\xi \,,$$

where ω_ϵ is that part of the bounding surface $\mathcal{F}V$ lying in the sphere Ω_ϵ and ω_ϵ' is that part of the surface of the sphere Ω_ϵ lying in the region V. As ϵ approaches zero, the integral on the left side of this equation approaches an improper integral over $\mathcal{F}V$. For its value, we take the integral on the right side, which we compute by using the reasoning of the preceding case, with the exception that now we have in formula (39), instead of the integral

$$\iint\limits_{\mathcal{F}\Omega_\epsilon} dS_\xi$$

the integral

$$\iint\limits_{\omega_\epsilon'} dS_\xi ,$$

which is equal to the area of that portion of the surface of the sphere Ω_ϵ that lies in the region V.

Let us introduce at the point x a local Cartesian coordinate system ζ_1, ζ_2, ζ_3, with axis 3 directed along the outward normal to the surface $\mathcal{F}V$ at the point x. By hypothesis (Chapter XVII, section 1), within some sphere with center at the point x the equation of the surface $\mathcal{F}V$ can be written in the form

$$\zeta_3 = f(\zeta_1, \zeta_2) ,$$

where the function f and its first-order derivatives are continuous and vanish at the point x. Therefore, from the definition of a differentiable function, throughout some neighbourhood of the point x,

$$\zeta_3 = h_1\zeta_1 + h_2\zeta_2 ,$$

where the quantities h_1 and h_2 vanish simultaneously with ζ_1, ζ_2. Let us introduce spherical coordinates r, θ, φ, by setting

$$\zeta_1 = r \sin\theta \cos\varphi , \qquad \zeta_2 = r\sin\theta\sin\varphi , \qquad \zeta_3 = r\cos\theta .$$

Substituting these expressions in the relations that we have found, we obtain

$$\cos\theta = h_1 \sin\theta\cos\varphi + h_2 \sin\theta \sin\varphi = h(r, \theta, \varphi) , \tag{41}$$

where h is a function that is bounded and that vanishes simultaneously with r, and where θ is the angular coordinate of the point on the surface $\mathcal{F}V$. By using this expression, we arrive at the following evaluation of the integral:

$$\begin{aligned} \frac{1}{4\pi\epsilon^2} \iint\limits_{\omega_\epsilon'} dS_\xi &= \frac{1}{4\pi\epsilon^2} \iint\limits_{\omega_\epsilon'} r^2 \sin\theta'\, d\theta'\, d\varphi' = \frac{1}{4\pi} \int_0^{2\pi} d\varphi' \int_0^{\theta} \sin\theta'\, d\theta' \\ &= \frac{1}{4\pi}\int_0^{2\pi} d\varphi' \int_0^{\frac{1}{2}\pi} \sin\theta'\, d\theta' + \frac{1}{4\pi}\int_0^{2\pi} d\varphi' \int_{\frac{1}{2}\pi}^{\theta} \sin\theta'\, d\theta' \\ &= \frac{1}{2} + \frac{1}{4\pi}\int_0^{2\pi} d\varphi' \left[-\cos\theta'\right]_{\frac{1}{2}\pi}^{\theta} = \frac{1}{2} + \frac{1}{4\pi}\int_0^{2\pi} h(\epsilon, \theta, \varphi')\, d\varphi' \\ &= \tfrac{1}{2} + H(\epsilon) , \end{aligned}$$

where

$$H(\epsilon) \equiv \frac{1}{4\pi}\int_0^{2\pi} h(\epsilon, \theta, \varphi')\, d\varphi'$$

is a bounded function that vanishes simultaneously with ϵ. Therefore,

$$\lim_{\epsilon\to 0} \iint_{\omega_\epsilon'} u \frac{dL}{dn}\, dS_\xi = \lim_{\epsilon\to 0} \frac{u_{av}}{4\pi\epsilon^2} \iint_{\omega_\epsilon'} dS_\xi = \lim_{\epsilon\to 0} u_{av}\,[\tfrac{1}{2} + H(\epsilon)] = \tfrac{1}{2}u(x)\ ,$$

which leads us to the relation

$$\iint_{\mathcal{F}V} \left(L\frac{du}{dn} - u\frac{dL}{dn}\right) dS_\xi = \iiint_V L\,\Delta u\, dV_\xi + \tfrac{1}{2}u(x) \qquad (x \in \mathcal{F}V)\ . \tag{42}$$

By combining formulae (38), (40), and (42), we may write

$$\iint_{\mathcal{F}V} \left(L\frac{du}{dn} - u\frac{dL}{dn}\right) dS_\xi = \iiint_V L\,\Delta u\, dV + \begin{cases} 0 & \text{when} \quad x \in R_E - V\ , \\ \tfrac{1}{2}u(x) & \text{when} \quad x \in \mathcal{F}V\ , \\ u(x) & \text{when} \quad x \in V - \mathcal{F}V\ . \end{cases} \tag{43}$$

If the function u is harmonic in the region V, then formula (43) takes the form

$$\iint_{\mathcal{F}V} \left(L\frac{du}{dn} - u\frac{dL}{dn}\right) dS_\xi = \begin{cases} 0 & \text{when} \quad x \in R_E - V\ , \\ \tfrac{1}{2}u(x) & \text{when} \quad x \in \mathcal{F}V\ , \\ u(x) & \text{when} \quad x \in V - \mathcal{F}V\ . \end{cases} \tag{44}$$

This relationship is called the *basic formula in the theory of harmonic functions*.

It can be extended to infinite regions. Suppose that V is an infinite region with a finite boundary $\mathcal{F}V$, and that V^* is that portion of the region V lying within a sphere Ω of finite radius r containing the boundary $\mathcal{F}V$. By applying formula (44) to the region V^*, we arrive at a formula whose left side will differ from that of formula (44) only in that the integral

$$\iint_{\mathcal{F}\Omega} \left(L\frac{du}{dn} - u\frac{dL}{dn}\right) dS_\xi$$

is added.

With unbounded increase in the radius of the sphere, this integral approaches zero because (on the basis of the lemma in section 3 on the behaviour of a harmonic function at infinity and from the definition of a fundamental solution $L(\xi,x)$), the integrand here decreases as $1/r^3$ whereas the area of the surface $\mathcal{F}V$ of the sphere Ω increases only as r^2. Taking the limit as r approaches ∞, we again obtain the formula

$$\iint_{\mathcal{F}V} \left(L\frac{du}{dn} - u\frac{dL}{dn}\right) dS_\xi = \begin{cases} 0 & \text{when} \quad x \in R_E - V\ , \\ \tfrac{1}{2}u(x) & \text{when} \quad x \in \mathcal{F}V\ , \\ u(x) & \text{when} \quad x \in V - \mathcal{F}V\ , \end{cases} \tag{45}$$

which coincides with formula (44) for bounded regions.

By using formulae (44) and (45), we shall show that, within the region where it is harmonic, an arbitrary harmonic function is differentiable infinitely many times. Let us set

$$L(\xi, x) = \frac{1}{4\pi}\frac{1}{r}.$$

The fundamental solution $(1/4\pi)(1/r)$ in an arbitrary region not containing the point $\xi = x$ is differentiable infinitely many times with respect to the coordinates of the point x, and each time, the result of the differentiation is a bounded function of the variable ξ. If x is an interior point of the region V, then ξ will be different from x when ξ is an element of $\mathcal{F}V$. Consequently, the integrals (44) and (45) can be differentiated with respect to the coordinates of the point x (treated as parameters) infinitely many times. This proves the assertion for the case when the harmonic function u and its first derivatives are continuous in the region V. If the first derivatives are not continuous, this assertion remains valid because, in formulae (44) and (45), we may change from integration over the surface $\mathcal{F}V$ to integration over the surface S, which lies entirely within the region V and which contains the point x. Since, within the region where it is harmonic, every harmonic function is differentiable twice, the formula containing the integral over the surface S will be meaningful and, consequently, it again implies differentiability infinitely many times of the function $u(x)$.

Suppose that Ω is a sphere of radius a with center at the point x and that Ω lies entirely within the region where the function u is harmonic. On the surface of the sphere Ω,

$$\mathrm{d}/\mathrm{d}n = \mathrm{d}/\mathrm{d}r .$$

As before, we set

$$L(\xi, x) = \frac{1}{4\pi}\frac{1}{r}.$$

Then, on the basis of eq. (34), formula (44) takes the form

$$\frac{1}{4\pi a^2}\iint_{\mathcal{F}\Omega} u \,\mathrm{d}S_\xi = u(x) ; \tag{46}$$

that is, the mean arithmetic value of a harmonic function on the surface of a sphere is equal to its value at the center of the sphere. This assertion is known as the *mean-value theorem* for harmonic functions.

Problems

1. By using formula (46), show that harmonic functions within the domains of their definitions are not only differentiable infinitely many times, but are also analytic.

2. Show that a function satisfying condition (46) is harmonic.

6. *Poisson's formula. The solution to Dirichlet's problem for a sphere*

Suppose that ζ is an arbitrary variable point, that u is a function which is harmonic in the sphere Ω defined by the equation

$$|\zeta| \leqslant 1 ,$$

that x is a point within the sphere Ω, and that ξ is the point harmonically conjugate to the point x (see section 3). We introduce the notations

$$r_0 \equiv |x| , \quad r \equiv |x - \zeta| , \quad r^* \equiv |\xi - \zeta| .$$

The functions $1/4\pi r$ and $1/4\pi r^*$ are the fundamental solutions to Laplace's equation with singular points inside and outside the sphere Ω respectively. Consequently, when we apply the basic formula (44), we obtain

$$\frac{1}{4\pi} \iint\limits_{|\zeta|=1} \left[\frac{1}{r^*} \frac{\mathrm{d}u}{\mathrm{d}n} - u \frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r^*}\right)\right] \mathrm{d}S_\zeta = 0 , \tag{47}$$

$$\frac{1}{4\pi} \iint\limits_{|\zeta|=1} \left[\frac{1}{r} \frac{\mathrm{d}u}{\mathrm{d}n} - u \frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r}\right)\right] \mathrm{d}S_\zeta = u(x) . \tag{48}$$

Recalling that

$$\xi_j = x_j/r_0^2$$

(for $j = 1, 2, 3$) and that

$$\sum_{\alpha=1}^{3} \zeta_\alpha^2 = 1$$

for $\zeta \in \mathcal{F}\Omega$, we obtain (for $\zeta \in \mathcal{F}\Omega$)

$$r^* = \sqrt{\sum_{\alpha=1}^{3} \left(\frac{x_\alpha}{r_0^2} - \zeta_\alpha\right)^2} = \sqrt{\frac{1}{r_0^2} - 2\sum_{\alpha=1}^{3} \frac{x_\alpha \zeta_\alpha}{r_0^2} + 1} = \frac{1}{r_0}\sqrt{\sum_{\alpha=1}^{3} (x_\alpha - \zeta_\alpha)^2} = \frac{r}{r_0} ,$$

or

$$r^* = r/r_0 \quad \text{when} \quad \zeta \in \mathcal{F}\Omega . \tag{49}$$

Multiplying eq. (47) by the quantity $-1/r_0$ and adding it to eq. (48), by eq. (49), we obtain

$$u(x) = \frac{1}{4\pi} \iint\limits_{|\zeta|=1} u \frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r_0 r^*} - \frac{1}{r}\right) \mathrm{d}S_\zeta . \tag{50}$$

Since the radius of the spherical surface $\mathcal{F}\Omega$ is equal to unity, the coordinates ζ_j (for $j = 1, 2, 3$) of the point ζ are numerically equal to the direction cosines of the outward normal to the surface $\mathcal{F}\Omega$ at the point ζ. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}n} = \sum_{\alpha=1}^{3} \zeta_\alpha \frac{\partial}{\partial \zeta_\alpha} .$$

From eq. (49), we obtain

$$\frac{d}{dn}\left(\frac{1}{r}\right) = \sum_{\alpha=1}^{3} \zeta_\alpha \frac{\partial}{\partial \zeta_\alpha}\left(\frac{1}{r}\right) = -\frac{1}{r^2} \sum_{\alpha=1}^{3} \zeta_\alpha \frac{\partial r}{\partial \zeta_\alpha}$$

$$= \frac{1}{r^2} \sum_{\alpha=1}^{3} \frac{\zeta_\alpha(x_\alpha - \zeta_\alpha)}{r} = \frac{1}{r^3} \sum_{\alpha=1}^{3} \zeta_\alpha x_\alpha - \frac{1}{r^3} \qquad ;$$

$$\frac{d}{dn}\left(\frac{1}{r^*}\right) = \sum_{\alpha=1}^{3} \zeta_\alpha \frac{\partial}{\partial \zeta_\alpha}\left(\frac{1}{r^*}\right) = \frac{1}{r^{*3}} \sum_{\alpha=1}^{3} \xi_\alpha \zeta_\alpha - \frac{1}{r^{*3}}$$

$$= \frac{1}{r^{*3} r_0^2} \sum_{\alpha=1}^{3} \zeta_\alpha x_\alpha - \frac{1}{r^{*3}} = \frac{r_0}{r^3} \sum_{\alpha=1}^{3} \zeta_\alpha x_\alpha - \frac{r_0^3}{r^3} ,$$

so that

$$\frac{d}{dn}\left(\frac{1}{r_0 r^*} - \frac{1}{r}\right) = \frac{1}{r_0} \frac{d}{dn}\left(\frac{1}{r^*}\right) - \frac{d}{dn}\left(\frac{1}{r}\right) = -\frac{r_0^2}{r^3} + \frac{1}{r^3} = \frac{1 - r_0^2}{r^3} .$$

Substituting this equation into eq. (50), we obtain *Poisson's formula*

$$u(x) = \frac{1}{4\pi} \iint_{|\zeta|=1} u \, \frac{1 - r_0^2}{r^3} \, dS_\zeta , \tag{51}$$

thus determining the value of the harmonic function u at points within the sphere $|x| \leq 1$ from the values of this function on the surface of the sphere.

If we substitute an arbitrary continuous function $\psi(\zeta)$ of the point ζ on the surface of the sphere $|x| \leq 1$ for u in Poisson's formula, we obtain some function

$$u(x) = \frac{1}{4\pi} \iint_{|\zeta|=1} \psi \, \frac{1 - r_0^2}{r^3} \, dS_\zeta . \tag{52}$$

Let us show that this function is a solution to Dirichlet's problem:

$$\begin{aligned} \Delta u &= 0 \quad \text{when} \quad |x| < 1 , \\ u &= \psi \quad \text{when} \quad |x| = 1 . \end{aligned} \tag{53}$$

We shall divide the proof into two stages. We shall first show that within the sphere $|x| \leq 1$. the function u is harmonic, and then we shall show that as $|x|$ approaches unity, the function u approaches ψ.

Consider the integrand

$$\psi \, \frac{1 - r_0^2}{r^3} \equiv \psi(\zeta) \frac{1 - x_1^2 - x_2^2 - x_3^2}{[(x_1 - \zeta_1)^2 + (x_2 - \zeta_2)^2 + (x_3 - \zeta_3)^2]^{\frac{3}{2}}} , \qquad |\zeta| = 1 . \tag{54}$$

If the point x lies within the sphere, it is continuous and bounded when the absolute value of $\zeta = 1$. Therefore, for $|x| < 1$, we may change the order of

integration with respect to ζ and differentiation with respect to the coordinates of the point x. Since the integrand, being a function of the point x, has, for $|x| < 1$, continuous second derivatives and also satisfies Laplace's equation (this can be seen by direct substitution into the equation), it follows that for $|x| < 1$ the integral (52) is a harmonic function.

Let us now show that on the surface $|\zeta| = 1$ the integral (52) assumes the same values as does the function ψ.

Let us consider some finite region containing the surface $|\zeta| = 1$. In this region and on its boundary, the function $u \equiv 1$ is harmonic. Therefore, we may apply Poisson's formula to it. This yields

$$\frac{1}{4\pi} \iint\limits_{|\zeta|=1} \frac{1 - r_0^2}{r^3} \, dS_\zeta = 1 .$$

Let us take the difference

$$u(x) - \psi(y) = \frac{1}{4\pi} \iint\limits_{|\zeta|=1} \frac{1 - r_0^2}{r^3} [\psi(\zeta) - \psi(y)] \, dS_\zeta ,$$

where y is an arbitrary point on the surface $|\zeta| = 1$. On the surface Σ defined by the equation $|\zeta| = 1$, let us take a small portion σ that lies within a sphere of radius η with center at the point y, and let us examine the integrals

$$J_1 = \frac{1}{4\pi} \iint\limits_{\sigma} \frac{1 - r_0^2}{r^3} [\psi(\zeta) - \psi(y)] \, dS_\zeta , \tag{55}$$

$$J_2 = \frac{1}{4\pi} \iint\limits_{\Sigma-\sigma} \frac{1 - r_0^2}{r^3} [\psi(\zeta) - \psi(y)] \, dS_\zeta . \tag{56}$$

We easily see that

$$|J_1| = \frac{1}{4\pi} \left| \iint\limits_{\sigma} \frac{1 - r_0}{r^3} [\psi(\zeta) - \psi(y)] \, dS_\zeta \right| \leq \frac{M}{4\pi} \iint\limits_{\sigma} \frac{1 - r_0^2}{r^3} \, dS_\zeta$$
$$< \frac{M}{4\pi} \iint\limits_{|\zeta|=1} \frac{1 - r_0^2}{r^3} \, dS_\zeta = M ,$$

where M is the least upper bound of the difference $\psi(\zeta) - \psi(y)$ for ζ belonging to σ. From the continuity of the function ψ, the radius η can always be chosen sufficiently small so that

$$|J_1| < \tfrac{1}{2}\epsilon , \tag{57}$$

where ϵ is an arbitrary positive number. Since the function ψ is continuous, it is bounded on Σ. Therefore, there exists a number A such that $|\psi| < A$ for ζ belonging to Σ. Therefore, for the integral J_2, we have

$$|J_2| = \frac{1}{4\pi} \left| \iint\limits_{\Sigma-\sigma} \frac{1 - r_0^2}{r^3} [\psi(\zeta) - \psi(y)] \, dS_\zeta \right| \leq \frac{2A}{4\pi} \iint\limits_{\Sigma-\sigma} \frac{1 - r_0^2}{r^3} \, dS_\zeta \leq 2AM^* ,$$

where M^* is the least upper bound of the expression $(1 - r_0^2)/r^3$ on $\Sigma - \sigma$. No matter what the radius η may be, by making the point x sufficiently close to the point y, the ratio of $1 - r_0^2$ to η can be made arbitrarily small because the distance

$$r \equiv |x - \zeta|$$

for

$$\zeta \in \Sigma - \sigma$$

is of the same order as η. Therefore, for arbitrary η, by taking x sufficiently close to the point y, we may arrange for

$$|J_2| < \tfrac{1}{2}\epsilon .$$

It follows from this that if the points x and y are sufficiently close to each other,

$$|u(x) - \psi(y)| = |J_1 + J_2| \leqslant |J_1| + |J_2| < \epsilon .$$

From the fact that ϵ is arbitrary, we conclude that when the point x, remaining within the sphere $|\zeta| \leqslant 1$, approaches the point y on its surface, the function $u(x)$ approaches $\psi(y)$, as asserted.

We note that we have succeeded in constructing a solution to the interior Dirichlet problem for the sphere $|\zeta| \leqslant 1$ for an arbitrary continuous boundary condition. In so doing, we have shown the *existence* of this solution.

This result can, by a linear transformation of coordinates, be generalized to the Dirichlet problem posed for an arbitrary sphere.

Problems

1. By using Poisson's formula, prove the mean-value theorem for harmonic functions (section 5).
2. By using the mean-value theorem, prove the extreme-value theorem for harmonic functions (section 3).

7. *Green's function*

In this section, we shall examine those solutions to boundary-value problems that belong to the class of functions that are continuous and that have continuous first derivatives in the region under consideration. This will make it possible for us to use the integral formulae (43) and (44) quite extensively.

Let us examine the Dirichlet problem

$$\begin{aligned} \Delta u &= f \quad \text{when} \quad x \in V - \mathcal{F}V , \\ u &= \psi \quad \text{when} \quad x \in \mathcal{F}V , \end{aligned} \tag{58}$$

where V is a bounded region and f and ψ are continuous functions. Let us suppose that

$$G(\xi, x) \equiv \frac{1}{4\pi}\left[\frac{1}{r} + \varphi(\xi, x)\right] \quad (r \equiv |\xi - x|) \tag{59}$$

is the fundamental solution to Laplace's equation in the region V which vanishes on its boundary $\mathcal{F}V$. The function $\varphi(\xi, x)$ must then be a solution to the boundary-value problem

$$\begin{aligned} \Delta_\xi \varphi(\xi,x) &= 0 \quad \text{when} \quad \xi, x \in V - \mathcal{F}V \\ \varphi(\xi, x) &= -\frac{1}{r} \quad \text{when} \quad \xi \in \mathcal{F}V,\ x \in V - \mathcal{F}V . \end{aligned} \tag{60}$$

Substituting into eq. (43) the values of the quantities given in the boundary-value problem (58), and setting $L(\xi, x) = G(\xi, x)$, we obtain

$$u(x) = -\iint\limits_{\mathcal{F}V} \psi \frac{\mathrm{d}G}{\mathrm{d}n} \mathrm{d}S_\xi - \iiint\limits_V fG \,\mathrm{d}V \quad (x \in V - \mathcal{F}V) . \tag{61}$$

If the fundamental solution $G(\xi, x)$ and its derivative $\mathrm{d}G/\mathrm{d}n$ exist, this formula will give a solution to Dirichlet's problem (58) (in integral form) that belongs to the class of functions that we are considering. Here, a solution to Dirichlet's problem (58) in its general form for a non-homogeneous equation can be obtained by finding the function $G(\xi, x)$. To do this, we need to find the solution to the special case of Dirichlet's problem (60) for a homogeneous equation. The fundamental solution $G(\xi, x)$ is called the Green's function for the problem (58) or the Green's function of the Laplacian operator.

The last result can be immediately extended to the exterior Dirichlet problem for Laplace's equation

$$\Delta u = 0 .$$

This follows from the coincidence of formulae (44) and (45) for bounded and infinite regions. As regards the exterior Dirichlet problem for Poisson's equation, considerations analogous to those for the interior problem require generalization of formula (43) to infinite regions. This is possible for solutions to Poisson's equation that satisfy the inequalities

$$u < \frac{A}{r}, \quad \left|\frac{\partial u}{\partial x_i}\right| < \frac{A}{r^2} \quad (i = 1, 2, 3,\ r > r_0) \tag{62}$$

(where A and r_0 are finite numbers) at infinity. These inequalities are analogous to inequality (24) for harmonic functions, with the additional condition that the integral

$$\iiint\limits_V fL \,\mathrm{d}V$$

be meaningful. Indeed, to generalize formula (43), we need only follow the same reasoning as in generalizing formula (44). The inequalities (62) are called the *conditions of regularity at infinity*. Thus, the solutions to the exterior Dirichlet problem for Poisson's equation for the class that we are examining are regular at infinity if the integral

$$\iiint\limits_V fG\,\mathrm{d}V$$

is meaningful. They can be represented in the form (61) provided the corresponding Green's functions exist.

Let us turn to the mixed problem

$$\begin{aligned} \Delta u = f \quad &\text{when} \quad x \in V - \mathcal{F}V\,, \\ \frac{\mathrm{d}u}{\mathrm{d}n} + \beta u = \psi \quad &\text{when} \quad x \in \mathcal{F}V \end{aligned} \tag{63}$$

By using the identity

$$L\left(\frac{\mathrm{d}u}{\mathrm{d}n} + \beta u\right) - u\left(\frac{\mathrm{d}L}{\mathrm{d}n} + \beta L\right) \equiv L\,\frac{\mathrm{d}u}{\mathrm{d}n} - u\,\frac{\mathrm{d}L}{\mathrm{d}n}$$

and introducing for brevity the notation

$$\mathcal{P} \equiv \frac{\mathrm{d}}{\mathrm{d}n} + \beta\,,$$

let us transform formula (43) into the form

$$u(x) = \iint\limits_{\mathcal{F}V} (L\mathcal{P}u - u\mathcal{P}L)\,\mathrm{d}S_\xi - \iiint\limits_V L\,\Delta u\,\mathrm{d}V \quad \text{when} \quad x \in V - \mathcal{F}V\,. \tag{64}$$

Suppose that $G(\xi, x)$ is the fundamental solution to Laplace's equation in the region V that satisfies the boundary condition

$$\mathcal{P}_\xi G(\xi, x) = 0 \quad \text{when} \quad \xi \in \mathcal{F}V\,,\ x \in V - \mathcal{F}V\,. \tag{65}$$

For this, the functions $\varphi(\xi, x)$ must be a solution to the boundary-value problem

$$\begin{aligned} \Delta_\xi \varphi = 0 \quad &\text{when} \quad \xi, x \in V - \mathcal{F}V \\ \mathcal{P}_\xi \varphi = -\mathcal{P}_\xi \frac{1}{r} \quad &\text{when} \quad \xi \in \mathcal{F}V\,,\ x \in V - \mathcal{F}V\,. \end{aligned} \tag{66}$$

Substituting, in formula (64), the values of the quantities given in the boundary-value problem (63) and setting $L = G$, we obtain an integral representation of the solutions to problem (63):

$$u(x) = \iint\limits_{\mathcal{F}V} G\psi\,\mathrm{d}S_\xi - \iiint\limits_V fG\,\mathrm{d}V_\xi \quad \text{when} \quad x \in V - \mathcal{F}V\,. \tag{67}$$

The fundamental solution $G(\xi, x)$ is called the Green's function of problem (63).

Finally, let us turn to Neumann's problem

$$\begin{aligned} \Delta u = f \quad &\text{when} \quad x \in V - \mathcal{F}V\,, \\ \mathrm{d}u/\mathrm{d}n = \psi \quad &\text{when} \quad x \in \mathcal{F}V \end{aligned} \tag{68}$$

By reasoning as we did in the case of the mixed problem, we conclude that

the solution to Neumann's problem would be expressed by a formula coinciding with formula (67) if the function $\varphi(\xi, x)$ were a solution to the boundary-value problem

$$\begin{aligned} &\Delta_\xi \varphi = 0 && \text{when} \quad \xi, x \in V - \mathcal{F}V \\ &\frac{d\varphi}{dn_\xi} = -\frac{d}{dn_\xi}\left(\frac{1}{r}\right) && \text{when} \quad \xi \in \mathcal{F}V, \ x \in V - \mathcal{F}V \end{aligned} \tag{69}$$

But such a function φ does not exist. For if, in formula (44), we set

$$u = 1, \quad L(\xi, x) = \frac{1}{4\pi}\frac{1}{r},$$

we find that

$$\iint_{\mathcal{F}V} \frac{d\varphi}{dn}\, dS = -\iint_{\mathcal{F}V} \frac{d}{dn}\left(\frac{1}{r}\right) dS = 4\pi \neq 0 \quad \text{when} \quad x \in V - \mathcal{F}V, \tag{70}$$

whereas, from formula (34), the integral of the normal derivatives of a harmonic function over a closed surface must be equal to zero.

Since there is no solution to problem (69), there is also no fundamental solution having a normal derivative equal to zero on the boundary of a finite region. Nonetheless, there may be a fundamental solution whose normal derivative on the boundary of the region is constant and which, therefore, may play a role analogous to the role of the Green's function of the mixed problem (63). To find this solution, let us change the boundary condition of problem (69) by setting

$$\frac{d\varphi}{dn} = -\frac{4\pi}{\bar{S}} - \frac{d}{dn}\left(\frac{1}{r}\right) \quad \text{when} \quad \xi \in \mathcal{F}V, \ x \in V - \mathcal{F}V.$$

Here,

$$\bar{S} \equiv \iint_{\mathcal{F}V} dS$$

is the area of the surface $\mathcal{F}V$. It is easy to see that eq. (34) applies here and, consequently, that the function φ may exist. Using it to determine the fundamental solution

$$G(\xi, x) = \frac{1}{4\pi}\left(\frac{1}{r} + \varphi\right),$$

we find that

$$\frac{dG}{dn_\xi} = -\frac{1}{\bar{S}}.$$

Setting $L = G$ in formula (43) and substituting the values of the quantities given in problem (68), we obtain

$$u(x) = \iint_{\mathcal{F}V} G\psi\, dS_\xi + \frac{1}{\bar{S}} \iint_{\mathcal{F}V} u\, dS_\xi - \iiint_V fG\, dV_\xi,$$

when

$$x \in V - \mathcal{F}V \, .$$

The integral

$$\frac{1}{\bar{S}} \iint_{\mathcal{F}V} u \, \mathrm{d}S_\xi$$

is the average value of the unknown function u on the surface $\mathcal{F}V$, which is, generally speaking, unknown. However, as we know, the solutions to the Neumann problem are determined only up to an additive constant. By a suitable selection of this constant, we can make the average value of the solution on the surface $\mathcal{F}V$ equal to any given value. Consequently, the integral in question must be regarded as an arbitrary constant.

Thus, by finding the solution φ to the problem

$$\begin{aligned} &\Delta_\xi \varphi = 0 && \text{when} \quad x, \xi \in V - \mathcal{F}V \quad ; \\ &\frac{\mathrm{d}\varphi}{\mathrm{d}n_\xi} = -\frac{4\pi}{\bar{S}} - \frac{\mathrm{d}}{\mathrm{d}n_\xi}\left(\frac{1}{r}\right) && \text{when} \quad x \in V - \mathcal{F}V \, , \ \xi \in \mathcal{F}V \, ; \end{aligned} \tag{71}$$

$$\bar{S} \equiv \iint_{\mathcal{F}V} \mathrm{d}S$$

and determining from formula (59) the fundamental solution $G(\xi, x)$, we may use formula (67) to construct that solution to the Neumann problem (68) whose mean value on the surface $\mathcal{F}V$ is equal to zero. All other solutions to the Neumann problem can be obtained by adding an arbitrary constant to this solution.

As regards the extension of formula (67) to the exterior mixed and Neumann problems, the same considerations are applicable as in the case of formula (61): it can be immediately extended to the exterior problems for Laplace's equation, and it can be extended to Poisson's equation — under the condition of regularity of the solution and convergence of the integral

$$\iiint_V fG \, \mathrm{d}V \, .$$

The exterior Neumann problem does not have any singularity in comparison with the exterior mixed problem, since condition (34) cannot be extended to functions that are harmonic in an infinite region.

Green s function has the simple physical interpretation of a field created by *point sources*. Let us clarify this with the example of an electrical point charge. According to Coulomb's law, the potential $u(\xi)$ of a single point charge located at a point x in free space is equal to $1/4\pi r$ (in the rationalized system of units), where

$$r = |\xi - x| \, .$$

Let us suppose, however, that this charge is located inside a hollow grounded conductor. Then charges will be induced on the boundary of the

shell. The potential $\varphi/4\pi$ of the field outside the shell must be offset by the field of the point charge, since the potential of the grounded conductor is equal to zero. Consequently, the potential φ on the boundary of the shell must satisfy the boundary condition $\varphi = -1/r$. It is clear from this that the potential of the total field in the conductor $(1/r+\varphi)/4\pi$ represents the Green's function of the Dirichlet problem.

Let us consider the question of the *existence* of the Green's functions. As is clear from their physical interpretation, we should expect Green's functions to exist under extremely general conditions. In the theory of differential equations of the elliptic type, it is shown that Green's functions exist if the solutions to the corresponding boundary-value problems exist and are unique. The solutions to these problems are given by formulae (61) and (67). For further details, see section 6 of Chapter XXVII.

Formulae (61) and (67) form the basis of Green's method of solving boundary-value problems, which we shall consider below.

Problems

1. Show that Green's function of Dirichlet's problem, set up for a region V, is positive inside this region.
   Method: Make use of the facts that Green's function is positive on the surface of a sufficiently small sphere with center at the pole, and that the function vanishes on the boundary of the region V. Use the extreme-value theorem.

2. Show that a Green's function $G(\xi, x)$ which is continuous and has continuous first derivatives in a region V is symmetric in this region with respect to the points ξ and x (that is, that $G(\xi, x) = G(x, \xi)$).
   Method: Apply Green's theorem (7) of Chapter XVII to the region
   $$V - \Omega_\epsilon(\xi) - \Omega_\epsilon(x) ,$$
   where $\Omega_\epsilon(\xi)$ and $\Omega_\epsilon(x)$ are spherical neighbourhoods of radius ϵ of the points ξ and $x \in V - \mathcal{F}V$. Take the limit as ϵ approaches zero.

8. *Harmonic functions in the plane*

Up to now, we have been examining harmonic functions in space. Although the theory of harmonic functions in the plane is almost analogous to the theory of harmonic functions in space, there are certain differences, which we shall now consider.

For functions that are harmonic in a bounded plane region, the extreme-value theorem (see section 3) remains completely valid.

By an inversion (in the two-dimensional case), we mean a transformation under which a point x in the plane is mapped into a point ξ with coordinates

$$\xi_i = x_i/|x|^2 \quad (i = 1, 2, \ |x|^2 = x_1^2 + x_2^2) . \tag{72}$$

Here, inversion with respect to a circle corresponds to inversion with respect to a sphere in the three-dimensional case.

Kelvin's theorem remains valid: If a function u is harmonic in a region S, the function

$$v(\xi) = u(x) \equiv u\left(\frac{\xi_1}{|\xi|^2}, \frac{\xi_2}{|\xi|^2}\right) \tag{73}$$

is harmonic in the region S' which is conjugate to the region S under inversion.

In formula (73), we note that there is no factor $|x|$ in front of u, as appears in formula (21). We shall leave the proof of Kelvin's theorem for a plane region to the reader.

It follows from eq. (73) that a function that is harmonic in an infinite plane region does not, in the general case, vanish at infinity. This is so since a function $v(\xi)$, obtained by the transformation of a function $u(x)$ that is harmonic in a bounded region, approaches zero at infinity only when $u(0) = 0$. However, it can be shown that the difference $v(\xi) - u(0)$ decreases in absolute value at infinity as $1/|\xi|$, and that the derivatives of a harmonic function decrease as $1/|\xi|^2$.

By means of the extreme-value theorem and Kelvin's theorem, and by using the same reasoning as in section 4, we can easily prove the uniqueness theorem for the solution to Dirichlet's problem for the Laplace and Poisson equations.

It is also easy to derive Dirichlet's formula:

$$\iint_S \left[\left(\frac{\partial w}{\partial x_1}\right)^2 + \left(\frac{\partial w}{\partial x_2}\right)^2\right] dS = \int_{\mathcal{F}S} w\mathcal{P}w \, dl - \tfrac{1}{2} \int_{\mathcal{F}S} \beta w^2 \, dl ,$$

where w is harmonic in the region S and

$$\mathcal{P}w \equiv \frac{dw}{dn} + \beta w .$$

Using this formula, one can prove the theorems on the uniqueness of the solutions to the Neumann and the mixed problems. These theorems are analogous to those of section 4, except that the solution to the exterior Neumann problem in a plane is determined up to an additive constant, as in the solution to the internal problem.

The fundamental solution to Laplace's equation in a plane region S is the function

$$L(\xi, x) = \frac{1}{2\pi}\left[\ln\frac{1}{r} + \varphi(\xi, x)\right] ,$$

where r is the distance between the points ξ and x, and $\varphi(\xi, x)$ is a function that is harmonic in the region S with respect to the coordinates of the point ξ. It is easy to see that when $\xi \neq x$, the function $\ln(1/r)$ is harmonic with respect to the coordinates of the points ξ and x.

By using Green's theorem (9) of Chapter XVII, we obtain

$$\int_{\mathcal{F}S}\left(L\frac{\mathrm{d}u}{\mathrm{d}n}-u\frac{\mathrm{d}L}{\mathrm{d}n}\right)\mathrm{d}l_\xi=\iint_S L\,\Delta u\,\mathrm{d}S+\begin{cases}0 & \text{when} \quad x\in R_2-S\,,\\ \tfrac12 u(x) & \text{when} \quad x\in\mathcal{F}S\,,\\ u(x) & \text{when} \quad x\in S-\mathcal{F}S\,,\end{cases}\tag{74}$$

where S is a bounded plane region and R_2 is the entire plane. Then, if $\Delta u = 0$, the "basic formula" of the theory of harmonic functions in a plane follows:

$$\int_{\mathcal{F}S}\left(L\frac{\mathrm{d}u}{\mathrm{d}n}-u\frac{\mathrm{d}L}{\mathrm{d}n}\right)\mathrm{d}l_\xi=\begin{cases}0 & \text{when} \quad x\in R_2-S\,,\\ \tfrac12 u(x) & \text{when} \quad x\in\mathcal{F}S\,,\\ u(x) & \text{when} \quad x\in S-\mathcal{F}S\,.\end{cases}\tag{75}$$

If the boundary $\mathcal{F}S$ is a circle C of unit radius, then, by transforming formula (75), we arrive at Poisson's integral formula for a harmonic function u in a plane:

$$u(x)=\frac{1}{2\pi}\int_C u(\xi)\,\frac{1-r_0{}^2}{r^2}\,\mathrm{d}l\,.\tag{76}$$

As in the three-dimensional case, it is easy to show that substitution of a continuous boundary condition into Poisson's integral formula gives a solution to the interior Dirichlet problem for a circle. The mean-value theorem also follows from Poisson's formula: the mean value of a harmonic function on a circle is equal to its value at the center of the circle. To prove this, it is sufficient to set $|x| = 0$ in Poisson's formula.

Setting $v = 1$ and $\Delta u = 0$ in Green's theorem (9) of Chapter XVII, we obtain the formula

$$\int_{\mathcal{F}S}\frac{\mathrm{d}u}{\mathrm{d}n}\,\mathrm{d}l=0\,,\tag{77}$$

which is analogous to formula (34).

Let us extend formulae (75) and (77) to functions that are harmonic in infinite regions with a finite boundary $\mathcal{F}S$. Suppose that u is such a function and that C is a circle of radius a (with center at the coordinate origin) containing the boundary $\mathcal{F}S$. Applying formula (75) to the function u in the region S^* contained between the curve $\mathcal{F}S$ and C, we obtain

$$\frac{1}{2\pi}\int_{\mathcal{F}S}\left(\frac{\mathrm{d}u}{\mathrm{d}n}\ln\frac1r-u\frac{\mathrm{d}}{\mathrm{d}n}\ln\frac1r\right)\mathrm{d}l$$

$$+\frac{1}{2\pi}\int_C\left(\frac{\mathrm{d}u}{\mathrm{d}n}\ln\frac1r-u\frac{\mathrm{d}}{\mathrm{d}n}\ln\frac1r\right)\mathrm{d}l=\begin{cases}0 & \text{when} \quad x\in R_2-S^*\,,\\ \tfrac12 u(x) & \text{when} \quad x\in\mathcal{F}S^*\,,\\ u(x) & \text{when} \quad x\in S^*-\mathcal{F}S^*\,.\end{cases}$$

Let us examine the second of the integrals on the left side of this equation as a approaches ∞. On the basis of the lemma on the behaviour of the derivatives of harmonic functions at infinity, the term $(\mathrm{d}u/\mathrm{d}n)\ln(1/r)$ approaches zero as a approaches ∞ at least as rapidly as does

$$\frac{\ln\,(x_1{}^2+x_2{}^2)}{x_1{}^2+x_2{}^2}\,,$$

and therefore the integral

$$\int_C \frac{\mathrm{d}u}{\mathrm{d}n} \ln \frac{1}{r}\, \mathrm{d}l$$

vanishes at infinity. As regards the integral

$$-\frac{1}{2\pi} \int_C u \frac{\mathrm{d}}{\mathrm{d}n} \ln \frac{1}{r}\, \mathrm{d}C ,$$

we note that

$$\frac{\mathrm{d}}{\mathrm{d}n} \ln \frac{1}{r} = -\frac{1}{r} .$$

Furthermore, no matter what the point x may be, when the radius a of the circle C increases without bound, the value of r approaches a. We then conclude that, as a approaches ∞, this integral approaches the "mean value of the function u at infinity":

$$u_\infty \equiv \lim_{a\to\infty} \frac{1}{2\pi a} \int_C u\, \mathrm{d}C .$$

Thus, we obtain the following basic formula for harmonic functions in an infinite plane region:

$$\frac{1}{2\pi} \int_{\mathcal{F}S} \left(\frac{\mathrm{d}u}{\mathrm{d}n} \ln \frac{1}{r} - u \frac{\mathrm{d}}{\mathrm{d}n} \ln \frac{1}{r}\right) \mathrm{d}l + u_\infty = \begin{cases} 0 & \text{when} \quad x \in R_2 - S , \\ \tfrac{1}{2}u(x) & \text{when} \quad x \in \mathcal{F}S , \\ u(x) & \text{when} \quad x \in S - \mathcal{F}S . \end{cases} \tag{78}$$

By applying the lemma on the derivatives of harmonic functions at infinity, and by examining the limit of the integrals over the bounded region S^* when its exterior contour increases without bound, we easily find that

$$\int_{\mathcal{F}S} \frac{\mathrm{d}u}{\mathrm{d}n}\, \mathrm{d}l = 0 . \tag{79}$$

We note that there is no formula analogous to this one for three dimensions. Thus, in a plane, the boundary condition for the exterior Neumann problem for Laplace's equation must satisfy the same integral relations as the boundary condition for the interior problem.

The reader should not experience any difficulty in defining the Green's functions in a plane or in setting up integral formulae analogous to the formulae of section 7.

In conclusion, we note that the theory of functions of a complex variable is an exceptionally powerful tool in the solution of boundary-value problems for Laplace's equation in a plane. Suppose that $w(z) = u + \mathrm{i}v$ is an analytic function of a complex argument $z = x_1 + \mathrm{i}x_2$. Then, it is known * that the functions u and v satisfy the Cauchy-Riemann equations

* See V.I. Smirnov [1], Vol. 3, Part 2, p. 2.

$$\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2}, \qquad \frac{\partial v}{\partial x_1} = -\frac{\partial u}{\partial x_2}. \tag{80}$$

Differentiating the first of these equations with respect to x_1 and the second with respect to x_2 and then adding the results, we obtain

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0 .$$

In an analogous fashion, we obtain

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} = 0 .$$

It then follows that

$$\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} = 0 ;$$

that is, an arbitrary analytic function of a complex variable satisfies Laplace's equation.

The following two propositions are proven in the theory of functions of a complex variable *

(a) Every analytic transformation of the variables x_1 and x_2 to the variables ξ_1 and ξ_2 is a conformal mapping, of the first kind, of the x-plane onto the ξ-plane and, conversely, an arbitrary conformal mapping of the first kind is analytic; (b) an arbitrary plane simply-connected region distinct from the entire plane, or from the plane with a deleted point, can be transformed into a circle by a conformal mapping.

Since a solution to the interior Dirichlet problem for a circle is given by Poisson's integral formula and since the exterior Dirichlet problem can be reduced to the interior by means of Kelvin's theorem, we conclude from the above that a solution to the Dirichlet problem for a plane region can be found by determining a transformation that maps the given region conformally into a circle. This last transformation always exists.

Let us now show that the Neumann problem in a plane can be reduced to the Dirichlet problem. We note that by the Cauchy-Riemann equations, the derivative of the real part u of the analytic function w with respect to an arbitrary direction n is equal to the derivative of the imaginary part of this function with respect to the direction τ perpendicular to n.

Let us now suppose that u is the desired solution to the Neumann problem and that

$$\mathrm{d}u/\mathrm{d}n = \psi \quad \text{when} \quad x \in \mathcal{F}S \tag{81}$$

is the given boundary condition. We introduce the harmonic functions v, defined on the boundary $\mathcal{F}S$ by

$$v(x) = \int_y^x \psi(\xi)\, \mathrm{d}l_\xi \qquad (x \in \mathcal{F}S) ,$$

where y is an arbitrary point of the contour $\mathcal{FS}$ and the integration is carried out along this contour. Here, we obviously have

$$dv/d\tau = \psi \ , \tag{82}$$

where $dv/d\tau$ is the derivative with respect to the tangent to the curve $\mathcal{FS}$. At points not belonging to the curve $\mathcal{FS}$, the function z can be defined as the solution to the corresponding Dirichlet problem.

If we define the function u in terms of v in such a way that eq. (80) is satisfied in the regions for which the solution to the Neumann problem is being sought, the function u will be harmonic and, on the basis of eq. (82) and the remark made above, it will have a normal derivative satisfying condition (81). This function will yield a solution to the corresponding Neumann problem. It is easy to see that the function u is determined by this construction up to an arbitrary constant.

Thus, the Dirichlet and Neumann problems for Laplace's equation in a plane can be reduced to some problem on conformal mapping. The reader can find a detailed exposition of conformal mappings in books on complex-variable theory.

Chapter XIX

POTENTIAL THEORY

1. *Newtonian potential*

Potential theory is one of the oldest branches of mathematical physics and is of great importance from the standpoint of physical applications.

By Newton's law, the gravitational potential at the point x due to a mass m concentrated at a point ξ is equal to $-\kappa(m/r)$, where κ is the gravitational constant and r is the distance between the points x and ξ. If the mass is distributed with density ρ in a region V, the resulting field potential will obviously be determined by the volume integral

$$-\kappa \iiint_V \frac{\rho}{r} \, \mathrm{d} V_\xi .$$

An expression of similar form, differing only by a constant factor, determines the Coulomb field potential of electric charges distributed with density ρ.

In both cases, the potential of the field is, up to a constant factor, equal to the integral

$$U(x) = \iiint_V \frac{\rho}{r} \, \mathrm{d} V_\xi , \tag{1}$$

which we shall call the *Newtonian potential*.

Let us emphasize the distinction between the Newtonian potential (1) on the one hand, and the potentials of gravitational and electric charges on the other.

In shifting to a gravitational field, we must put a *negative* factor in front of the integral (1). This takes into account the nature of the interaction between masses (attraction). The absolute value of this factor depends on the choice of the units of measurement and therefore it is not important for us. Furthermore, the density ρ of the masses, as distinct from the density of electric charges, is *always non-negative*. Therefore, in examining a gravitational field, we are always dealing with a special case of a Newtonian potential (1).

In the case of a field of electric charges, we must put a *positive* factor in front of the integral (1) because electric charges of like kind *repel* each other. The density ρ can be of variable sign.

Thus, in studying a Newtonian potential (1), we leave aside the specific nature of the interaction (attraction or repulsion) and our conclusions are not dependent on this nature. They will always be applicable to a field of

electric charges. They will also be applicable to a gravitational field if the requirement of non-negative density ρ is met.

Let us study the properties of a Newtonian potential.

If the density ρ is a bounded function with continuous first derivatives that decreases at infinity at least as rapidly as does $1/|\xi|^2$, where

$$|\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 ,$$

we may show that the Newtonian potential satisfies Poisson's equation

$$\Delta U' = -4\pi\rho \tag{2}$$

and has continuous first and second derivatives; also, the first derivatives can be obtained by differentiation under the integral sign. We shall not give the proof of these assertions, which is simple, though somewhat tedious *. A proof of eq. (2) under considerably more general assumptions than those stated will be given in Chapter XXXIX.

Below we shall examine the Newtonian potential at points outside the region V in which the masses or charges are distributed. We shall assume that the region V is bounded and that the density ρ is continuous. When x belongs to $R_E - V$, where R_E is all of space, the integrand in the integral (1) is continuous and infinitely many times differentiable with respect to the coordinate of the point x. Consequently, when x belongs to $R_E - V$, the derivatives of all orders of the Newtonian potential can be obtained by differentiating under the integral sign.

Since the function $1/r$ is harmonic when ξ belongs to V and x belongs to $R_E - V$, the Newtonian potential satisfies Laplace's equation. As x increases without bound, the integrand becomes infinitesimal. Since the region V is bounded, the Newtonian potential then approaches zero. Consequently, outside the region of distribution of masses (or charges), *the Newtonian potential is a harmonic function.*

Problems

1. Find the Newtonian potential created by a homogeneous disk of radius R at a point on the axis of the disk and a distance h from the center of the disk.
   Answer:
   $$U = \frac{2m}{R^2}\left(\frac{1}{h} - \frac{1}{\sqrt{R^2+h^2}}\right),$$
   where m is the total mass of the disk.

2. The density of a massive sphere of radius R varies as the square of the distance from a plane passing through the center. Find the potential of a point on the perpendicular passing through the center of this plane at a distance a from the center of the sphere.
   Answer:

* See V.I.Smirnov [1)], Vol. 2, pp. 200-201.

Answer:

$$U = \frac{4\pi}{15} R^4 \left(\frac{R}{h} + \frac{2}{7}\frac{R^3}{h^3}\right) .$$

2. *Potentials of different orders*

Let us examine the triangle $x\xi\zeta$ shown in fig. 44. We use the following notation for the length of the sides of the triangle:

$$r \equiv |\xi - x| , \quad r_0 \equiv |\xi - \zeta| , \quad R \equiv |x - \zeta| .$$

By a well-known formula,

$$r = R\sqrt{1 + \frac{r_0^{\,2}}{R^2} - 2\frac{r_0}{R}\cos\gamma} ,$$

where γ is the angle $x\zeta\xi$. We denote by Ω that portion of space outside the sphere $|x - \zeta| \leq r_{\max}$, where $r_{\max}$ is the greatest distance between ξ and ζ for ξ belonging to V. If x belongs to Ω, r_0 will be less than R, so that the function $1/r$ can be expanded in an absolutely and uniformly convergent series

$$\frac{1}{r} = \frac{1}{R} + \frac{1}{R}\sum_{\alpha=1}^{\infty}\left(\frac{r_0}{R}\right)^{\alpha} P_\alpha(\cos\gamma) \qquad (x \in \Omega) , \tag{3}$$

where, from section 5 of Chapter XV, the $P_\alpha(\cos\gamma)$ are the Legendre polynomials.

On the other hand, if we note that

$$r \equiv \sqrt{\sum_{\alpha=1}^{3}[(x_\alpha - \zeta_\alpha) - (\xi_\alpha - \zeta_\alpha)]^2} ,$$

we can expand the function $1/r$ in a Taylor series in powers of the distances $|\xi_j - \zeta_j|$, which yields

$$\frac{1}{r} = \frac{1}{R} + \sum_{\alpha=1}^{\infty}\frac{1}{\alpha!}\left\{\left[\sum_{\beta=1}^{3}(\xi_\beta - \zeta_\beta)\frac{\partial}{\partial\xi_\beta}\right]^{\alpha}\frac{1}{r}\right\}_{\xi=\zeta} .$$

The subscript equation $\xi = \zeta$ following the brace indicates that after the differentiation is performed the coordinates ξ_1, ξ_2, and ξ_3 of the point ξ should be replaced by the coordinates ζ_1, ζ_2, and ζ_3 of the point ζ. When x belongs to Ω, this series also converges absolutely and uniformly. Since

$$\frac{\partial}{\partial\xi_i}\left(\frac{1}{r}\right) = -\frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) \qquad (i = 1, 2, 3) , \tag{4}$$

the differentiation with respect to ξ_i can be replaced with differentiation with respect to x_i. Here, we may make the substitution $\xi = \zeta$ before the differentiation. On the basis of the identity

$$\left.\frac{1}{r}\right|_{\xi=\zeta} \equiv \frac{1}{R} ,$$

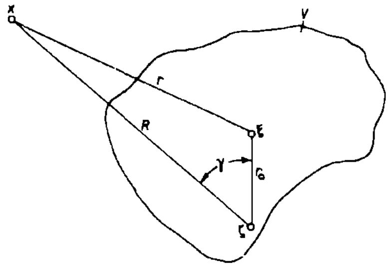

Fig. 44.

this will yield

$$\frac{1}{r} = \frac{1}{R} + \sum_{\alpha=1}^{\infty} \frac{(-1)^\alpha}{\alpha!} r_0{}^\alpha \left[\sum_{\beta=1}^{3} \frac{\xi_\beta - \zeta_\beta}{r_0} \frac{\partial}{\partial x_\alpha}\right] \frac{1}{R}. \tag{5}$$

Noting that the ratios $(\xi_\beta - \zeta_\beta)/r_0$ are equal to the direction cosines of the segment $\overline{\zeta\xi}$ (see fig. 44), as a result of which the expression

$$\sum_{\beta=1}^{3} \frac{\xi_\beta - \zeta_\beta}{r_0} \frac{\partial}{\partial x_\beta} \equiv \frac{\partial}{\partial r_0}$$

denotes differentiation in the direction of the segment $\overline{\zeta\xi}$, we obtain

$$\frac{1}{r} = \frac{1}{R} + \sum_{\alpha=1}^{\infty} (-1)^\alpha \frac{r_0{}^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial r_0{}^\alpha}\left(\frac{1}{R}\right) \quad (r_0 < R) . \tag{6}$$

By equating the series (3) and (6), we obtain the following representations of the Legendre polynomials:

$$P_n(\cos\gamma) = (-1)^n \frac{R^{n+1}}{n!} \frac{\partial^n}{\partial r_0{}^n}\left(\frac{1}{R}\right), \tag{7}$$

from which, in particular, it follows that the product

$$R^{n+1} \frac{\partial^n}{\partial r_0{}^n}\left(\frac{1}{R}\right)$$

does not depend on R but only on the angle γ.

Let us multiply the series (3) and (6) by $\rho(\xi)$ and differentiate them termwise (which is permissible because of the uniform convergence) over the region V of distribution of the masses. As a result, we obtain the expansion of the Newtonian potential (1) as an infinite series:

$$U(x) = U_0(x) + U_1(x) + U_2(x) + \dots \quad (x \in \Omega) , \tag{8}$$

where

$$U_n(x) \equiv \frac{1}{R^{n+1}} \iiint_V \rho\, r_0{}^n P_n(\cos\gamma)\, dV = \frac{(-1)^n}{n!} \iiint_V \rho\, r_0{}^n \frac{\partial^n}{\partial r_0{}^n}\left(\frac{1}{R}\right) dV \tag{9}$$

$$(n = 0, 1, 2, 3, \dots) .$$

The functions U_n are called the n-th order potentials. It is easy to show that they are harmonic.

At a sufficiently great distance from the region V, the Newtonian potential $U(x)$ can be described with any desired degree of accuracy by the first of the n-th order potentials that is not equal to zero. For, in accordance with formula (14) of Chapter XV,

$$|P_n(\cos\gamma)| \leqslant 1 .$$

Therefore,

$$\left|\iiint_V \rho\, r_0^n\, P_n(\cos\gamma)\,\mathrm{d}V\right| \leqslant \iiint_V |\rho|\, r_{\max}^n\,\mathrm{d}V \leqslant r_{\max}^n\, q^* ,$$

where

$$q^* = \iiint_V |\rho|\,\mathrm{d}V ,$$

and $r_{\max}$ is the greatest value of

$$r_0 \equiv |\xi - \zeta|$$

in the region V. By combining this inequality with eq. (9), we obtain

$$|U_n| \leqslant \frac{q^*}{R}\left(\frac{r_{\max}}{R}\right)^n ,$$

so that

$$\sum_{\alpha=m}^{\infty} |U_\alpha| \leqslant \frac{q^*}{R}\sum_{\alpha=m}^{\infty}\left(\frac{r_{\max}}{R}\right)^\alpha = \frac{q^*\, r_{\max}}{R^{m+1}}\,\frac{R}{R - r_{\max}} . \tag{10}$$

Suppose that

$$U_{m-1} \equiv \frac{1}{R^m}\iiint_V \rho\, r_0^{\,m-1}\, P_m(\cos\gamma)\,\mathrm{d}V$$

is the first of the potentials that is not identically equal to zero. As x increases without bound, this potential approaches zero as $1/R^m$ whereas, from relationship (10), the sum of all the remaining potentials decreases as $1/R^{m+1}$. This proves the above assertion.

For a gravitational field, the density ρ does not change sign. Therefore, if we choose the center of gravity of the masses as the point ζ, we can arrange for the first order potential $U_1(x)$ to vanish. Consequently, at a distance that is great in comparison with the transverse dimension of the region in which the masses are distributed, their Newtonian potential will coincide, up to third-order terms, with the Newtonian potential of a mass point situated at the center of gravity of the masses and having a mass equal to the entire mass distributed throughout the region V.

By setting $m = 0$ in eq. (10), we see that the Newtonian potential decreases at infinity at least as rapidly as does $1/R$.

Problem

Show that if the density ρ changes sign in the region V it will, generally speaking, be impossible to choose the point ζ such that the first-order potential will vanish.

3. *Multipoles*

Let us turn to the expansion of a Newtonian potential in series (8). From the relationship (9), the first term of the series (8) (the zeroth-order potential) is equal to

$$U_0 \equiv \frac{1}{R} \iiint\limits_V \rho \, dV .$$

This expression formally coincides with the potential of a point charge

$$q = \iiint\limits_V \rho \, dV ,$$

situated at the point ξ. It turns out that the subsequent terms of the series (8) can be regarded as the potentials of point objects.

Let us consider two point charges $-q$ and q, where q is positive, that are located, respectively, at the points ζ and η (fig. 45). The potential of the field created by this pair of charges at the point x is equal to

$$U(x) = -\frac{q}{R} + \frac{q}{R_1} ,$$

where R and R_1 are the lengths of the segments $\overline{\zeta x}$ and $\overline{\zeta\eta}$. Assuming that the length r_1 of the interval $\overline{\zeta\eta}$ is less than R, we expand the function $1/R_1$ in a series of the form (3), which gives

$$U(x) = \frac{q}{R} \sum_{\alpha=1}^{\infty} \left(\frac{r_1}{R}\right)^{\alpha} P_\alpha(\cos\gamma_1) , \tag{11}$$

where γ_1 is the angle between the segments $\overline{\zeta x}$ and $\overline{\zeta\eta}$.

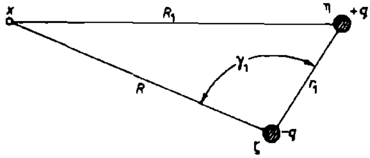

Fig. 45.

Let us bring the points ζ and η close together by displacing the point η (with charge q) along the segment $\overline{\eta\zeta}$. At the same time, let us increase the absolute value q of the charges so that the product

$$p_1 \equiv q r_1 \tag{12}$$

will remain constant. Then, as we can easily see, all the terms in the expansion (11) except the first will approach zero. When we take the limit, we obtain

$$\lim_{r_1 \to 0} U(x) = p_1 \frac{P_1(\cos \gamma)}{R^2} \equiv U^{(1)}(x) . \tag{13}$$

On the basis of eq. (7), we can also write

$$U^{(1)}(x) = - p_1 \frac{\partial}{\partial r_1} \left(\frac{1}{R}\right) , \tag{14}$$

where $\partial / \partial r_1$ denotes differentiation along the segment $\overline{\zeta\eta}$.

The point object obtained as a result of this process of having the points ζ and η approach each other is called a dipole. More precisely, a dipole is a singular point of a field characterized by a potential (13) or (14). The potential of a dipole decreases inversely as the square of the distance, as does the first-order potential in expansion (8).

The quantity t_1 that appears in expressions (13) and (14) is called the *dipole moment* and the direction of the segment $\overline{\zeta\eta}$ taken from the negative charge to the positive charge is called a dipole axis.

Let us now construct a point object whose potential decreases inversely as R^3, that is, as the second-order potential in the expansion (8).

To do this, we put a dipole with moment p_1 and axis oriented along the direction r_1 at an arbitrary point η, and we put a similar dipole at the point ρ but with axis oriented in the direction $-r_1$ (fig. 46). According to formula (14), the potential of the field created by this system at the point x is equal to

$$U(x) = p_1 \left[\frac{\partial}{\partial r_1} \left(\frac{1}{R}\right) - \frac{\partial}{\partial r_1} \left(\frac{1}{R_1}\right)\right] = p_1 \frac{\partial}{\partial r_1} \left(\frac{1}{R} - \frac{1}{R_1}\right) .$$

Let us expand the function $1/R_1$ in a series of the form (5), remembering that, in this case, the distance r_0 is equal to the length of the segment $\overline{\zeta\eta}$, which we denote by r_2. We obtain

$$U(x) = - p_1 \frac{\partial}{\partial r_1} \sum_{\alpha=1}^{\infty} (-1)^{\alpha} \frac{r_2^{\alpha}}{\alpha!} \frac{\partial^{\alpha}}{\partial r_2^{\alpha}} \left(\frac{1}{R}\right)$$
$$= \sum_{\alpha=1}^{\infty} (-1)^{\alpha+1} \frac{p_1 r_2^{\alpha}}{\alpha!} \frac{\partial^{1+\alpha}}{\partial r_1 \, \partial r_2^{\alpha}} \left(\frac{1}{R}\right) , \tag{15}$$

where $\partial / \partial r_2$ denotes differentiation in the direction of the segment $\overline{\zeta\eta}$.

Keeping the direction of the segment $\overline{\zeta\eta}$ constant, let us move the point η toward the point ζ, at the same time increasing the value of the moment p_1 so that the product

$$p_2 \equiv 2! \, p_1 \, r_2$$

remains constant. Then, all terms in the series (15), with the exception of the first, will approach zero, so that

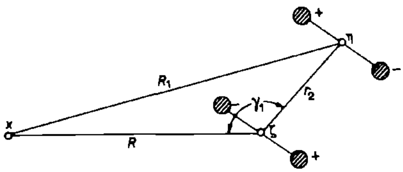

Fig. 46.

$$\lim_{r_2 \to 0} U(x) = \frac{p_2}{2!} \frac{\partial^2}{\partial r_1 \partial r_2} \left(\frac{1}{R}\right) \equiv U^{(2)}(x) . \tag{16}$$

The point object obtained by taking this limit is called a *quadrupole*. The quantity p_2 is called the *quadrupole moment* and the directions r_1 and r_2 are its *axes*. The potential of a quadrupole decreases with increasing R in inverse proportion to R^3 as does the third-order potential in the expansion (8).

By having two quadrupoles approach each other, we can construct an *octupole* – the point source of a field characterized by a potential

$$U^{(3)}(x) = \frac{p_3}{3!} \frac{\partial^3}{\partial r_1 \partial r_2 \partial r_3} \left(\frac{1}{R}\right) .$$

Continuing this process, we obtain a *multipole* of order n – a point source of a field characterized by the potential

$$U^{(n)}(x) = \frac{p_n}{n!} \frac{\partial^n}{\partial r_1 \partial r_2 \dots \partial r_n} \left(\frac{1}{R}\right) , \tag{17}$$

where $\partial/\partial r_k$ (for $k = 1, 2, \dots, n$) denotes differentiation in the direction r_k. The directions r_k are called the *axes of the multipole* and the quantity p_n is its *moment*.

As the order of a multipole is increased, an ever greater number of parameters are necessary to characterize it. A "multipole of zeroth order" (a point charge) is completely determined by a single algebraic number – the quantity of charge. A dipole is characterized by three parameters – the dipole moment and the two quantities determining the direction of its axis. An n-th order multipole is characterized by $2n+1$ parameters – the multipole moment p_n and $2n$ parameters determining the direction of its n-axes (the n directions of the differentiation in formula (17)).

In a particular case, all the axes of a multipole may coincide, having some common direction r_0. Such a multipole is said to be axial. Its potential is

$$U_0^{(n)}(x) = \frac{p_n}{n!} \frac{\partial^n}{\partial r_0^n} \left(\frac{1}{R}\right) .$$

Remembering the relationship (7), the potential of a multipole axis can be represented in the form

$$U_0^{(n)}(x) = (-1)^n \frac{p_n}{R^{n+1}} P_n(\cos\gamma) , \tag{18}$$

where γ is the angle between the axis of the multipole and the direction from the multipole to the point x.

Problems

1. Show that a dipole can be uniquely characterized by the *dipole-moment vector* $\boldsymbol{P}$, equal in magnitude to the dipole moment p_1 and directed along the dipole axis.
   Method: Take the direction cosines between the dipole axis and the coordinate axes

   $$\cos(\boldsymbol{P}, X_i) = \frac{x_i - \zeta_i}{r_0} \quad (i = 1, 2, 3) ,$$

   and express the potential of the dipole in terms of the scalar product of the vector whose components are $p_1 \cos(\boldsymbol{P}, X_i)$ with the unit vector directed from the position vector of the dipole to the point at which the potential is being determined.

2. Show that two multipoles of the same order that are situated at the same point can be replaced by a single multipole of the same order, with a moment and directions of the axes such that the field will not be changed.
   Method: Use the identity

   $$\frac{\partial^n}{\partial r_1 \, \partial r_2 \ldots \partial r_n} = \sum_{\alpha+\beta+\gamma=n} a_{\alpha\beta\gamma} \frac{\partial^n}{\partial x_1^\alpha \, \partial x_2^\beta \, \partial x_3^\gamma} ,$$

   where the $a_{\alpha\beta\gamma}$ are constant.

3. Express the potential of the field of a point charge q located at a point ξ in terms of the potential of a system of multipoles of different order situated at a point ζ.

4. *Analysis of a potential in terms of multipoles. Spherical functions*

Let us introduce spherical coordinates R, θ, and φ with the origin at the point ζ and let us examine the expansion (8). As has been shown (see the statement following formula (7)), the products

$$R^{n+1} \frac{\partial}{\partial r_0{}^n} \left(\frac{1}{R}\right)$$

do not depend on the value of R. Therefore, the potentials

$$U_n = \frac{(-1)^n}{n} \iiint\limits_V \rho\, r_0{}^n \frac{\partial^n}{\partial r_0{}^n}\left(\frac{1}{R}\right) \mathrm{d}V$$
$$\equiv \frac{1}{R^{n+1}} \frac{(-1)^n}{n!} \iiint\limits_V \rho\, r_0{}^n R^{n+1} \frac{\partial^n}{\partial r_0{}^n}\left(\frac{1}{R}\right) \mathrm{d}V \tag{19}$$

can be represented as the products of two factors; the first $1/R^{n+1}$ depends only on R, and the second

$$Y_n(\theta,\varphi) \equiv \frac{(-1)^n}{b!} \iiint\limits_V \rho\, r_0{}^n R^{n+1} \frac{\partial^n}{\partial r_0{}^n}\left(\frac{1}{R}\right) \mathrm{d}V \tag{20}$$

does not depend on R and, therefore, can depend only on the angular coordinates θ and ξ. In other words, *the distribution of values of the potentials* U_n *over all the spherical surfaces* R = *constant* (for x belonging to Ω) *is similar*. Consequently, an arbitrary n-th order potential can be uniquely characterized by the factor $Y_n(\theta, \varphi)$, which depends only on the coordinates θ and φ. This factor is called a second-order spherical function.

Since the potentials of a multipole are harmonic, the spherical functions are continuous and infinitely many times differentiable with respect to θ and φ.

Let us derive certain relations between potentials of multipoles and spherical functions.

Let us denote by ν_1, ν_2, and ν_3 the direction cosines of the segment $\overline{\zeta\xi}$ (see fig. 44). Noting that

$$\frac{\partial}{\partial r_0} = \sum_{\alpha=1}^{3} \nu_\alpha \frac{\partial}{\partial x_\alpha},$$

we obtain

$$\frac{\partial^n}{\partial r_0{}^n} = \left(\sum_{\alpha=1}^{3} \nu_\alpha \frac{\partial}{\partial x_\alpha}\right)^n = \sum_{\alpha+\beta+\gamma=n} \frac{n!}{\alpha!\,\beta!\,\gamma!} \nu_1{}^\alpha \nu_2{}^\beta \nu_3{}^\gamma \frac{\partial^n}{\partial x_1{}^\alpha\, \partial x_2{}^\beta\, \partial x_3{}^\gamma}. \tag{21}$$

Substituting this expression into eq. (19) (and remembering that the derivatives

$$\frac{\partial^n}{\partial x_1{}^\alpha\, \partial x_2{}^\beta\, \partial x_3{}^\gamma}\left(\frac{1}{R}\right)$$

do not depend on the coordinates of points of the region V over which the integration is taken), we obtain, after some manipulations,

$$U_n = \sum_{\alpha+\beta+\gamma=n} \frac{1}{n!} e_{\alpha\beta\gamma} \frac{\partial^n}{\partial x_1{}^\alpha\, \partial x_2{}^\beta\, \partial x_3{}^\gamma}\left(\frac{1}{R}\right), \tag{22}$$

where

$$e_{\alpha\beta\gamma} \equiv (-1)^n \frac{n!}{\alpha!\,\beta!\,\gamma!} \iiint\limits_V \rho\, \nu_1{}^\alpha \nu_2{}^\beta \nu_3{}^\gamma \,\mathrm{d}V. \tag{23}$$

The constants e_{jkl} (where $j+k+l=n$) are called *n-th order moments*.

By equating the terms of the sum (22) with the expressions (17), we conclude that the relationship (22) gives the sum of the potentials of multipoles of order n situated at the point ζ and having multipole moments $e_{\alpha\beta\gamma}$. It is easy to show that this sum is equivalent to a single multipole situated at that point; that is, we may construct an n-th order multipole by choosing its moment and the direction of its axes in such a way that its potential will coincide with U_n. Specifically, we take the expression

$$L_n \equiv \sum_{\alpha+\beta+\gamma=n} e_{\alpha\beta\gamma} \frac{\partial^n}{\partial x_1^{\alpha}\, \partial x_2^{\beta}\, \partial x_3^{\gamma}}$$

in the form of the product

$$L_n = p_n \prod_{\alpha=1}^{n} \left(\sum_{\beta=1}^{3} a_{\alpha\beta} \frac{\partial}{\partial x_\beta} \right),$$

where p_n is a constant *. The coefficients $a_{\alpha 1}$, $a_{\alpha 2}$, and $a_{\alpha 3}$ are assumed to satisfy the condition

$$\sum_{\beta=1}^{3} a_{\alpha\beta}^2 = 1 .$$

Indeed, if for some α this is not the case, we divide the corresponding trinomial by

$$\sqrt{\sum_{\beta=1}^{3} a_{\alpha\beta}^2}$$

and change the value of p_n correspondingly; we then obtain numbers $a_{\alpha 1}$, $a_{\alpha 2}$, and $a_{\alpha 3}$ that satisfy the stated conditions. Therefore, the coefficients $a_{\alpha 1}$, $a_{\alpha 2}$, and $a_{\alpha 3}$ can be taken as the direction cosines of some direction r_α, so that

$$L_n = p_n \prod_{\alpha=1}^{n} \frac{\partial}{\partial r_\alpha} = p_n \frac{\partial^n}{\partial r_1\, \partial r_2 \dots \partial r_n} .$$

This proves the assertion made above.

Thus, expression (19) is the potential of some multipole located at the point ζ. Consequently, the series (8) represents the expansion of a Newtonian potential in a series of potentials of multipoles of different orders located at the point ζ.

Multiplying both sides of expression (22) by R^{n+1} and recalling formulae (19) and (20), we find that

$$Y_n(\theta, \varphi) = \sum_{\alpha+\beta+\gamma=n} e_{\alpha\beta\gamma}\, Y_{\alpha\beta\gamma} , \tag{24}$$

* The possibility of such a representation can be shown directly if, in the last expression, we expand the expression in parenthesis, perform the multiplication, and collect similar terms. Then $a_{\alpha\beta}$ can be chosen so that the expression obtained will coincide with the original expression for arbitrary p_n.

where

$$Y_{\alpha\beta\gamma} \equiv \frac{1}{n!} R^{n+1} \frac{\partial^n}{\partial x_1{}^\alpha \, \partial x_2{}^\beta \, \partial x_3{}^\gamma} \left(\frac{1}{R}\right) \tag{25}$$

are spherical functions of a special type. Since the moments $e_{\alpha\beta\gamma}$ do not depend on the coordinates R, θ, and φ, we conclude from formula (24) that an arbitrary n-th order spherical function can be expressed as a linear combination of spherical functions of the special type (25). The number of functions can be shown to be equal to

$$1 + 2 + \ldots + n + (n+1) = \tfrac{1}{2}(n+2)(n+1) .$$

However, not all the functions $Y_{\alpha\beta\gamma}$ (where $\alpha+\beta+\gamma = n$) are linearly independent; that is, some of them can be represented as a linear combination of the other functions $Y_{\alpha\beta\gamma}$ of the same order.

This might be expected in view of the fact that the function Y_n/R^{n+1} represents the potential of some n-th order multipole. As was shown in section 3, an n-th order multipole is completely determined by a total of only $2n+1$ parameters. Therefore, we may expect that there will be no more than $2n+1$ linearly independent spherical functions of order n.

To prove this, we note that the function $1/R$ satisfies Laplace's equation:

$$\left(\frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial x_3{}^2}\right) \frac{1}{R} = 0 ,$$

and therefore some of the partial derivatives of $1/R$ can be expressed in terms of the others. For example, we may eliminate all the derivatives with respect to x_3 of higher order than the first by means of the identity

$$\frac{\partial}{\partial x_3{}^2} \left(\frac{1}{R}\right) = -\left(\frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2}\right) \frac{1}{R} ,$$

thus expressing all functions of the form (25) with $\gamma > 1$ in terms of functions with $\gamma \leq 1$. The number of values that the exponent β can take for a given value of γ is equal to $n-\gamma+1$. Then, to each given set of values of γ and β, there corresponds a definite value of α such that $\alpha+\beta+\gamma = n$. If $\gamma \leq 1$, the number of values that the exponent β can take is equal to $(n+1) + n = 2n + 1$; that is, among the functions (25) there are no more than $2n + 1$ that are linearly independent.

A method for the systematic construction of $2n+1$ mutually orthogonal (and hence linearly independent) spherical functions for a given value of n will be shown in Chapter XXI, and the completeness of the entire system of spherical functions constructed by this method will be proven.

We note that a question may arise as to whether the expansion in multipoles (or, more generally, whether the field outside a region V occupied by masses or charges) uniquely determines the density ρ of mass (or charge) distribution in the region V. In the general case, the answer is negative. This can be seen, for example, from the fact that a Taylor series expansion of functions of three variables contains, as we know, $\frac{1}{2}(n+\ell)(n+1)$ lin-

early independent terms of order n, whereas, in the expansion of the potential of a field, as we have seen, there are no more than $2n + 1$ linearly independent terms of n-th order. Thus, the same field can be created by different distributions of masses or charges. Only the integrals (20), in particular, the total mass or charge in the region V, are uniquely determined.

Problems

1. Find linearly independent spherical functions of the first order.
2. Express the potential of a multipole of order n in terms of a spherical function of order n.

5. *The potentials of single and double layers*

Let us suppose that a certain mass (or charge) is distributed over a surface S with a surface density $\bar{\rho}(\xi)$. The potential of the field formed by this distribution of mass (charge) is, except for a constant factor, equal to the integral

$$\bar{U}(x) \equiv \iint_S \frac{\bar{\rho}}{r}\,\mathrm{d}S_\xi \qquad (r \equiv |x - \xi|)\ , \tag{26}$$

which is called a *single-layer potential*. The density $\bar{\rho}(\xi)$ is called a *single-layer density*.

Let us now suppose that a layer of dipoles (with axes directed along the outward normal n to the surface S) is distributed on the surface S. The dipole moment of an element $\mathrm{d}S$ of the surface S is set equal to

$$\bar{\bar{\rho}}(\xi)\ \mathrm{d}S_\xi\ .$$

Formula (14) implies that the potential at a point x of the field created by the dipoles in the element $\mathrm{d}S_\xi$ is equal to

$$-\ \bar{\bar{\rho}}(\xi)\ \frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r}\right)\ ,$$

where r is the distance between the points ξ and x. It is clear from this that the potential of the field created by this distribution of dipoles can be characterized by the integral

$$\bar{\bar{U}}(x) \equiv \iint_S \bar{\bar{\rho}}\,\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r}\right)\mathrm{d}S_\xi\ , \tag{27}$$

which is called the *double-layer potential*. The function $\bar{\bar{\rho}}$ is called the *double-layer density*.

In what follows the densities $\bar{\rho}$ and $\bar{\bar{\rho}}$ will be assumed continuous.

Let us note certain properties of the integrals (26) and (27). If a point x does not belong to the layer, differentiation with respect to the coordinates

of the point x can be carried out under the integral sign. Since the function $1/r$ is harmonic outside the points of the layer (that is, for $x \neq \xi \in S$), the single- and double-layer potentials satisfy Laplace's equation everywhere outside the points of the layer. We shall consider the surface S closed and bounded. In this case, as x increases without bound, the integrals (26) and (27) are of the same order as the functions $1/r$ and $1/r^2$, respectively. These functions tend to zero at an infinitely distant point. (The detailed proofs are left to the reader.) Consequently, the single- and double-layer potentials outside the points of the layer are everywhere harmonic.

Conversely, if, in formulae (44) and (45) of Chapter XVIII, we set

$$L(\xi, x) = \frac{1}{4\pi}\,\frac{1}{r}, \qquad \frac{\partial u}{\partial n} \equiv \bar{\rho}, \qquad -u \equiv \bar{\bar{\rho}},$$

we conclude that every harmonic function can be represented as a sum of a single-layer and a double-layer potential.

The basic difficulty lies in the investigation of the behaviour of the integrals (26) and (27) in the neighbourhood of points of the layer and at these points themselves. To conduct a rigorous investigation of this behaviour, we need to make certain assumptions concerning the properties of the surface S on which the layers in question are distributed and concerning the properties of the densities $\bar{\rho}$ and $\bar{\bar{\rho}}$.

6. *Lyapunov surfaces*

A. M. Lyapunov, who derived a number of results in potential theory, assumed that surfaces S on which a single or double layer is distributed satisfy the conditions:

(a) At each point of the surface there is a unique normal.

(b) A sufficiently small radius ϵ can be found such that, for any point ζ on the surface, that portion of the surface within a sphere of radius ϵ having center at ζ will be intersected at no more than one point by any given straight line parallel to the normal at the point ζ.

(c) The angle between the normals at two arbitrary points ζ and ξ does not exceed the quantity

$$A|\xi - \zeta|^{\lambda},$$

where $|\xi - \zeta|$ is the distance between these points, and A and λ are constants with $0 < \lambda \leq 1$.

Surfaces satisfying these conditions are called *Lyapunov surfaces*.

The assumptions of section 1, Chapter XVII as to the properties of bounding surfaces assure the satisfaction of the first two Lyapunov conditions. That the first of these is satisfied for smooth surfaces is obvious. Let us show that the second is satisfied.

Let us set up a local Cartesian coordinate system at an arbitrarily chosen point ζ on a surface S, directing axis 3 along the outward normal to the surface S at the point ζ. By the hypothesis of section 1, Chapter XVII,

regarding boundary surfaces, the equation for the surface S within some sphere $|\xi| \leqslant a$ may be rewritten in the form

$$\xi_3 = f(\xi_1, \xi_2) ,$$

where the functions f and its first derivative are continuous within the sphere $|\xi| \leqslant a$ and vanish at the point ζ.

From the well-known formulae of analytic geometry, the direction cosines of the normal to S at the point ξ on S are equal to

$$\cos \gamma_1 = \frac{f_1}{\sqrt{1 + f_1^2 + f_2^2}} , \qquad \cos \gamma_2 = \frac{f_2}{\sqrt{1 + f_1^2 + f_2^2}} ,$$
$$\cos \gamma_3 = \frac{1}{\sqrt{1 + f_1^2 + f_2^2}} , \tag{28}$$

where f_1 and f_2 are the partial derivatives of the function f with respect to ξ_1 and ξ_2 at the points in question. We note that the angle γ_3 is equal to the angle between the normals at the points ζ and ξ. The radius a of the sphere $|\xi| \leqslant a$ can be chosen sufficiently small so that, for an arbitrary choice of the points ξ on S within the sphere, we would have

$$f_1^2 + f_2^2 < 1$$

and therefore

$$\cos \gamma_3 > C > 0 . \tag{29}$$

Here, the bending of the surfaces S within the sphere $|\xi| \leqslant a$ does not exceed $\frac{1}{2}\pi$ and, consequently, an arbitrary straight line parallel to the normal at the point ζ intersects that portion of the surface S that lies within this sphere at no more than one point.

Let us now show the connection between the assumptions of section 1 of Chapter XVII and the third Lyapunov condition. From formulae (28), we find that

$$\sin \gamma_3 = \frac{\sqrt{f_1^2 + f_2^2}}{\sqrt{1 + f_1^2 + f_2^2}} .$$

Since, by assumption, the derivatives f_1 and f_2 are continuous, a radius a of a sphere $|\xi| \leqslant a$ can be chosen sufficiently small so that $\sin \gamma_3$, for an arbitrary choice of the point ζ on S, will be less than an arbitrary given number. Since

$$0 < \frac{\sin \gamma_3}{\gamma_3} \leqslant 1$$

for small values of γ_3 we can find a positive constant A_1 such that, within the sphere $|\xi| \leqslant a$,

$$\gamma_3 < A_1 \sqrt{f_1^2 + f_2^2} .$$

If the derivatives f_1 and f_2 satisfy (within the sphere) not only the conditions of section 1 of Chapter XVII but also *Hölder's condition*

$$\frac{|f_i(\xi_1,\xi_2) - f_i(0,0)|}{|\xi|^\lambda} < A_2 \quad (i = 1,2) , \tag{30}$$

where A_2 is a bounded positive number and the exponent λ satisfies the inequality $0 < \lambda \leqslant 1$, then since $f_i(0,0) = 0$, we obtain the inequality

$$\gamma_3 < A|\xi|^\lambda . \tag{31}$$

Thus, if the first derivatives of the function f satisfy Hölder's condition with $0 < \lambda \leqslant 1$, the third Lyapunov condition will be satisfied within the sphere $|\xi| \leqslant a$. But then it must be satisfied for the entire surface S. To see this, suppose that ζ and ξ are two arbitrary points on the surface S, and that ψ is the angle between the normals at these two points. We connect the points ζ and ξ by a line lying entirely on the surface S. Let us partition this line into segments $1,2,\ldots,\alpha,\ldots,n$ by points that are at a distance no greater than a from each other, whereupon the inequalities of the form (31) are satisfied for two neighbouring points. We denote by $\gamma_{3\alpha}$ the angle between the normals at the end of the segment whose number is α, and we denote by $|\xi|_\alpha$ the difference between the ends of the segment. We can then write the obvious inequality

$$\sum_{\alpha=1}^{n} \gamma_{3\alpha} \geqslant \psi , \qquad \sum_{\alpha=1}^{n} |\xi|_\alpha^\lambda \geqslant |\zeta - \xi|^\lambda \quad \text{when} \quad 0 < \lambda \leqslant 1 ,$$

from which, when we take into account the inequalities of the form (31) for each of the segments α, the validity of the third Lyapunov condition follows.

Thus, to satisfy the third Lyapunov condition, we need to add to the assumptions of section 1 of Chapter XVII the assumption that the first derivatives satisfy Hölder's condition (30) with $0 < \lambda \leqslant 1$. We shall assume below that this assumption is satisfied.

Let us determine certain inequalities that follow from Hölder's condition for the derivatives f_1 and f_2.

It follows from formulae (28) that by choosing positive numbers a and A suitably, we can arrange to have

$$|\cos\gamma_1| < A|\xi - \zeta|^\lambda , \qquad |\cos\gamma_2| < A|\xi - \zeta|^\lambda \quad \text{when} \quad |\xi - \zeta| < a . \tag{32}$$

We note that, from the formula for finite increments,

$$\xi_3 = f(\xi_1,\xi_2) = f_1(\theta\xi_1,\theta\xi_2)\xi_1 + f_2(\theta\xi_1,\theta\xi_2)\xi_2 , \tag{33}$$

where θ is a number lying between zero and unity. Since, obviously, $|\xi_1| \leqslant |\xi|$, $|\xi_2| \leqslant |\xi|$, for suitably chosen positive a and A we obtain

$$|\xi_3| < A|\xi|^{1+\lambda} . \tag{34}$$

Problem

Suppose that S is a two-sided surface satisfying the Lyapunov conditions, and that S is not closed. Let us consider as being positive the solid angles subtended by sections of one of the sides of the surface, and as negative those subtended by sections of the other side. Show that

$$\iint\limits_S \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS_\xi = -\omega \qquad (r \equiv |x-\xi|) ,$$

where ω is the solid angle subtended by the surface S at the point x.

7. *The convergence and the continuous dependence of improper integrals on parameters*

Suppose that $F(\xi, x)$ is a function of the coordinates of the point ξ on a Lyapunov surface S that is parametrically dependent on the coordinates of some point x. Let us assume that for $\xi \neq x$, the function $F(\xi, x)$ is continuous and that, in some neighbourhood of the point $\xi = x$,

$$|F(\xi,x)| \leqslant \frac{B}{r^{2-\lambda}} ,$$

where $r \equiv |\xi - x|$ is the distance between the points ξ and x, and where λ and B are positive constants. Let us show that under this assumption the integral

$$\iint\limits_S F(\xi, x)\, dS_\xi$$

converges absolutely. It is sufficient to show that the integral

$$\iint\limits_S \frac{dS}{r^{2-\lambda}} \tag{35}$$

converges, from which the absolute convergence of the above integral obviously follows.

If the point x does not lie on the surface S, the integral (35) is proper and hence finite. However, if $x = \zeta$, where ζ is a point on S, we set up a coordinate system with origin at the point ζ and with axis 3 directed along the normal to the surface S at the point ζ. As we have seen, on an element s of the surface S that lies within the sphere $|\xi| \leqslant a$ (whose radius a is sufficiently small), we have the inequality (29):

$$\cos \gamma_3 > C > 0 ,$$

where γ_3 is the angle between the normals at the points ζ and $\xi \in s$. Therefore, on the element s,

$$dS_\xi = \frac{d\xi_1\, d\xi_2}{\cos \gamma_3} < \frac{1}{C}\, d\xi_1\, d\xi_2 .$$

Therefore,

$$\iint_S \frac{dS}{r^{2-\lambda}} \leqslant \frac{1}{C} \iint_S \frac{d\xi_1\, d\xi_2}{r^{2-\lambda}} ,$$

For positive λ, we know from a well-known convergence test * that the integral on the right side converges. Hence, the integral on the left side also converges. The convergence of the integral (35) then follows from the fact that the corresponding integral with respect to $S - s$ is proper and therefore bounded. This proves our assertion.

Let us now suppose that x is a point in some region G that intersects a Lyapunov surface S. The region G may have an arbitrary number of dimensions.

The integral

$$\iint_S F(\xi, x)\, dS_\xi$$

over a bounded surface S is said to *converge uniformly* at a point $x = \zeta$ in S if, for every positive ϵ, there exist regions $m(\epsilon) \in G$ and $s(\epsilon) \in S$ of the point ζ such that

$$\iint_{s(\epsilon)} |F(\xi, x)|\, dS_\xi < \epsilon \qquad \text{when} \qquad x \in m(\epsilon) .$$

Let us show that the integral

$$\iint_S F(\xi, x)\, dS_\xi$$

(if it converges at the point $x = \zeta$), defines a function of x that is continuous at that point. In other words, let us show that a sufficiently small neighbourhood $m_1(\epsilon)$ of the point ζ belonging to the region G exists such that for x belonging to $m_1(\epsilon)$, the difference

$$\iint_S F(\xi, x)\, dS_\xi - \iint_S F(\xi, \zeta)\, dS_\xi$$

will not exceed (in absolute value) an arbitrarily small positive number.

We use the inequality

$$\left| \iint_S F(\xi, x)\, dS_\xi - \iint_S F(\xi, \zeta)\, dS_\xi \right|$$

$$\leqslant \left| \iint_{S-s(\epsilon)} F(\xi,x)\, dS_\xi - \iint_{S-s(\epsilon)} F(\xi,\zeta)\, dS_\xi \right| + \iint_{s(\epsilon)} |F(\xi,x)|\, dS_\xi + \iint_{s(\epsilon)} |F(\xi,\zeta)|\, dS_\xi .$$

Because we have assumed the uniform convergence of the integral that we are considering, a neighbourhood $s(\epsilon) \in S$ of the point ζ can be chosen suffi-

* See V.I.Smirnov [1)], Vol. 2. p. 86.

ciently small, and a neighbourhood $m(\epsilon) \in G$ of the point ζ can be found, such that the last two integrals on the right side of the above inequality do not exceed an arbitrarily small positive number ϵ. Furthermore, because of the continuity of the function $F(\xi, x)$ for $\xi \neq x$, a neighbourhood $m_1(\epsilon)$ of the point ζ (constituting a portion of the neighbourhood $m(\epsilon) \in G$) exists such that, for x belonging to $m_1(\epsilon)$ and ξ belonging to $S - s(\epsilon)$,

$$|F(\xi, x) - F(\xi, \zeta)| < \frac{\epsilon}{\bar{S}},$$

where $\bar{S}$ is the area of the surface S. From the inequality

$$\iint_{S-s(\epsilon)} [F(\xi,x) - F(\xi,\zeta)] \, dS_\xi$$

$$\leqslant \iint_{S-s(\epsilon)} |F(\xi,x) - F(\xi,\zeta)| \, dS_\xi \leqslant |F(\xi,x) - F(\xi,\zeta)| \, \bar{S},$$

it follows that the first term on the right side of the original inequality does not exceed ϵ. By using the results that we have obtained, we see that for x belonging to $m_1(\epsilon)$, we have

$$\left| \iint_S F(\xi, x) \, dS_\xi - \iint_S F(\xi, \zeta) \, dS_\xi \right| < 3\epsilon .$$

Since ϵ is arbitrary, this proves our assertion.

Let us now verify the following continuity test:

The integral

$$\iint_S F(\xi, x) \, dS_\xi$$

is a continuous function of x at a point ζ belonging to the surface S if for every positive ϵ a neighbourhood $n(\epsilon)$ of the point ζ exists such that

$$|F(\xi, x)| < \frac{B}{|\xi - x|^{2-\lambda}}, \tag{36}$$

where B and λ are positive numbers and $|\xi - x|$ is the distance between the points ξ and x.

From what was said above, the integral (36) implies the absolute convergence of the integral that we are considering. Consequently, for any positive number ϵ, there exists a neighbourhood $s(\epsilon) \in S$ of the point ζ such that for all x,

$$\iint_{s(\epsilon)} |F(\xi,x)| \, dS_\xi < \epsilon .$$

This proves the uniform convergence of the integral

$$\iint_S F(\xi, x) \, dS_\xi$$

at the point $x = \zeta$. But the uniform convergence implies the continuity of the integral at the point $x = \zeta$, as was asserted.

8. *The behaviour of a single-layer potential and of its normal derivatives upon crossing the layer*

If we apply the test that we have just verified for the continuity of improper integrals to the single-layer potential

$$\overline{U}(x) = \iint_S \frac{\overline{\rho}}{r}\,\mathrm{d}S_\xi\ ,$$

we see that this potential is continuous at all points of the layer and, consequently, throughout all space.

Let us find the derivative of the potential $\overline{U}(x)$ with respect to an arbitrary direction ν. Formally differentiating $1/r$ under the integral sign, we obtain

$$\frac{\partial \overline{U}}{\partial \nu} = \iint_S \overline{\rho}\,\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{1}{r}\right)\mathrm{d}S_\xi = \iint_S \frac{\rho}{r^2}\cos\varphi\,\mathrm{d}S_\xi\ , \tag{37}$$

where ξ is the angle between the direction of ν and the segment $\overline{\xi x}$. For all x not lying on the surface S, the integrand is continuous. Therefore, for these values of x, differentiation under the integral sign is permissible and gives the corresponding derivative of the single-layer potential.

Let us first study the behaviour of the integral (37) as the point x approaches the surface S, under the assumption that the point x approaches the point ζ on S by being displaced along the normal n at the point ζ. We denote by

$$\frac{\mathrm{d}\overline{U}(\zeta)}{\mathrm{d}n_\mathrm{e}}\ ,\qquad \frac{\mathrm{d}\overline{U}(\zeta)}{\mathrm{d}n_\mathrm{i}}$$

the limiting values of the derivative of $\overline{U}(x)$ with respect to this normal as x approaches ζ from outside S and from inside S, respectively. We denote by

$$\frac{\mathrm{d}\overline{U}(\zeta)}{\mathrm{d}n_\mathrm{o}}$$

the value of the integral (37) at the point $x = \zeta$. The quantities $\mathrm{d}\overline{U}(\zeta)/\mathrm{d}n_\mathrm{e}$ and $\mathrm{d}\overline{U}(\zeta)/\mathrm{d}n_\mathrm{i}$ are called, respectively, the outer and inner normal derivatives of the single-layer potential at the point ζ, and the quantity $\mathrm{d}\overline{U}(\zeta)/\mathrm{d}n_\mathrm{o}$ is called the direct value of the normal derivative at this point.

Let us show that the outer and inner normal derivatives and the direct value of the normal derivative of the single-layer potential exist, are uniquely defined, and are related to each other by the equations

$$\frac{\mathrm{d}\overline{U}(\zeta)}{\mathrm{d}n_\mathrm{e}} = \frac{\mathrm{d}\overline{U}(\zeta)}{\mathrm{d}n_\mathrm{o}} - 2\pi\overline{\rho}(\zeta)\ ,\qquad \frac{\mathrm{d}\overline{U}(\zeta)}{\mathrm{d}n_\mathrm{i}} = \frac{\mathrm{d}\overline{U}(\zeta)}{\mathrm{d}n_\mathrm{o}} + 2\pi\overline{\rho}(\zeta)\ . \tag{38}$$

Consider the difference

$$\iint_S \bar\rho \frac{d}{dn_0}\left(\frac{1}{r}\right) dS_\xi - \bar\rho_0 \iint_S \frac{d}{dn}\left(\frac{1}{r}\right) dS_\xi$$

$$\equiv \iint_S \bar\rho \left(\frac{d}{dn_0} - \frac{d}{dn}\right)\frac{1}{r}\, dS_\xi + \iint_S (\bar\rho - \bar\rho_0)\frac{d}{dn}\left(\frac{1}{r}\right) dS_\xi\ , \tag{39}$$

where d/dn_0 denotes differentiation with respect to the direction of the outward normal to S at the point ζ, d/dn denotes differentiation with respect to the outward normal at a variable point ξ of the surface S, and

$$\bar\rho \equiv \bar\rho(\xi)\ , \qquad \bar\rho_0 \equiv \bar\rho(\zeta)\ .$$

Let us show that the difference (39) is continuous at the point ζ. It will then follow that the integral

$$\iint_S \bar\rho \frac{d}{dn_0}\left(\frac{1}{r}\right) dS_\xi$$

in which we are interested has the same discontinuities as does the function

$$\bar\rho_0 \iint_S \frac{d}{dn}\left(\frac{1}{r}\right) dS\ .$$

Differentiating $1/r$ with respect to the directions of the normals n_0 and n, we obtain

$$\frac{d}{dn_0}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\cos\alpha\ , \qquad \frac{d}{dn}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\cos\beta\ , \tag{40}$$

where α and β are the angles between the segment $\overline{\xi x}$ and the direction of the normals n_0 and n, respectively (fig. 47). Hence,

$$\left(\frac{d}{dn_0} - \frac{d}{dn}\right)\frac{1}{r} = -\frac{1}{r^2}(\cos\alpha - \cos\beta) = \frac{2}{r^2}\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta)\ . \tag{41}$$

Let us consider the trihedral angle with vertex at the point ξ formed by the rays n_0 and n and by the segment $\overline{\xi x}$. From a well-known property of trihedral angles,

$$|\alpha-\beta| \leq |\psi|\ , \tag{42}$$

where ψ is the angle between the rays n_0 and n (the equality sign being applicable when the rays n and n_0 and the segment $\overline{\xi x}$ all lie in the same plane). But from property (3) of Lyapunov's surfaces,

$$\psi \leq A|\xi - \zeta|^\lambda\ ,$$

where A is positive and λ is a positive number not exceeding unity. Combining this inequality with relations (41) and (42), we conclude that there exists a constant A^* such that

$$\left|\left(\frac{d}{dn_0} - \frac{d}{dn}\right)\frac{1}{r}\right| \leq A^* \frac{|\xi-\zeta|^\lambda}{r^2} = A^* \frac{1}{r^{2-\lambda}}\left(\frac{|\xi-\zeta|}{|x-\xi|}\right)^\lambda\ . \tag{43}$$

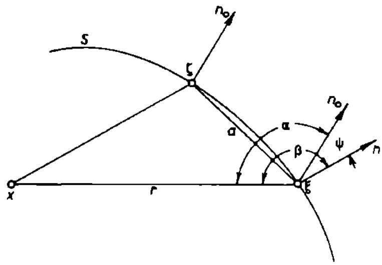

Fig. 47.

Let us show that the ratio

$$\frac{|\xi - \zeta|}{|x - \xi|}$$

remains bounded as x approaches ζ for all ξ. Let us set up a local coordinate system with origin at the point ζ and with axis 3 directed along the normal n_0. Since the point x lies on axis 3,

$$|x - \xi| \geqslant \sqrt{\xi_1^2 + \xi_2^2} = \sqrt{|\xi|^2 - \xi_3^2}\ .$$

Noting that

$$|\xi - \zeta| = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}\ ,$$

we obtain

$$\frac{|\xi - \zeta|}{|x - \xi|} \leqslant \sqrt{1 + \frac{\xi_3^2}{|\xi|^2 - \xi_3^2}}\ .$$

From formula (34), we have

$$\xi_3^2 < A^2 |\xi|^{2(1+\lambda)}\ ,$$

where λ is positive. Therefore,

$$\frac{|\xi - \zeta|}{|x - \xi|} < \sqrt{1 + \frac{\xi_3^2}{|\xi|^2\,(1 - |\xi|^{2\lambda})}} < \sqrt{1 + \frac{1}{1 - |\xi|^{2\lambda}}}\ .$$

The right side of this inequality can be made arbitrarily close to $\sqrt{2}$ for sufficiently small $|\xi|$. On the basis of inequality (43), we now conclude that there exists a positive number B such that

$$\left|\left(\frac{d}{dn_0} - \frac{d}{dn}\right)\frac{1}{r}\right| < \frac{B}{r^{2-\lambda}}\ ,$$

and it then follows, on the basis of the continuity test for improper integrals (section 7), that the first of the integrals on the right side of eq. (39) is continuous at the point ζ.

Let us consider the second integral on the right side of this equation. We again use the point ζ for the coordinate origin. Suppose that s is an element of the surface S lying within a sphere $|\xi| \leqslant a$ of sufficiently small radius. We easily see that

$$\left|\iint_S (\bar{\rho}-\bar{\rho}_0)\frac{d}{dn}\left(\frac{1}{r}\right) dS\right| \leq \iint_S \left|(\bar{\rho}-\bar{\rho}_0)\frac{d}{dn}\left(\frac{1}{r}\right)\right| dS$$

$$\leq |\bar{\rho}-\bar{\rho}_0|_s \iint_S \left|\frac{d}{dn}\left(\frac{1}{r}\right)\right| dS = |\bar{\rho}-\bar{\rho}_0|_s \iint_S \frac{|\cos\varphi|}{r^2}\, dS\,, \tag{44}$$

where φ is the angle between the normal n at the point ξ on s and the segment $\overline{\xi x}$ and where $|\bar{\rho}-\bar{\rho}_0|_s$ is the maximum value of $|\bar{\rho}-\bar{\rho}_0|$ on s. When $x = \zeta$,

$$|\cos\varphi| = \left|\frac{\xi_1}{r}\cos\gamma_1 + \frac{\xi_2}{r}\cos\gamma_2 + \frac{\xi_3}{r}\cos\gamma_3\right| \leq |\cos\gamma_1| + |\cos\gamma_2| + \frac{|\xi_3|}{|\xi|}$$

where $\cos\gamma_1$, $\cos\gamma_2$, and $\cos\gamma_3$ are the direction cosines of the normal n. By using the inequalities (32) and (34), we find that

$$|\cos\varphi| < 3A|\xi|^{\lambda}\,,$$

and hence,

$$\frac{|\cos\varphi|}{r^2} < \frac{3A}{|\xi|^{2-\lambda}}\,.$$

Therefore, it follows on the basis of the convergence test proved in section 7 that the integral

$$\iint_S \frac{|\cos\varphi|}{r^2}\, dS$$

is bounded when $x = \zeta$. When x is not equal to ζ, this integral is bounded because the integrand is bounded. Consequently, by a suitable choice of the radius a of the sphere $|\xi| < a$, we can make the quantity $|\bar{\rho}-\bar{\rho}_0|_s$ sufficiently small so that the right side of the inequality (44) is less than an arbitrarily small positive number ϵ for any position of the point x within the sphere $|\xi| \leq \epsilon$. We conclude from this that there exists a neighbourhood $s(\epsilon)$ of the point ζ, belonging to the surface S, and another neighbourhood $m(\epsilon)$ of the point ζ, belonging to the sphere $|\xi| \leq a$, such that

$$\iint_{s(\epsilon)} \left|(\bar{\rho}-\bar{\rho}_0)\frac{d}{dn}\left(\frac{1}{r}\right)\right| dS < \epsilon \qquad \text{when} \qquad x \in m(\epsilon)\,.$$

Here, the function $(\bar{\rho}-\bar{\rho}_0)d(1/r)/dn$ is continuous on $S - s(\epsilon)$. It then follows that the second of the integrals on the right side of (39) converges uniformly and is therefore continuous at the point ζ.

Thus, the difference (39) is a continuous function of the point x belonging to n, as a result of which the integrals

$$\iint_S \bar{\rho}\frac{d}{dn_0}\left(\frac{1}{r}\right) dS \quad \text{and} \quad \rho_0 \iint_S \frac{d}{dn}\left(\frac{1}{r}\right) dS$$

change their value by the same amount when x crosses the surface S.

To compute the last integral, we use the fundamental formula of the theory of harmonic functions (44) of Chapter XVIII. Setting

$$u = 1 \,, \qquad L(\xi, x) = \frac{1}{4\pi}\,\frac{1}{r} \,, \qquad FV = S$$

in this formula, we obtain *Gauss' formula*

$$\iint_S \frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{1}{r}\right) \mathrm{d}S = \begin{cases} -4\pi & \text{when} \quad x \text{ is inside } S \,, \\ -2\pi & \text{when} \quad x \text{ is on } S \,, \\ 0 & \text{when} \quad x \text{ is outside } S \,, \end{cases} \tag{45}$$

from which formulae (38) follow directly.

Problem

Show that the direct value of the derivative of the single-layer potential on S is a continuous function on S.

9*. *The tangential derivatives of the single-layer potential and its derivatives in an arbitrary direction*

Let us suppose that the direction τ is parallel to the tangent plane to a Lyapunov surface S at the point ζ, and that n_0 is the normal to S at this point. Suppose that a point x, which is being displaced along the normal n_0 in one direction, approaches S. The limit of the integral

$$\frac{\mathrm{d}U}{\mathrm{d}\tau} \equiv \iint_S \bar{\rho}\, \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{r}\right) \mathrm{d}S \tag{46}$$

as x approaches ζ is called the *tangential derivative* of the single-layer potential at the point ζ. If the point x approaches the surface S from its inner side, we shall call this derivative the inner derivative and, in the opposite case, we shall call it the outer derivative.

It is easy to show that the inner and outer tangential derivatives of the single-layer potential both exist or both fail to exist. To prove their existence, apart from the assumption that the density $\bar{\rho}$ is continuous, we need to make additional assumptions as to the behaviour of the density in the neighbourhood of the point ζ. These assumptions are of various types and are sufficient. The necessity of some of them has not been proved.

Let us show that if the single-layer density in the neighbourhood of a point ζ satisfies the Hölder condition

$$\frac{|\bar{\rho}(\zeta) - \bar{\rho}(\eta)|}{|\zeta - \eta|^{\lambda_1}} < A_1 \,, \tag{47}$$

where A_1 and λ_1 are positive constants and ζ and η are two arbitrary points in the neighbourhood in question, then the outer and inner tangential derivatives at the point ζ coincide and are continuous functions of ζ on the surface S.

To prove this, we draw the normal n_0 at the point ζ and we construct a cylinder C, of radius and altitude b, whose axis is the normal n_0 and whose

geometric center is the point ζ. From property (c) of Lyapunov's surfaces, the quantity b can be chosen sufficiently small that straight lines parallel to the normal n_0 intersect an element s (cut from the surface S by the cylinder in question) only once, since this cylinder can be completely contained in a sphere of radius $b\sqrt{2}$ with center at ζ. Under this condition, integration over the element s can be replaced by integration over the element s' cut by the cylinder in question from the plane tangent to S at the point ζ. We choose the radius b sufficiently small that the angle ψ between the normals to S at the point ζ and at an arbitrary point on the element s does not exceed $\frac{1}{8}\pi$. Then, the surface S intersects the cylinder C on its lateral surface but not on its base, and the element s' on the tangential plane is a circle.

When we replace the integration over the element s with integration over the circle s', we obtain

$$\iint_S \bar{\rho} \frac{d}{d\tau}\left(\frac{1}{r}\right) dS = \iint_{s'} \bar{\rho} \frac{d}{d\tau}\left(\frac{1}{r}\right) \frac{dS'}{\cos\psi}. \tag{48}$$

Let us show that when the element s is plane ($\cos\psi = 1$) and when the density $\bar{\rho}$ has a constant value ρ_0 on the element s, this integral (for $x \neq \zeta \in S$) is identically equal to zero, and that it is conditionally convergent at the point $x = \zeta$.

Let us take the point ζ for the origin of a rectangular coordinate system, and let us direct axis 1 in the direction τ and axis 3 along the axis of the cylinder C. The plane of axes 1 and 2 will then coincide with the plane K that is tangent to the surface S at the point ζ. Remembering that

$$r = |\xi - x| = \sqrt{\xi_1{}^2 + \xi_2{}^2 + x_3{}^2} \quad \text{when} \quad \xi \in K,\ x \in n_0\,,$$

$$\frac{d}{d\tau}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x_1}\left(\frac{1}{r}\right) = \frac{\xi_1 - x_1}{r^3} = \frac{\xi_1}{(\xi_1{}^2 + \xi_2{}^2 + x_3{}^2)^{\frac{3}{2}}}\,,$$

we obtain

$$\iint_{s'} \rho_0 \frac{d}{d\tau}\left(\frac{1}{r}\right) dS = \rho_0 \iint_{s'} \frac{\xi_1}{(\xi_1{}^2 + \xi_2{}^2 + x_3{}^2)^{\frac{3}{2}}}\, b\xi\,,\ d\xi_2\,.$$

Let us suppose that x_3 is not zero. If we partition the circle s' into two halves, with ξ_1 positive on one half, and negative on the other, and if we note that the integrals over these semicircles are equal in absolute value but opposite in sign, we conclude that, for all $x_3 \neq 0$, the integral in question is equal to zero. Therefore, in particular, the integral

$$\lim_{x_3 \to 0} \iint_{s'} \frac{\xi_1}{(\xi_1{}^2 + \xi_2{}^2 + x_3{}^2)^{\frac{3}{2}}}\, d\xi_1\, d\xi_2 = 0\,. \tag{49}$$

When $x = \zeta$, we have $x_3 = 0$ and we must examine the improper integral

$$\iint_{s'} \frac{\xi_1}{(\xi_1{}^2 + \xi_2{}^2)^{\frac{3}{2}}}\, d\xi_1\, d\xi_2\,.$$

Suppose that s'' is some neighbourhood of the point $\xi = 0$ that belongs to the circle s'. From the definition of an improper integral, the integral in question is equal to the limit of the sequence of proper integrals

$$\iint\limits_{s'-s''} \frac{\xi_1}{(\xi_1^2 + \xi_2^2)^{\frac{3}{2}}} \, d\xi_1 \, d\xi_2$$

when the neighbourhood s'' is contracted in an arbitrary manner to the point $\xi = 0$. But this limit does not exist because the integral

$$\iint\limits_{s'} \frac{|\xi_1|}{(\xi_1^2 + \xi_2^2)^{\frac{3}{2}}} \, d\xi_1 \, d\xi_2$$

diverges and, consequently, the value (even the existence) of the limit that we are interested in will depend on the choice of the way in which the neighbourhood s'' is contracted to the point $\xi = 0$. However, for any *one* way, we can arrive at a definite finite value for the limits that we are interested in. In this sense, we shall say that the integral in question is *conditionally convergent*. In particular, by choosing for boundaries of the neighbourhood s'' those neighbourhoods whose centers are at the point $\xi = 0$, we see, without difficulty, that the integral converges conditionally to zero.

Let us now turn to the general case ($\cos \psi \neq 1$) and let us denote by r, as before, the distance between the points $x \in n_0$ and $\xi \in S$, and by r' the distance between the point x and the projection ξ' of the point ξ on the plane formed by axes 1 and 2. Let us consider the difference

$$\iint\limits_{s'} \bar{\rho} \frac{d}{d\tau}\left(\frac{1}{r}\right) \frac{dS}{\cos\psi} - \bar{\rho}_0 \iint\limits_{s'} \frac{d}{d\tau}\left(\frac{1}{r'}\right) dS$$

$$= \iint\limits_{s'} (\bar{\rho} - \bar{\rho}_0) \frac{d}{d\tau}\left(\frac{1}{r}\right) \frac{dS}{\cos\psi} + \iint\limits_{s'} \frac{1-\cos\psi}{\cos\psi} \bar{\rho}_0 \frac{d}{d\tau}\left(\frac{1}{r}\right) dS$$

$$+ \bar{\rho}_0 \iint\limits_{s'} \frac{d}{d\tau}\left(\frac{1}{r} - \frac{1}{r'}\right) dS\,, \tag{50}$$

where

$$\bar{\rho}_0 \equiv \bar{\rho}(\zeta)\,.$$

Recalling that

$$\frac{d}{d\tau}\left(\frac{1}{r}\right) = -\frac{\cos\gamma}{r^2}\,,$$

(where γ is the angle between the direction τ and the segment $\overline{\xi x}$) and also considering inequality (47) and the condition for the angle ψ on a Lyapunov surface, the reader can show without difficulty that the first two integrals on the right side of the relationship (50) are continuous at the point ζ (using the continuity test of section 7). Let us turn to the last of the integrals on the right side. We easily see that

$$\frac{d}{d\tau}\left(\frac{1}{r} - \frac{1}{r'}\right) = \frac{\partial}{\partial x_1}\left(\frac{1}{r} - \frac{1}{r'}\right) = (\xi_1 - x_1)\left(\frac{1}{r^3} - \frac{1}{r'^3}\right) = \xi_1 \frac{r'-r}{r'r}\left(\frac{1}{r^2} + \frac{1}{r'^2} + \frac{1}{rr'}\right).$$

From inequality (34) we obtain, for sufficiently small values of the radius b,

$$r, r' \geqslant |\xi'| = \sqrt{\xi_1^2 + \xi_2^2} = \sqrt{|\xi|^2 - \xi_3^2} = |\xi| \sqrt{1 - \frac{\xi_3^2}{|\xi|^2}} > |\xi|\sqrt{1 - A|\xi|^2} > \frac{|\xi|}{A_1},$$

where A and A_1 are finite positive numbers. It also follows from the triangle $\zeta\xi\xi'$ (fig. 48) that $|r - r'| \leqslant |\xi_3|$, from which, on the basis of inequality (34), we have the inequality

$$|r - r'| \leqslant A|\xi|^{1+\lambda}.$$

Finally, when we recall the obvious inequality

$$|\xi_1| \leqslant |\xi|, \qquad |\xi'| \leqslant |\xi|,$$

we obtain, for sufficiently small b,

$$\left|\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{r} - \frac{1}{r'}\right)\right| < |\xi| \frac{AA_1^2|\xi|^{1+\lambda}}{|\xi|^2} \frac{3A_1^2}{|\xi|^2} < \frac{B}{|\xi|^{2-\lambda}} \leqslant \frac{B}{|\xi'|^{2-\lambda}},$$

where B is a finite positive number. Thus,

$$\iint_{s'} \left|\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{r} - \frac{1}{r'}\right)\right| \mathrm{d}S < B\iint_{s'} \frac{\mathrm{d}S}{|\xi|^{2-\lambda}} = B\iint_{s'} \frac{\mathrm{d}\xi_1\, \mathrm{d}\xi_2}{(\xi_1^2 + \xi_2^2)^{2-\lambda}}.$$

The integral on the right side of this inequality converges absolutely. Therefore, for any positive ϵ, there is a neighbourhood $s'(\epsilon)$ of the point ζ such that

$$\iint_{s'(\epsilon)} \left|\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{r} - \frac{1}{r'}\right)\right| \mathrm{d}S < \epsilon$$

for an arbitrary position of the point $x \in n_0$. Since the functions $1/r$ and $1/r'$ have a discontinuity only at the point ζ, the function $\mathrm{d}(1/r - 1/r')/\mathrm{d}\tau$ is continuous outside the neighbourhood $s'(\epsilon)$ and on its boundary. It then follows that the third integral on the right side of the relationship (50) con-

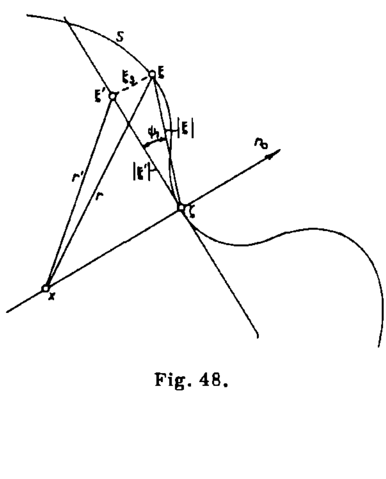

Fig. 48.

verges uniformly at the point ζ and is therefore continuous at that point. Thus, the difference (50) is continuous, so that the integral

$$\iint_S \bar{\rho} \frac{d}{d\tau}\left(\frac{1}{r}\right) dS \tag{51}$$

as s approaches $\zeta \in S$ can have only the same discontinuities as does the integral

$$\bar{\rho}_0 \iint_{S'} \frac{d}{d\tau}\left(\frac{1}{r'}\right) dS\,. \tag{52}$$

Since, for all x_3 different from zero, the last integral exists and is equal to zero, and since it approaches zero as x_3 approaches zero, it follows that the values of the integral (51) and, hence, of the integral (46) become arbitrarily close to each other and approach a common limit as x approaches ζ from within and from without the surface S.

Let us finally note that the above estimates of the integrals depend on the properties of the Lyapunov surface and the density $\bar{\rho}$, independently of the position of the point ζ on the surface; that is, they are uniform with respect to the position of the point ζ on S. But, as was shown in section 7 of Chapter XIX, the convergence of an improper integral depending on a parameter implies its continuity on a surface S. Therefore, we may assert that the tangential derivatives are continuous functions of the point ζ on the surface S. This proves the assertion made.

We note that at the point $x = \zeta$, the integral (51) converges conditionally, since at this point the integral (52) converges conditionally. Therefore, at $x = \zeta$, the value of the integral (51), regarded as the limit

$$\lim_{s\to 0} \iint_{S-s} \bar{\rho} \frac{d}{d\tau}\left(\frac{1}{r}\right) dS$$

(s being a neighbourhood of the point ζ), depends on the way in which s approaches zero. Therefore, the *direct* value of the tangential derivative does not exist, although definite outer and inner tangential derivatives do exist and, as we have shown, are equal to each other. In other words, the integral (46) has a removable discontinuity when the point x crosses the surface S.

Let us now suppose that the point x approaches the surface S while remaining inside S and that, at each of its positions, the derivative $\partial\overline{U}/\partial l$ is defined in a constant direction l. The limit $\partial\overline{U}/\partial l_1$ of the derivative $\partial\overline{U}/\partial l$ is called the inner derivative in the direction l (at the given point of the surface). The outer derivative in the direction l at the point on the surface is defined analogously.

Let us show that if the single-layer density on a Lyapunov surface S satisfies Hölder's condition (47), then, as the surface S is approached, the derivatives of the single-layer potential in an arbitrary fixed direction approach definite limits, independently of the way in which the surface is approached. (However, they may differ for the cases of approach from within and from without the surface S.)

We shall prove this theorem for the case in which the layer is approached from outside S. The proof is analogous in the case of approach from within S.

Suppose that x is an arbitrary point outside S and that n is the normal to S that passes through the point x. Let us write the outer derivatives of the potential of the point x in an arbitrary fixed direction l in the form

$$\frac{\partial \overline{U}}{\partial l} = \frac{d\overline{U}}{dn} \cos(n, l) + \frac{d\overline{U}}{d\tau} \cos(\tau, l) ,$$

where the symbols d/dn and $d/d\tau$ denote differentiation in the direction n and in a suitably chosen direction τ perpendicular to n. On the basis of property (b) of Lyapunov surfaces, this representation is unique close to S. Therefore, from the properties proven above for the derivatives $d\overline{U}/dn$ and $d\overline{U}/d\tau$, and also from the problem at the end of section 4, it follows that as the surface S is approached along its normal, the limits of the derivatives in the arbitrary fixed direction l exist and are a continuous function of the point $\xi \in S$.

Let us now denote by n_ξ the normal to S at the point $\xi \in S$. Suppose that $D(\xi)$ is the limit that the derivative $\partial\overline{U}/\partial l$ approaches as the point ξ is approached along the normal n_ξ. For any positive ϵ, there exists a positive number η such that

$$\left|\frac{\partial \overline{U}(x)}{\partial l} - D(\xi)\right| \leq \epsilon \quad \text{when} \quad |x - \xi| < \eta . \tag{53}$$

For if we denote by s that part of the surface S lying within the sphere $|\xi - \zeta| \leq \eta$, we can choose η sufficiently small so that

$$\left|\left(\frac{\partial \overline{U}(x)}{\partial l}\right)_{x \in S} - D(\xi)\right| \leq \epsilon \quad \text{when} \quad x, \xi \in S .$$

Since the function $d(\xi)$ is continuous, the radius η can be chosen sufficiently small so that

$$|D(\xi) - D(\zeta)| \leq \tfrac{1}{2}\epsilon \quad \text{when} \quad \xi, \zeta \in s .$$

The inequality (53) follows from the last two inequalities, which completes the proof of the theorem.

Problem

Prove the formula

$$\frac{\partial \overline{U}(\zeta)}{\partial l_i} - \frac{\partial \overline{U}(\zeta)}{\partial l_e} = 4\pi\overline{\rho}(\zeta) \cos(n,l) .$$

10. The behaviour of the double-layer potential when the layer is crossed

Suppose that ζ is an arbitrary fixed point of a double layer situated on a Lyapunov surface S and that

$$\bar{\bar{\rho}}_0 = \bar{\bar{\rho}}(\zeta)$$

is its density at the point ζ. Let us consider the difference

$$\bar{\bar{\rho}}_0 \iint_S \frac{d}{dn}\left(\frac{1}{r}\right) dS - \iint_S \bar{\bar{\rho}} \frac{d}{dn}\left(\frac{1}{r}\right) dS = \iint_S (\bar{\bar{\rho}}_0 - \bar{\bar{\rho}}) \frac{d}{dn}\left(\frac{1}{r}\right) dS \tag{54}$$

when the point x coincides with the point ζ. Let us suppose that the density $\bar{\bar{\rho}}$ of the layer is continuous. Then, the integral on the right side is known to be continuous. For if s is a neighbourhood of the point ζ and is contained in S,

$$\left|\iint_s (\bar{\bar{\rho}}_0 - \bar{\bar{\rho}}) \frac{d}{dn}\left(\frac{1}{r}\right) dS\right| \leqslant |\bar{\bar{\rho}}_0 - \bar{\bar{\rho}}|_s \iint_s \left|\frac{d}{dn}\left(\frac{1}{r}\right)\right| dS ,$$

where

$$|\bar{\bar{\rho}}_0 - \bar{\bar{\rho}}|_s$$

is the maximum absolute value of the difference

$$\bar{\bar{\rho}}_0 - \bar{\bar{\rho}}$$

on s. As is shown in section 8, the integral on the right side of this inequality is bounded. Therefore, because of the assumption that the function $\bar{\bar{\rho}}$ is continuous, for any positive ϵ, the neighbourhood s can always be chosen sufficiently small so that

$$\left|\iint_{s(\epsilon)} (\bar{\bar{\rho}}_0 - \bar{\bar{\rho}}) \frac{d}{dn}\left(\frac{1}{r}\right) dS\right| < \epsilon .$$

Since the function

$$(\bar{\bar{\rho}}_0 - \bar{\bar{\rho}}) \frac{d}{dn}\left(\frac{1}{r}\right)$$

is continuous on the boundaries of the neighbourhood $s(\epsilon)$, the integral that we are concerned with converges uniformly at the point ζ and is therefore continuous at that point. Consequently, the discontinuities that the integrals on the left side of eq. (27) may have when the point x crosses the layer must be the same. By using Gauss' formula (45), we conclude that the double-layer potential, as an arbitrary point ζ of the layer is approached from without and within S, approaches, respectively, the values

$$\bar{\bar{U}}_{2e}(\zeta) = \bar{\bar{U}}(\zeta) + 2\pi\bar{\bar{\rho}}_0 , \qquad \bar{\bar{U}}_{2i}(\zeta) = \bar{\bar{U}}(\zeta) - 2\pi\bar{\bar{\rho}}_0 , \tag{55}$$

where

$$\bar{\bar{U}}(\zeta) \equiv \iint_S \bar{\bar{\rho}} \frac{d}{dn}\left(\frac{1}{r}\right) dS ,$$

which is a function that is continuous on S. We shall call this function the *direct value* of the double-layer potential.

Let us note a formal property of formulae (55) and (38). Investigation of the behaviour of the derivatives of the double-layer potential on Lyapunov surfaces is considerably more complicated than in the single-layer case. Therefore, we confine ourselves to noting that, under the assumption that we have been making with regard to the smoothness of the function $\bar{\bar{\rho}}$, it can be shown * that, when a Lyapunov surface S on which there is a double layer is crossed, the *normal* derivatives of the double-layer potential remain continuous, whereas the *tangential* derivatives undergo a discontinuity. The value of this discontinuity is such that the outer tangential derivative is less by an amount $2\pi\partial\bar{\bar{\rho}}(\zeta)/\partial\tau$ (and the inner tangential derivative greater by the same amount) than the direct value of the tangential derivative of the point ζ.

Problem (Lyapunov's example)

Show that satisfaction of the Hölder condition

$$\frac{|\bar{\bar{\rho}}(\xi)-\bar{\bar{\rho}}(\zeta)|}{|\xi-\zeta|^{\lambda_1}} < A_1\,, \qquad A_1 > 0\,, \qquad \lambda_1 > 0\,, \qquad \xi,\eta \in S$$

is not sufficient for the existence of normal derivatives of a double layer. To do this, examine a Lyapunov surface S, a part of which, S_1, is *plane*. Take a circle Σ on S_1 with center at the point ζ and assume that

$$\bar{\bar{\rho}} = A_1|\zeta-\xi|^{\lambda_1} \quad \text{when} \quad \xi \in \Sigma\,.$$

Then, integrate over Σ to find an expression for the double-layer at points close to Σ and, by differentiating this expression, show that the normal derivatives of the potential close to Σ increase without bound.

* See Gyunter [18)], Chapter II, section 10.

Chapter XX

ELEMENTS OF THE THEORY OF LOGARITHMIC POTENTIAL

1. *Logarithmic potential*

In this chapter, we shall examine very briefly the development of potential theory in a plane.

Let us examine a mass point having unit mass which is attracted to a segment $\overline{ab}$ according to Newton's law. Let us take rectangular coordinate axes as shown in fig. 49 and let us denote by r the distance from the point x to the segment $\overline{ab}$.

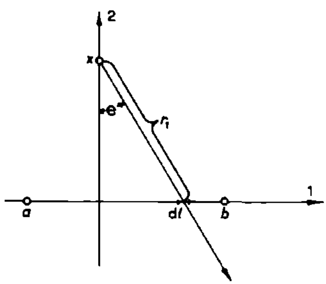

Fig. 49.

It is easy to see that an element dl of the straight line segment attracts the point x with a force dF given by the formula

$$dF = k\,\frac{\rho_1\,dl}{r_1{}^2} = k\,\frac{\rho_1\,d\theta}{r_1\cos\theta} = k\,\frac{\rho_1\,d\theta}{r}\,,$$

where k is a proportionality constant, ρ_1 is the mass density along the straight line, and r_1 is the distance from the point x to the element dl. It follows immediately that the projections of the force dF onto the coordinate axes are given by the formulae

$$dF_1 = \frac{k\rho_1\sin\theta\,d\theta}{r}\,,\qquad dF_2 = -\,\frac{k\rho_1\cos\theta\,d\theta}{r}\,.$$

We denote by θ_0 and θ_1 those values of the angle θ that correspond to the ends of the segment $\overline{ab}$. Then, the projections of the force of attraction of the *entire* segment $\overline{ab}$ are given by the equations

$$F_1 = \frac{k\rho_1}{r} \int_{\theta_0}^{\theta_1} \sin\theta \, d\theta = \frac{k\rho_1}{r} (\cos\theta_0 - \cos\theta_1) ,$$
$$F_2 = -\frac{k\rho_1}{r} \int_{\theta_0}^{\theta_1} \cos\theta \, d\theta = \frac{k\rho_1}{r} (\sin\theta_0 - \sin\theta_1) . \tag{1}$$

Let us now suppose that the segment $\overline{ab}$ is sufficiently long that we may consider it as extending infinitely far in both directions. Evidently, in this case,

$$\theta_0 = -\tfrac{1}{2}\pi , \qquad \theta_1 = \tfrac{1}{2}\pi ,$$

and formulae (1) become

$$F_1 = 0 , \qquad F_2 = -\frac{2k\rho_1}{r} .$$

It is obvious from this that an infinitely long heavy straight line attracts a unit positive mass with a force F given by

$$F = -\frac{2k\rho_1}{r} .$$

This example shows that in certain cases *the interaction between mutually attracting bodies is inversely proportional to the distance.* In other words, if a mass m attracts a unit mass according to this law, the force of attraction F is expressed by the formula

$$F = m/r ,$$

where r is the distance between the two attracting masses.

It is easy to see that in this case the components of the attracting force are partial derivatives of the function

$$U = m \ln \frac{1}{r} ,$$

which is called the *logarithmic potential*.

Let us now suppose that attracting masses of constant density ρ_1 are uniformly distributed along a certain curve L. If these masses attract a unit mass situated at a point x with a force inversely proportional to the distance, the gravitational field thus formed will have a potential that can be computed from the formula

$$U = \int_L \rho_1 \ln \frac{1}{r} \, dL , \tag{2}$$

where r is the distance from the point x to a variable point ξ on the curve L.

This potential is called the single-layer logarithmic potential, and its properties are entirely analogous to the properties of the single-layer Newtonian potential. Thus, for example, in an arbitrary region containing no point of the curve referred to, the potential (2) is a *harmonic* function.

Furthermore, it is easy to show that if we make certain assumptions concerning the curve L and the density ρ_1, this potential will be continuous even in the case in which the point x coincides with some point of the curve L.

We note, however, that as the point x is displaced to infinity, the logarithmic potential changes in a different way than does the single-layer Newtonian potential. For, denoting by R the distance from the point x to the coordinate origin, and recalling the reasoning that we used in studying a Newtonian potential at infinity, we can see that, for high values of R, the logarithmic single-layer potential is expressed by the following approximate equation:

$$U(x) = m \ln \frac{1}{R}, \tag{3}$$

where m denotes the entire mass of the attracting layer. It is clear from this formula that the logarithmic potential $U(x)$ approaches infinity as the point x is displaced infinitely far, whereas the Newtonian potential $U(x)$ in this case approaches zero.

2. *The double-layer logarithmic potential*

Let us denote by n the direction of the outward normal to the curve L and let us take the following line integral:

$$U_1(x) = \int_L \rho_2 \frac{\mathrm{d}}{\mathrm{d}n}\left(\ln \frac{1}{r}\right) \mathrm{d}L\,. \tag{4}$$

This integral is called *the double-layer logarithmic potential* and the function ρ_2 appearing in it is called the *density* of the double layer.

In its definition and properties, the logarithmic potential is completely analogous to the double-layer Newtonian potential. Obviously, in every region not containing points of the curve L, the function $U_1(x)$ is *harmonic.* Furthermore, if we note that

$$\frac{\mathrm{d}}{\mathrm{d}n}\left(\ln \frac{1}{r}\right) = \frac{\cos \varphi}{r}, \tag{5}$$

where φ is the angle between the directions n and r (fig. 50), we may also give this potential in the form

$$U_1(x) = \int_L \frac{\cos \varphi}{r} \mathrm{d}L \tag{6}$$

and show that, on the curve, the potential U_1 has a discontinuity completely analogous to the discontinuity of the double-layer Newtonian potential. If L is a closed curve satisfying conditions analogous to the Lyapunov conditions for surfaces and, in particular, if L has a definite tangent at each point, then the discontinuity referred to above is characterized by the following equations:

$$U_{1i} = U_{10} - \pi\rho_{20}\,, \qquad U_{1e} = U_{10} + \pi\rho_{20}\,, \tag{7}$$

where U_{10} and ρ_{20} are the direct value of the potential U_1 and the value of the density ρ_2 at some point ζ lying on the curve L, and where U_{1i} and U_{1e} are the limiting values of the same potential in those cases in which the point x tends to the point ζ, approaching it from within or from without the curve L, respectively.

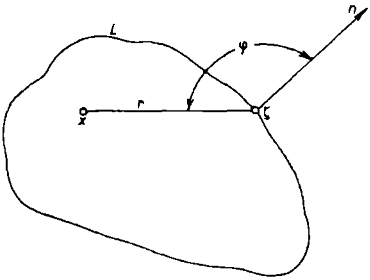

Fig. 50.

In this particular case, the integral

$$\int_L \frac{\cos\varphi}{r}\,\mathrm{d}L\,, \tag{8}$$

which is analogous to the integral in Gauss' formula, has three different values

$$-2\pi\,, \qquad 0\,, \qquad -\pi$$

depending on whether x is inside, outside, or on the curve L.

Let us apply formula (8) to the solution of the interior Dirichlet problem for Laplace's equations in a circle of radius r_0. We place the pole of a polar coordinate system in the center of the circle and we direct the polar axis along axis 1. We denote by r_0, θ_0 and r', θ' the polar coordinates of the points ζ and x, and we denote by $f(r_0,\theta)$ a given function that varies continuously on the circle $r' = r_0$. The double-layer potential is then

$$U_1(r,\theta) = \int_C f(r_0,\theta)\,\frac{\cos\varphi}{r}\,\mathrm{d}C\,, \tag{9}$$

where C is the circle referred to above. Observe fig. 51. It is clear from the figure that in the case in which the point x belongs to the curve

$$-\frac{\cos\varphi}{r} = \frac{1}{2r_0}\,.$$

It immediately follows that the potential (9) has a constant value on the circle, given by

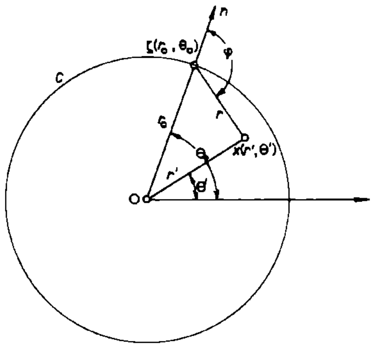

Fig. 51.

$$U_1(r_0, \theta) = -\frac{1}{2r_0} \int_C f(r_0, \theta)\, dC , \tag{10}$$

on the basis of which it is easy to show that the formula

$$U(r, \theta) = \frac{1}{\pi} \int_C f(r_0, \theta) \left(\frac{1}{2r_0} - \frac{\cos\varphi}{r}\right) dC \tag{11}$$

gives the solution to the interior Dirichlet problem. In other words, this formula defines a *harmonic* function $U(r, \theta)$ that assumes the given value $f(r_0, \theta)$ on the circle C. For if we rewrite eq. (11) in the form

$$U(r, \theta) = \frac{1}{\pi}\left[U_1(r_0, \theta) - U_1(r, \theta)\right]$$

we can easily see that $U(r, \theta)$ is a harmonic function. If we now let the point x approach the point ζ, we see that

$$\lim_{x \to \zeta} U(r, \theta) = \frac{1}{\pi}\left[U_1(r_0, \theta) - \lim_{x \to \zeta} U_1(r, \theta)\right] .$$

But from the first of formulae (7),

$$\lim_{x \to \zeta} U_1(r, \theta) = U_1(r_0, \theta) - \pi f(r_0, \theta) ,$$

and, consequently,

$$\lim_{x \to \zeta} U(r, \theta) = f(r_0, \theta) ,$$

which proves our assertion.

We now note that on the circle C, $dl = r_0 d\theta$. We may therefore write formula (11) in the form

$$U(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(r_0, \theta_0) \frac{r^2 - r_0^2}{r_0^2 - 2rr_0 \cos(\theta - \theta_0) + r^2}\, d\theta_0 . \tag{12}$$

This is Poisson's integral formula. It can be easily transformed to the form (76), Chapter XVIII.

3. *Discontinuity in the normal derivative of the logarithmic potential on a curve*

As in the case of the double-layer potential, the normal derivatives of the single-layer logarithmic potential are discontinuous on a curve. This discontinuity is characterized by the formulae

$$\frac{dU}{dn_i} = \pi\rho_{10} + \int_L \rho_1 \frac{\cos\psi}{r'} dL , \tag{13}$$

$$\frac{dU}{dn_e} = -\pi\rho_{10} + \int_L \rho_1 \frac{\cos\psi}{r'} dL , \tag{14}$$

which are analogous to the corresponding formulae in the theory of Newtonian potential. In these formulae, ρ_{10}, dU/dn_i, and dU/dn_e denote the density and the normal derivatives at some point ζ on the curve L. The notations r' and ψ are clear from fig. 52, on which ξ denotes a variable point of the curve L.

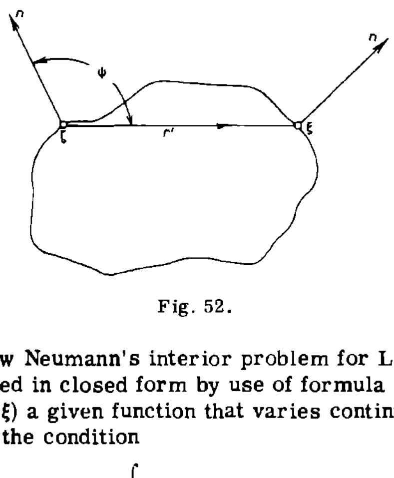

Fig. 52.

Let us show how Neumann's interior problem for Laplace's equation in a circle can be solved in closed form by use of formula (13).

We denote by $f(\xi)$ a given function that varies continuously on the circle C and that satisfies the condition

$$\int_C f(\xi)\, dC = 0 . \tag{15}$$

Let us use this function to set up the integral

$$U(x) = \frac{1}{\pi} \int_C f(\xi) \ln\frac{1}{r} dC_\xi , \tag{16}$$

where r is the distance from the point x to some point ξ on the circle C. Obviously, this integral is a harmonic function within the given circle, since it can clearly be regarded as the single-layer potential with density given by the equation

$$\rho_1(\xi) = \frac{1}{\pi} f(\xi) \; .$$

Let us now show that as x approaches the point ζ on C, the normal derivative $\mathrm{d}U/\mathrm{d}n_{\mathrm{i}}$ of the potential (16) approaches the value $f(\zeta)$.

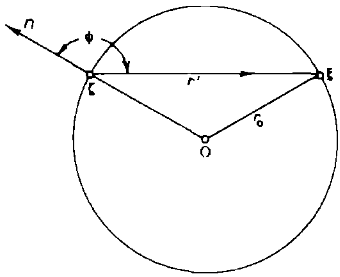

Fig. 53.

It is clear from fig. 53 that

$$\frac{\cos \psi}{r'} = - \frac{1}{2r_0} \, ,$$

where r_0 is the radius of the given circle. By using this equation and formula (13), we can easily see that

$$\frac{\mathrm{d}U}{\mathrm{d}n_{\mathrm{i}}} = f(\zeta) - \frac{1}{2r_0} \int_C f(\xi) \, \mathrm{d}C \; .$$

With eq. (15) in mind, we finally obtain

$$\frac{\mathrm{d}U}{\mathrm{d}n_{\mathrm{i}}} = f(\zeta) \; ,$$

which proves our assertion.

Thus, formula (16), which was discovered by Dini, gives the solution to the problem posed.

4. *The logarithmic potential of masses distributed over an area*

Let us suppose that some portion of a plane is filled with attracting masses of density ρ. The gravitational field caused by these masses has a potential

$$U(x) = \iint_S \rho \ln \frac{1}{r} \, d\xi_1 \, d\xi_2 , \tag{17}$$

where r is the distance from the point x to a variable point ξ in the region S filled by the attracting masses.

This potential has properties analogous to the properties that we have already investigated for the Newtonian potential of space masses. Thus, for example, at every point outside the region S, it is a harmonic function with respect to the variables x_1 and x_2. However, if the point x lies within the area of attraction, this potential will satisfy Poisson's equation:

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} = -2\pi\rho . \tag{18}$$

If L denotes some closed curve, then it is easy to show that

$$\int_L \frac{dU}{dn} \, dL = -2\pi m , \tag{19}$$

where dU/dn denotes the derivative of the potential (17) in the direction of the *outward* normal to the curve L, and m denotes that part of the entire attracting mass that is contained *within* the contour L.

Problem

Derive Poisson's integral formula (12) from the expansion of the function $f(r_0, \theta)$ in a Fourier series:

$$f(r_0, \theta) = \tfrac{1}{2}a_0 + \sum_{\alpha=1}^{\infty} (a_\alpha \cos \alpha\theta + b_\alpha \sin \alpha\theta) .$$

Chapter XXI

SPHERICAL FUNCTIONS

1. *The construction of a system of linearly independent spherical functions*

In studying the Newtonian potential in section 4 of Chapter XIX, we introduced spherical functions, defining them as the factor $Y_n(\theta,\varphi)$ in the expression for a potential of order n:

$$U_n(R,\theta,\varphi) = \frac{Y_n(\theta,\varphi)}{R^{n+1}}, \tag{1}$$

where R, θ, and φ are spherical coordinates. We showed that to every value of n there correspond no more than $2n+1$ linearly independent spherical functions, in terms of which the remaining spherical functions of order n can be expressed.

Since the potential (1) is harmonic, in order to construct a system of linearly independent spherical functions, we shall seek the solution to Laplace's equation (written in spherical coordinates):

$$\frac{\partial}{\partial R}\left(R^2\frac{\partial u}{\partial R}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = 0, \tag{2}$$

having the form (1).

We use the method of separation of variables. When we substitute the expression

$$u = v(R)\, Y(\theta,\varphi),$$

into eq. (2), we obtain

$$\frac{1}{v}\frac{\partial}{\partial R}\left(R^2\frac{\partial v}{\partial R}\right) + \frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right] = 0.$$

The first of the terms on the left side is independent of θ and φ and the second is independent of R. Therefore, the equation can be satisfied for all R, θ, and φ only if each of the terms on the left side is a constant, that is, only if

$$\frac{1}{v}\frac{\partial}{\partial R}\left(R^2\frac{\partial v}{\partial R}\right) = \lambda, \quad \frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right] = -\lambda,$$

where λ is a constant. We then obtain the two equations:

$$R^2\frac{\partial^2 v}{\partial R^2} + 2R\frac{\partial v}{\partial R} - \lambda v = 0, \tag{3}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}+\lambda Y=0\,. \tag{4}$$

The general solution to eq. (3) is equal to

$$v(R)=A_nR^n+\frac{B_n}{R^{n+1}}\,, \tag{5}$$

where the number n satisfies the equation

$$n(n+1)=\lambda\,. \tag{6}$$

For integral n and $A_n = 0$, we obtain the solutions to the Laplace equation in the form

$$u=\frac{Y_n(\theta,\varphi)}{R^{n+1}}\,, \tag{7}$$

where $Y_n(\theta,\varphi)$ is some solution to eq. (4) for $\lambda = n(n+1)$, with integral n. In our new approach, we obviously exhaust all functions $Y_n(\theta,\varphi)$ that appear in the solutions to Laplace's equation that are of the form (7). Consequently, spherical functions are solutions to the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_n}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y_n}{\partial\varphi^2}+n(n+1)\,Y_n=0 \tag{8}$$

that have continuous derivatives through the second order. These solutions we shall call *regular* and eq. (8) itself we shall call the *equation for spherical functions*.

We shall also seek solutions to eq. (8) by using the method of change of variables. By making the substitution

$$Y_n(\theta,\varphi)=P(\theta)\,Q(\varphi) \tag{9}$$

we reduce eq. (8) to the system of equations

$$\frac{d^2Q_m}{d\varphi^2}+m^2Q_m=0\,, \tag{10}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_{nm}}{d\theta}\right)+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]P_{nm}=0\,, \tag{11}$$

where m is an arbitrary number.

Solutions to eq. (10) that are single-valued and continuous in some neighbourhood are obtained for integral values of m. To every such value of m, there correspond the two linearly independent solutions

$$Q_m=\cos m\varphi \quad\text{and}\quad Q_m=\sin m\varphi \qquad (m=0,1,2,\dots)\,. \tag{12}$$

By making the substitution

$$\zeta=\cos\theta\,,$$

we reduce eq. (12) to the form

$$\frac{d}{d\zeta}(1-\zeta^2)\frac{dP_{nm}}{d\zeta}+\left[n(n+1)-\frac{m^2}{1-\zeta^2}\right]P_{nm}=0\,. \tag{13}$$

In particular, for $m = 0$, we obtain the equation

$$\frac{d}{d\zeta}(1-\zeta^2)\frac{dP_n}{d\zeta} + n(n+1)P_n = 0 , \tag{14}$$

which, from formula (4) of Chapter XV, is the equation for the Legendre polynomials $P_n(\zeta)$. In eq. (13), let us make the substitution

$$P_{nm}(\zeta) = (1-\zeta^2)^{\frac{1}{2}m} y .$$

The function y will satisfy the equation

$$(1-\zeta^2)\frac{d^2y}{d\zeta^2} - 2(m+1)\zeta\frac{dy}{d\zeta} + (n-m)(n+m+1)y = 0 . \tag{15}$$

To find the particular solutions, let us differentiate the equation for the Legendre polynomials (14) m times with respect to ζ. By applying Leibnitz' formula, we obtain

$$(1-\zeta^2)\frac{d^{m+2}P_n}{d\zeta^{m+2}} - 2(m+1)\zeta\frac{d^{m+1}P_n}{d\zeta^{m+1}} + (n-m)(n+m+1)\frac{d^mP_n}{d\zeta^m} = 0 . \tag{16}$$

By comparing this equation with eq. (15), we see that the functions

$$y = \frac{d^mP_n(\zeta)}{d\zeta^m}$$

are particular solutions to the latter equation. It is then clear that the functions

$$P_{nm}(\zeta) = (1-\zeta^2)^{\frac{1}{2}m}\frac{d^mP_n(\zeta)}{d\zeta^m} \tag{17}$$

are particular solutions to eq. (13). Returning to the variable θ, we obtain the desired particular solution to eq. (11):

$$P_{nm}(\cos\theta) = \sin^m\theta\,\frac{d^m}{d\cos\theta^m}P_n(\cos\theta) . \tag{18}$$

Since the Legendre polynomials $P_n(\cos\theta)$ are polynomials of degree n in $\cos\theta$, the functions $P_{nm}(\cos\theta)$ are also polynomials and

$$P_{nm}(\cos\theta) = 0 \quad \text{for} \quad m > n .$$

The functions $P_{nm}(\cos\theta)$ are called *associated Legendre polynomials*. Like all polynomials, they are continuous and differentiable infinitely many times.

Thus, for every n, we have $n+1$ particular solutions to eq. (11), namely,

$$P_n(\cos\theta), P_{n1}(\cos\theta), \ldots, P_{nn}(\cos\theta) ,$$

corresponding to the values $m = 0, 1, 2, \ldots$. Combining these solutions with the solutions (12) to eq. (10), we obtain the $2n+1$ spherical functions:

$$P_n(\cos\theta) , \quad P_{nm}(\cos\theta)\cos m\varphi , \quad P_{nm}(\cos\theta)\sin m\varphi \tag{19}$$

$$(m = 1, 2, 3, \ldots, n, \; n = 0, 1, 2, \ldots) ,$$

which are particular solutions to eq. (8). The $2n+1$ spherical functions are linearly independent, since the factors $\cos m\varphi$, $\sin m\varphi$ $(m = 0, 1, \ldots, n)$ are linearly independent. The functions $P_n(\cos\theta)$ are called zonal functions and the functions $P_{nm}(\cos\theta)\cos m\varphi$ and $P_{nm}(\cos\theta)\sin m\varphi$ are called *tesseral* spherical functions. See problem 4 for the origin of these terms.

As we know, there can be only $2n+1$ linearly independent spherical functions of order n. Therefore, we can represent an arbitrary spherical function $Y_n(\theta, \varphi)$ in the form of a linear combination of the linearly independent solutions (19) that we have found:

$$Y_n(\theta, \varphi) = a_0 P_n(\cos\theta) + \sum_{k=1}^{n} (a_k \cos k\varphi + b_k \sin k\varphi)\, P_{nk}(\cos\theta) ,$$

where a_0, a_k, and b_k are constants.

Problems

1. Show that the integral

$$\int_{-\pi}^{\pi} f(x_3 + \mathrm{i}\, x_1 \cos\zeta + \mathrm{i}\, x_2 \sin\zeta, \zeta)\, d\zeta ,$$

where $f(\xi, \zeta)$ is an arbitrary function that can be twice differentiated under the integral sign with respect to the parameters x_1, x_2, and x_3, is a solution to Laplace's equation.

2. Show that all second-order tesseral spherical functions can be obtained by twice differentiating $1/R$ in the directions of the coordinate axes.

3. Show that all tesseral spherical functions can be obtained from the function $1/R$ by differentiating it $n - m$ times in the direction x_3 and m times in the directions lying in the 1-2 plane at an angle π/m to each other.

4. The curves of a spherical surface along which the value of a spherical function is equal to zero are called the *nodal curves* of that spherical function.

(a) Show that the nodal lines of the Legendre polynomial $P_n(\cos\theta)$ represent the parallels dividing the spherical surface into $n+1$ zones characterized by the fact that, in each of these zones, $P_n(\cos\theta)$ retains its sign, changing sign only upon crossing a nodal line.

(b) Show that the nodal lines of the tesseral spherical functions

$$P_{nm}(\cos\theta)\cos m\varphi \quad \text{and} \quad P_{nm}(\cos\theta)\sin m\varphi$$

constitute n parallels and m equally spaced meridians which partition the spherical surface into cells (tesserae) characterized by the fact that these functions retain their sign throughout each of the cells but change it on crossing the boundary of a cell (that is, a nodal line) (see fig. 54).

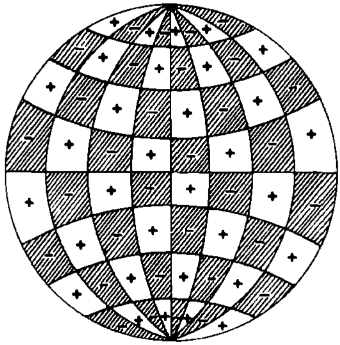

Fig. 54.

2. *The orthogonality of spherical functions*

Let us show that the spherical functions defined in the preceding section are orthogonal on the surface Σ of an arbitrary space with center at the coordinate origin (that is, that the integral of the product of two different functions (19), when integrated over the surface Σ, is equal to zero).

Let us begin with spherical functions of different orders. Suppose that $Y_k(\theta, \varphi)$ and $Y_m(\theta, \varphi)$ (with $k \neq m$) are two such functions. The functions

$$u_k = R^k Y_k(\theta, \varphi) \quad \text{and} \quad u_m = R^m Y_m(\theta, \varphi)$$

are harmonic in an arbitrary bounded neighbourhood of the coordinate origin. This is true because they are regular in an arbitrary finite region and, from formula (5) (for $B_n = 0$), they satisfy Laplace's equation. Therefore, it follows from Green's theorem (7) of Chapter XVIII that

$$\iint_\Sigma \left(u_k \frac{\partial u_m}{\partial n} - u_m \frac{\partial u_k}{\partial n}\right) dS = 0 .$$

In this case, differentiation along the normal to Σ coincides with differentiation with respect to R; that is, $\partial/\partial n = \partial/\partial R$. Therefore,

$$\iint_\Sigma \left(u_k \frac{\partial u_m}{\partial n} - u_m \frac{\partial u_k}{\partial n}\right) dS = \frac{1}{R} \iint_\Sigma (k u_m u_k - m u_m u_k)\, dS$$

$$= \frac{k-m}{R} \iint_\Sigma u_m u_k \, dS = (k-m)\, R^{k+m-1} \iint_\Sigma Y_m(\theta, \varphi)\; Y_k(\theta, \varphi)\, dS = 0 ,$$

and, since $k \neq m$,

$$\iint_\Sigma Y_m(\theta, \varphi)\; Y_k(\theta, \varphi)\, dS = 0 \qquad (m \neq k) .$$

Turning now to spherical functions (19) of the same order, we note that the surface integral over Σ can be represented in the form of an iterated integral, with the integration performed with respect to φ from 0 to 2π. But the angle φ appears in the function (19) of a single order in the factors

$$1,\ \cos\varphi,\ \sin\varphi,\ \cos 2\varphi,\ \sin 2\varphi,\ \ldots,\ \cos n\varphi,\ \sin n\varphi,$$

which form an orthogonal system in the interval $(0, 2\pi)$. Therefore, the integral of the product of an arbitrary pair of these functions from 0 to 2π is equal to zero. Consequently, the integral over Σ is equal to zero.

The following are the integrals of the squares of the spherical functions, which we present without derivation:

$$\iint_\Sigma [P_n(\cos\theta)]^2\, dS = \frac{4\pi R^2}{2n+1},$$

$$\iint_\Sigma [P_{nm}(\cos\theta)\cos m\varphi]^2\, dS = \iint_\Sigma [P_{nm}(\cos\theta)\sin m\varphi]^2\, dS = \frac{2\pi R^2}{2n+1}\,\frac{(n+m)!}{(n-m)!} \tag{20}$$

where R is the radius of the spherical surface Σ.

Let us finally derive the integral formulae that contain an arbitrary spherical function and a Legendre polynomial.

Suppose, as above, that Σ is a spherical surface with center at the coordinate origin and that $x(R_0, \theta_0, \varphi_0)$ is a point within Σ. By applying formula (44) of Chapter XVIII to the harmonic function

$$u_n(R, \theta, \varphi) = R^n Y_n(\theta, \varphi),$$

we obtain

$$u_n(R_0, \theta_0, \varphi_0) = \frac{1}{4\pi}\iint_\Sigma \left[\frac{1}{r}\frac{\partial u_n}{\partial n} - u_n\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] dS,$$

where r is the distance between the point x and a variable point $\xi(R, \theta, \varphi)$ on Σ. In this case,

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial R}, \qquad r = \sqrt{R^2 + R_0^{\,2} - 2RR_0\cos\gamma},$$

where γ is a variable angle between the radius vectors of the points x and ξ. Let us expand the function $1/r$ into the uniformly convergent series

$$\frac{1}{r} = \sum_{k=0}^{\infty} \frac{R_0^{\,k}}{R^{k+1}} P_k(\cos\gamma) \qquad (R_0 < R)$$

and let us note that, on Σ,

$$\frac{\partial u_n}{\partial n} = \frac{\partial}{\partial R} R^n Y_n(\theta, \varphi) = nR^{n-1} Y_n(\theta, \varphi),$$

$$\frac{\partial}{\partial n}\left(\frac{1}{r}\right) = \frac{\partial}{\partial R}\left(\frac{1}{r}\right) = -\sum_{k=0}^{\infty} (k+1)\frac{R_0^{\,k}}{R^{k+2}} P_k(\cos\gamma).$$

From these relationships,

$$\frac{1}{r}\frac{\partial u_n}{\partial n} - u_n \frac{\partial}{\partial n}\left(\frac{1}{r}\right) = nR^n Y_n(\theta,\varphi) \sum_{k=0}^{\infty} \frac{R_0{}^k}{R^{k+2}} P_k(\cos\gamma)$$

$$+ R^n Y_n(\theta,\varphi) \sum_{k=0}^{\infty} (k+1) \frac{R_0{}^k}{R^{k+2}} P_k(\cos\gamma)$$

$$= R^{n-2} Y_n(\theta,\varphi) \sum_{k=0}^{\infty} (n+k+1) \left(\frac{R_0}{R}\right)^k P_k(\cos\gamma).$$

When we substitute this expression into Green's theorem, we obtain

$$u_n(R_0,\theta_0,\varphi_0) = R_0{}^n Y_n(\theta_0,\varphi_0)$$

$$= \frac{R^n}{4\pi} \iint_\Sigma \sum_{k=0}^{\infty} (n+k+1) \left(\frac{R_0}{R}\right)^k Y_n(\theta,\varphi)\, P_k(\cos\gamma) \frac{ds}{R^2}\,.$$

When we integrate the uniformly convergent series on the right side of this equation termwise, we obtain the series

$$\left(\frac{R_0}{R}\right)^n Y_n(\theta_0,\varphi_0) = \sum_{k=0}^{\infty} C_k \left(\frac{R_0}{R}\right)^k , \tag{21}$$

the coefficients of which,

$$C_k = \frac{n+k+1}{4\pi} \iint_\Sigma Y_n(\theta,\varphi)\, P_k(\cos\gamma) \frac{ds}{R^2}\,, \tag{22}$$

do not depend either on R_0 or on R, since the ratio ds/R^2 remains invariant with change in R. When we equate the coefficients of like powers of the ratio R_0/R in the series (21) and apply formula (22), we obtain the formulae

$$\begin{aligned} &\iint_\Sigma Y_n(\theta,\varphi)\, P_k(\cos\gamma)\, dS = 0 \qquad (n \neq k)\,, \\ &\iint_\Sigma Y_n(\theta,\varphi)\, P_n(\cos\gamma)\, dS = \frac{4\pi R^2}{2n+1} Y_n(\theta_0,\varphi_0)\,. \end{aligned} \tag{23}$$

3. *Expansions in spherical functions*

Suppose that $f(\theta,\varphi)$ is a function of bounded variation* on the surface Σ

* A function $f(x)$ is said to be of bounded variation in an interval (a, b) in the domain of x if all sums of the form

$$\sum_{\alpha=1}^{n} |f(x_{\alpha+1}) - f(x_\alpha)|$$

remain bounded for every partition $x_1 = a$, $x_2, x_3, \ldots, x_n, x_{n+1} = b$, where $x_1, x_2, \ldots$ are increasing values of x. In particular, the function $f(x)$ is of bounded variation in the interval (a,b) when Dirichlet's conditions are satisfied: (1) The interval (a,b)

of a unit sphere and that f is absolutely integrable on Σ. Let us show that, at points of continuity, it can be expanded in a uniformly convergent series of spherical functions:

$$f(\theta,\varphi) = \sum_{k=0}^{\infty} Y_k(\theta,\varphi) . \tag{24}$$

This series is sometimes called *Laplace's series.*

To prove that the expansion (24) is possible, we rely on the theorem † on the expansion of a function in a series of Legendre polynomials.

If, in the closed interval $0 \leqslant \gamma \leqslant \pi$, the function $\psi(\gamma)\sin\gamma$ is absolutely integrable with respect to γ, and if the function $\psi(\gamma)$ is of bounded variation, then, in an arbitrary open interval which is contained within the closed interval in question and throughout which the function $\psi(\gamma)$ is continuous, the function $\psi(\gamma)$ can be expanded in a uniformly convergent series of Legendre polynomials:

$$\psi(\gamma) = \sum_{k=0}^{\infty} a_k P_k(\cos\gamma) ,$$

where

$$a_k = \frac{2n+1}{2}\int_0^{\pi} \psi(\gamma')\; P_k(\cos\gamma')\; \sin\gamma'\; d\gamma' .$$

Let us denote by θ' and φ' the coordinates of a variable point on Σ and by γ the angle between the radii drawn from the center of the sphere to the points (θ',φ') and (θ,φ).

Let us first assume that the series (24) converges and that it can be integrated termwise. If we multiply this series by the Legendre polynomial $P_k(\cos\gamma)$ and integrate it over Σ, we obtain, on the basis of the orthogonality relations (23),

$$\iint_{\Sigma} f(\theta',\varphi')\, P_m(\cos\gamma)\, dS = \sum_{k=0}^{\infty} \iint_{\Sigma} Y_k(\theta',\varphi')\, P_m(\cos\gamma)\, dS = \frac{4\pi}{2m+1}\, Y_m(\theta,\varphi) \quad ,$$

so that

$$Y_m(\theta,\varphi) = \frac{2m+1}{4\pi}\iint_{\Sigma} f(\theta',\varphi')\; P_m(\cos\gamma)\; dS . \tag{25}$$

can be partitioned into a finite number of subintervals in each of which the function $f(x)$ is monotone; and (2), in the interval (a, b), the function $f(x)$ either is continuous or has a finite number of discontinuities of the first kind (that is, a discontinuity where right- and left-hand limits of the function exist but are not both equal to the value of the function). When a function depends on several variables that vary in a region V, it is said to be of bounded variation in V if it is of bounded variation for each of these variables for all choices of fixed values for the remaining variables.

† See, for example, Hobson [19]. p. 319. A similar theorem is valid with fewer requirements on $f(\theta,\varphi)$.

We introduce new spherical coordinates (γ, ω) with pole at the point (θ, φ). Noting that

$$dS = \sin\gamma \, d\gamma \, d\omega \ ,$$

we rewrite eq. (25) in the form

$$Y_m = \frac{2m+1}{2} \int_0^{\pi} P_m(\cos\gamma) \sin\gamma \, d\gamma \, \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\gamma, \omega) \, d\omega$$

$$= \frac{2m+1}{2} \int_0^{\pi} \Phi(\gamma) \, P_m(\cos\gamma) \sin\gamma \, d\gamma \ , \tag{26}$$

where

$$\Phi(\gamma) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\gamma, \omega) \, d\omega$$

is the mean value of the function $f(\theta', \varphi')$ on the circle γ = constant with center at the point (θ, φ).

The function $\Phi(\gamma)$ is of bounded variation and is absolutely integrable for $0 \leqslant \gamma \leqslant \pi$. This is true because the function $f(\theta, \varphi)$ possesses these properties by hypothesis and they are obviously maintained when it is averaged. Furthermore, the function $\Phi(\gamma)$ is continuous in a neighbourhood of the point $\gamma = 0$, since the function f is continuous at $\gamma = 0$. Consequently, at the point $\gamma = 0$, the function $\Phi(\gamma)$ can be expanded in the series of Legendre polynomials

$$\Phi(0) = \sum_{k=0}^{\infty} \frac{2k+1}{2} \int_0^{\pi} \Phi(\gamma) \, P_k(\cos\gamma) \sin\gamma \, d\gamma \ . \tag{27}$$

Here, we used the fact that $P_k(1) = 1$. But from formula (26), the series on the right side of eq. (27) is equal to

$$\sum_{k=0}^{\infty} Y_k \ ,$$

where the values of Y_k are defined by the relationship (25). On the other hand, because of the continuity of the function $f(\theta, \varphi)$ at the point (θ, φ), for any positive number ϵ, there exists a number η depending only on ϵ such that, for arbitrary ω,

$$| f(\gamma, \omega) - f(\theta, \varphi) | < \epsilon \quad \text{when} \quad \gamma < \eta, \ 0 \leqslant \omega \leqslant 2\pi \ .$$

Since the value of $| \Phi(\gamma) |$ lies between the largest and the smallest values of $| f(\gamma, \omega) |$ on the circle γ = constant, it follows from the preceding inequality that, for $\gamma < \eta$,

$$| \Phi(\gamma) - f(\theta, \varphi) | < \epsilon \ .$$

Since ϵ can be made arbitrarily small,

$$\Phi(0) = f(\theta, \varphi) \ .$$

Substituting the expressions obtained into eq. (27), we obtain an expansion of the form (24), which completes the proof.

Since an arbitrary spherical function can be represented in the form of a linear combination of linearly independent orthogonal spherical functions belonging to the system (19), this theorem implies the completeness of the system.

If we represent each of the spherical functions $Y_k(\theta, \varphi)$ as a linear combination of the spherical functions of the system (19)

$$Y_k(\theta, \varphi) = a_{0k} P_k(\cos\theta) + \sum_{m=1}^{k} (a_{mk} \cos m\varphi + b_{mk} \sin m\varphi) P_{km}(\cos\theta) \quad (28)$$

and substitute these expressions into the relationship (24), we obtain the expansion of an arbitrary function $f(\theta, \varphi)$ as a linear combination of functions in the system of spherical functions (19):

$$f(\theta, \varphi) = a_0 + \sum_{k=1}^{\infty} \Big[a_{0k} P_k(\cos\theta) + \sum_{m=1}^{k} (a_{mk} \cos m\varphi + b_{mk} \sin m\varphi) P_{km}(\cos\theta) \Big] . \quad (29)$$

The reader should experience no difficulty in verifying by means of the formulae (20) that the coefficients of the series (29) are determined by

$$\begin{aligned} a_{0k} &= \frac{2k+1}{4\pi} \iint_{\Sigma} f(\theta', \varphi') P_k(\cos\theta') \, dS \\ a_{mk} &= \frac{2k+1}{2\pi} \frac{(k-m)!}{(k+m)!} \iint_{\Sigma} f(\theta', \varphi') P_{km}(\cos\theta') \cos m\varphi' \, dS , \quad (30) \\ b_{mk} &= \frac{2k+1}{2\pi} \frac{(k-m)!}{(k+m)!} \iint_{\Sigma} f(\theta', \varphi') P_{km}(\cos\theta') \sin m\varphi' \, dS . \end{aligned}$$

Problem

Suppose that γ is the angle between two radii drawn from the center of a spherical surface Σ to the points (θ_0, φ_0) and (θ, φ) on Σ. Assuming that $\gamma = \gamma(\theta, \varphi)$, show that

$$P_n(\cos\gamma) = P_n(\cos\theta) P_n(\cos\theta_0) + \sum_{k=0}^{n} 2 \frac{(n-k)!}{(n+k)!} P_{nk}(\cos\theta) P_{nk}(\cos\theta_0) \cos k(\varphi - \varphi_0) ,$$

which is known as the composition theorem for Legendre polynomials.
Method: Expand $P_n(\cos\gamma)$ in a series of the form (30) and use formulae (23) for calculating the coefficients in the series.

4. *The use of spherical functions for solving boundary problems*

Let us examine the application of the theory of spherical functions to the solution of Dirichlet's and Neumann's problems.

Suppose that Σ is a spherical surface, defined in a spherical coordinate system (R, θ, φ) by the equation $R = R_0$, and that $f(\theta, \varphi)$ is a function defined on Σ and represented as a series of spherical functions:

$$f(\theta, \varphi) = \sum_{k=0}^{\infty} Y_k(\theta, \varphi) . \tag{31}$$

As we know (see section 1), the function $Y_n(\theta, \varphi)/R^{n+1}$, and hence the function

$$(R_0/R)^{n+1}\, Y_n(\theta, \varphi) ,$$

which differs from it only by a constant factor, is harmonic in some region not containing the point $R = 0$. Therefore, the function

$$u(R, \theta, \varphi) = \sum_{k=0}^{\infty} Y_k(\theta, \varphi) \left(\frac{R_0}{R}\right)^{k+1} \qquad (R_0 \leqslant R) , \tag{32}$$

is harmonic outside the spherical surface Σ. On the basis of eq. (31), it coincides with $f(\theta, \varphi)$ on Σ and, therefore, it represents a solution to the exterior Dirichlet problem for the region lying outside the spherical surface Σ, with boundary condition

$$u|_{R=R_0} = f(\theta, \varphi) .$$

By using Kelvin's theorem (chapter XVIII, section 3), we find that the function

$$u(R, \theta, \varphi) = \sum_{k=0}^{\infty} Y_k(\theta, \varphi) \left(\frac{R}{R_0}\right)^{k} \qquad (R_0 \geqslant R) \tag{33}$$

is harmonic inside Σ and therefore represents a solution to the corresponding interior Dirichlet problem for the same boundary conditions.

Let us now examine the function

$$u(R, \theta, \varphi) = \sum_{k=0}^{\infty} \frac{R_0}{k+1}\, Y_k(\theta, \varphi) \left(\frac{R_0}{R}\right)^{k+1} , \tag{34}$$

which is harmonic in the region outside Σ. Let us take the direction of the normal within Σ for the positive direction. Then,

$$\frac{\partial}{\partial n} = - \frac{\partial}{\partial R} ,$$

since the normal derivative of the function $u(R, \theta, \varphi)$ on Σ is equal to

$$\left.\frac{\partial u}{\partial n}\right|_{R=R_0} = - \left.\frac{\partial u}{\partial R}\right|_{R=R_0} = \sum_{k=0}^{\infty} Y_k(\theta, \varphi) = f(\theta, \varphi) .$$

Thus, the series (34) gives the solution to the exterior Neumann problem for the regions outside Σ, with the boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{R=R_0} = f(\theta, \varphi) \ .$$

According to section 4 of Chapter XVIII, the interior Neumann problem has a solution only when the boundary condition $f(\theta, \varphi)$ satisfies the equation

$$\iint_{\Sigma} f(\theta', \varphi') \, dS = 0 \ . \tag{35}$$

Because of the orthogonality of spherical functions of different orders,

$$\iint_{\Sigma} Y_k(\theta', \varphi') \, dS = \begin{cases} 4\pi R_0^2 Y_0 & \text{for } k = 0, \\ 0 & \text{for } k \neq 0, \end{cases}$$

where Y_0 is a constant. We conclude from this that, for the requirement (35) to be satisfied, there must be no term of zero order in the expansion (31); that is,

$$f(\theta, \varphi) = \sum_{k=1}^{\infty} Y_k(\theta, \varphi) \ . \tag{36}$$

In this case, the series

$$u(R, \theta, \varphi) = C + \sum_{k=1}^{\infty} \frac{R_0}{k} Y_k(\theta, \varphi) \left(\frac{R}{R_0}\right)^k \qquad (R_0 \geqslant R) \ , \tag{37}$$

where C is an arbitrary constant, satisfies the interior Neumann problem for the sphere, with the boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{R=R_0} = f(\theta, \varphi) \ .$$

This is true because the function $u(R, \theta, \varphi)$ is harmonic within Σ and its normal derivatives coincide on Σ with $f(\theta, \varphi)$. (We assume the normal to be directed toward the region outside Σ.)

Problems

1. Find the solution to the interior Dirichlet problem in the form (33), starting with Poisson's integral (51) of Chapter VIII.
   Method: Use the expansion

$$\frac{1 - h^2}{(1 - 2h \cos\theta + h^2)^{\frac{3}{2}}} = \sum_{k=0}^{\infty} (2k+1) \, h^k P_k(\cos\theta) \qquad (h < 1) \ .$$

2. Find the solution to the interior Dirichlet problem with the boundary condition

$$u(R, \theta, \varphi)|_{R=R_0} = \sin 3\theta \, \cos\varphi \ .$$

Answer:

$$u(R, \theta, \varphi) = \frac{8}{15}\left(\frac{R}{R_0}\right)^3 P_{31}(\cos\theta) - \frac{1}{5}\frac{R}{R_0} P_{11}(\cos\theta)\cos\varphi .$$

3. Demonstrate the possibility of solving the mixed boundary problem for a spherical surface by means of spherical functions.

4. Solve Dirichlet's problem for the region between two concentric spherical surfaces with radii R_1 and R_2 with the condition that the desired solution becomes the given function $f(\theta, \varphi)$ on the first spherical surface and the function $F(\theta, \varphi)$ on the second.
Answer: The desired solution is given by the system of equations

$$u = \sum_{k=0}^{\infty} \sum_{\beta=0}^{k} \left[\left(A_{\beta k}R^k + B_{\beta k}\frac{1}{R^{k+1}}\right)\cos\beta\varphi + \left(C_{\beta k}R^k + D_{\beta k}\frac{1}{R^{k+1}}\right)\sin\beta\varphi\right] P_{k3}(\cos\theta) ,$$

$$A_{\beta k}R_1{}^k + B_{\beta k}\frac{1}{R_1{}^{k+1}} = a_{\beta k} , \qquad C_{\beta k}R_1{}^k + D_{\beta k}\frac{1}{R_1{}^{k+1}} = b_{\beta k} ,$$

$$A_{\beta k}R_2{}^k + B_{\beta k}\frac{1}{R_2{}^{k+1}} = \bar{a}_{\beta k} , \qquad C_{\beta k}R_2{}^k + D_{\beta k}\frac{1}{R_2{}^{k+1}} = \bar{b}_{\beta k} ,$$

where $a_{\beta k}$ and $b_{\beta k}$ are the coefficients in the expansion of the function $f(\theta, \varphi)$ in a series of spherical functions of the form (29) and $\bar{a}_{\beta k}$ and $\bar{b}_{\beta k}$ are the coefficients of the expansion of the function $F(\theta, \varphi)$ in the same type of series.

5. Solve the preceding problem under the assumption that the functions f and F depend only on the angle θ.
Answer:

$$u = \sum_{k=0}^{\infty} \left\{A_k\left[\left(\frac{R}{R_2}\right)^k - \left(\frac{R_2}{R}\right)^{k+1}\right] + B_k\left[\left(\frac{R}{R_1}\right)^k - \left(\frac{R_1}{R}\right)^{k+1}\right]\right\} P_k(\cos\theta) ,$$

where

$$A_k = \frac{2k+1}{2}\,\frac{1}{(R_1/R_2)^k - (R_2/R_1)^{k+1}}\int_0^{\pi} f(\theta')\,P_k(\cos\theta')\sin\theta'\,d\theta' ,$$

$$B_k = \frac{2k+1}{2}\,\frac{1}{(R_2/R_1)^k - (R_1/R_2)^{k+1}}\int_0^{\pi} F(\theta')\,P_k(\cos\theta')\sin\theta'\,d\theta' .$$

5. *Green's function of the Dirichlet problem for a sphere*

Let us use spherical functions to find the Green's function of the Dirichlet problem

$$\Delta u = f \quad \text{when} \quad x \in V - \mathcal{F}V\,, \qquad u = \psi \quad \text{when} \quad x \in \mathcal{F}V\,, \tag{38}$$

in the particular case in which the region V is a sphere or an infinite region located outside some sphere.

As we know (section 7 of Chapter XVIII), the Green's function of Dirichlet's problem is equal to

$$G(\xi, x) = \frac{1}{4\pi}\left[\frac{1}{r(\xi, x)} + \varphi(\xi, x)\right], \tag{39}$$

where $r(\xi, x)$ is the distance between the points ξ and x and $\varphi(\xi, x)$ is the solution to the boundary problem (60) of Chapter XVIII:

$$\Delta_\xi \varphi = 0 \qquad \text{when} \qquad \xi, x \in V - \mathcal{F}V\,, \tag{40}$$

$$\varphi = -\frac{1}{r} \qquad \text{when} \qquad \xi \in \mathcal{F}V\,,\; x \in V - \mathcal{F}V\,. \tag{41}$$

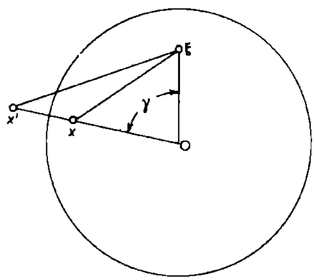

Fig. 55.

Let us begin with the interior problem, assuming that the region V is a sphere of radius a. Let us denote by $|x|$ and $|\xi|$ the distances of the points x and ξ from the center of the sphere, and by γ the angle between the radius vectors of the points x and ξ (fig. 55), and let us expand the function $\varphi(\xi, x)$ in a series of Legendre polynomials:

$$\varphi(\xi, x) = \sum_{k=0}^{\infty} b_k P_k(\cos\gamma)\left(\frac{|x|\,|\xi|}{a^2}\right)^k \qquad (|x| < a,\ |\xi| \leq a)\,.$$

To determine the coefficients b_k, we use the representation of the function

$$\frac{1}{r} = \frac{1}{\sqrt{|x|^2 + |\xi|^2 - 2|x|\,|\xi|\,\cos\gamma}}$$

in the form of a series in powers of $|x|/|\xi|$. In accordance with formula (15) of Chapter XV, the coefficients of this series will be the quantities

$$\frac{1}{|x|}\,P_n(\cos\gamma)\,.$$

Therefore, for the case in which the point ξ lies on the surface $\mathcal{F}V$, we obtain

$$\frac{1}{r} = \sum_{k=0}^{\infty} P_k(\cos\gamma)\frac{|x|^k}{a^{k+1}} \quad (\xi \in \mathcal{F}V) .$$

Comparing the expansions for φ and $1/r$, we conclude that the boundary condition (41) will be satisfied identically if we set

$$b_k = 1/a .$$

Consequently,

$$\varphi(\xi, x) = -\frac{1}{a}\sum_{k=0}^{\infty} P_k(\cos\gamma)\left(\frac{|x|\,|\xi|}{a^2}\right)^k .$$

Comparing this series with the series (15) of Chapter XV, we see that

$$\varphi = -\frac{1}{a\sqrt{1 + (|x|\,|\xi|/a^2)^2 - 2(|x|\,|\xi|/a^2)\cos\gamma}}$$

$$= -\frac{a}{|x|}\frac{1}{\sqrt{(a^2/|x|)^2 + |\xi|^2 - 2|\xi|(a^2/|x|)\cos\gamma}}$$

Substituting this expression into formula (39), we obtain the desired Green's function:

$$G(\xi, x) = \frac{1}{4\pi}\left(\frac{1}{r} - \frac{a}{|x|}\frac{1}{r_1}\right), \tag{42}$$

where

$$r_1 = \sqrt{\left(\frac{a^2}{|x|}\right)^2 + |\xi|^2 - 2a^2\frac{|\xi|}{|x|}\cos\gamma} . \tag{43}$$

It is easy to see that the quantity r_1 represents the distance from the point ξ to the point x', which is the harmonic conjugate (Chapter XVIII, section 3) of the point x with respect to the surface $\mathcal{F}V$ of the sphere in question. For, by definition of harmonically conjugate points, the point x' lies on the same ray with origin at the center of the sphere as does the point x and it is at a distance

$$|x'| = \frac{a^2}{|x|}$$

from the center (fig. 55). When we write the expressions for the distance $|x' - \xi|$, we obtain formula (43).

For the exterior Dirichlet problem, the form of the expression (42) defining the Green's function does not change. To prove this assertion, it is sufficient to show that the function

$$-\frac{a}{|x|}\frac{1}{r_1}$$

is harmonic with respect to the coordinates of the point ξ in an infinite region outside the sphere, and that it satisfies the boundary condition (41). The first requirement is obviously satisfied since r_1 is the distance from the

point ξ (lying outside the sphere V or on its surface) to the point x', which is the harmonic conjugate of the point x and, hence, lies within the sphere. Therefore, the pole of the function $1/r_1$ lies within the sphere. Hence, this function is harmonic in the region outside the sphere. The second condition follows from formula (43). For if the point ξ is on the surface of the sphere, then $|x| = a$ and

$$r_1 = \frac{a}{|x|} \sqrt{a^2 + |x|^2 - 2|x|\,|\xi| \cos\gamma} = a \frac{r}{|x|}$$

Thus, formula (42) gives an expression for the Green's function that satisfies both the exterior and the interior Dirichlet problem for a sphere. Here, in contrast with formulae (32) and (33), the solution is represented in closed form by means of the Green's function and it encompasses Dirichlet's problem not only for Laplace's equation but also for Poisson's (see section 7 of Chapter XVIII).

Problem

Starting with expression (42) for the Green's function, derive Poisson's integral formula (51) of Chapter XVIII.

6. *Green's function for the Neumann problem for a sphere*

Let us now find Green's function for the interior Neumann problem:

$$\Delta u = 0 \quad \text{when} \quad x \in V - \mathcal{F}V\,, \qquad \frac{\partial u}{\partial n} = \psi \quad \text{when} \quad x \in \mathcal{F}V\,,$$

when the region V is a sphere. Let us represent the Green's function in the form (39). In determining the function $\varphi(\xi,x)$, we are led to a boundary problem that differs from the problem (40) - (41) in its boundary condition

$$\frac{\partial \varphi}{\partial n} = \frac{4\pi}{\bar{S}} - \frac{\partial}{\partial n}\left(\frac{1}{r}\right) \quad \text{when} \quad x \in \mathcal{F}V\,,\ x \in V - \mathcal{F}V\,, \tag{44}$$

where $\bar{S}$ is the area of the surface $\mathcal{F}V$.

Using the notation of the preceding section, we again represent the function $\varphi(\xi,x)$ in the form of a series of Legendre polynomials:

$$\varphi(\xi,x) = \sum_{k=0}^{\infty} b_k P_k(\cos\gamma)\left(\frac{|x|\,|\xi|}{a^2}\right)^k \qquad (|x| < a,\ |\xi| \leqslant a)\,.$$

Noting that differentiation along the outward normal to the surface $\mathcal{F}V$ is equivalent to differentiation with respect to $|\xi|$ and that $|\xi| = a$ if the point ξ lies on the surface $\mathcal{F}V$, we obtain

$$\frac{\partial \varphi}{\partial n} = \sum_{k=1}^{\infty} k b_k P_k(\cos\gamma) \frac{|x|^k}{a^{k+1}} \quad \text{when} \quad \xi \in \mathcal{F}V\,.$$

Furthermore, according to formula (15) of Chapter XV, for $|\xi| = a$,

$$\frac{1}{r} = \frac{1}{\sqrt{a^2 + |x|^2 - 2a|x| \cos\gamma}} = \sum_{k=0}^{\infty} P_k(\cos\gamma) \frac{|x|^k}{a^{k+1}} \quad \text{when} \quad \xi \in \mathcal{F}V ,$$

so that

$$\frac{\partial}{\partial n}\left(\frac{1}{r}\right) = \frac{\partial}{\partial a}\left(\frac{1}{r}\right) = - \sum_{k=0}^{\infty} \frac{k+1}{a} P_k(\cos\gamma) \frac{|x|^k}{a^{k+1}} \quad \text{when} \quad \xi \in \mathcal{F}V .$$

Substituting these expressions into the boundary condition (44) and remembering that, in the present case, $\bar{S} = 4\pi a^2$, we obtain

$$\sum_{k=0}^{\infty} \left(kb_k - \frac{k+1}{a}\right) P_k(\cos\gamma) \frac{|x|^k}{a^{k+1}} = - \frac{1}{a^2} .$$

This equation will be satisfied identically if

$$b_k = \frac{k+1}{k} \frac{1}{a} \quad (k > 0) . \tag{45}$$

The coefficient b_0 can be chosen arbitrarily. Taking $b_0 = 1/a$, we obtain

$$\varphi = \frac{1}{a} \sum_{k=0}^{\infty} P_k(\cos\gamma) \left(\frac{|x|\,|\xi|}{a^2}\right)^k + \frac{1}{a} \sum_{k=1}^{\infty} \frac{P_k(\cos\gamma)}{k} \left(\frac{|x|\,|\xi|}{a^2}\right)^k . \tag{46}$$

By comparing the first of these series with the series (15) of Chapter XV, we easily see that

$$\frac{1}{a} \sum_{k=0}^{\infty} P_k(\cos\gamma) \left(\frac{|x|\,|\xi|}{a^2}\right)^k = \frac{a}{\sqrt{a^4 + |x^2|\,|\xi|^2 - 2|x|\,|\xi| a^2 \cos\gamma}} . \tag{47}$$

The second of the series appearing in eq. (46) can also be summed easily. To do this we divide both sides of the equation

$$\frac{1}{\sqrt{1 + \rho^2 - 2\rho \cos\gamma}} = 1 + \sum_{k=1}^{\infty} \rho^k P_k(\cos\gamma) ,$$

where $|\rho| < 1$, by ρ and integrate the expressions obtained with respect to ρ. Since

$$\int \frac{d\rho}{\rho\sqrt{1 + \rho^2 - 2\rho \cos\gamma}} = - \ln \tfrac{1}{2}(1 - \rho \cos\gamma + \sqrt{1 + \rho^2 - 2\rho \cos\gamma}) + C ,$$

we obtain

$$\int \sum_{k=1}^{\infty} \frac{P_k(\cos\gamma)}{k} \rho^k = - \ln \tfrac{1}{2}(1 - \rho \cos\gamma + \sqrt{1 + \rho^2 - 2\rho \cos\gamma}) .$$

Substituting for ρ the ratio $|x|\,|\xi|/a^2$, we find that

$$\frac{1}{a}\sum_{k=1}^{\infty}\frac{P_k(\cos\gamma)}{k}\left(\frac{|x|\,|\xi|}{a^2}\right)^2$$
$$= \ln\frac{2a^2}{a^2 - |x|\,|\xi|\,\cos\gamma + \sqrt{a^4 + |x|^2|\xi|^2 - 2a|x|\,|\xi|\,\cos\gamma}}. \tag{48}$$

Eqs. (47) and (48) can be simplified if we introduce the distance r_1 between the point ξ and the point x', which is the harmonic conjugate of the point x with respect to the surface of the sphere in question. Then, as can easily be seen, we obtain, on the basis of formulae (46), (47), and (48),

$$\varphi = \frac{a}{|x|r_1} + \frac{1}{a}\ln\frac{2a^2}{a^2 + |x|r_1 - |x|\,|\xi|\,\cos\gamma}$$

and, on the basis of formula (39), we find that

$$G(\xi,x) = \frac{1}{r} + \frac{a}{|x|r_1} + \frac{1}{a}\ln\frac{2a^2}{a^2 + |x|r_1 - |x|\,|\xi|\,\cos\gamma}. \tag{49}$$

If the point ξ lies on the surface $\mathcal{F}V$, then, as we saw in the preceding section,

$$r_1 = \frac{r}{|x|}\,a\,,$$

as a consequence of which,

$$G(\xi,x) = \frac{1}{4\pi}\left(\frac{2}{r} + \frac{1}{a}\ln\frac{2a}{a + r - |x|\cos\gamma}\right)$$
$$= \frac{1}{4\pi}\left[\frac{2}{r} - \frac{1}{a}\ln\,(a + r - |x|\cos\gamma) + \frac{\ln 2a}{a}\right] \quad \text{when} \quad \xi \in \mathcal{F}V\,.$$

Putting this expression into formula (67) of Chapter XVIII and remembering, on the basis of formula (35) of Chapter XVIII, that

$$\iint\limits_{\mathcal{F}V}\frac{\partial u}{\partial n}\,dS = 0\,,$$

we obtain the solution to the Neumann problem for a sphere in the form found by Neumann himself from physical considerations:

$$u(x) = \frac{1}{4\pi}\iint\limits_{\mathcal{F}V}\left[\frac{2}{r} - \frac{1}{a}\ln\,(a + r - |x|\cos\gamma)\right]\psi\,dS\,. \tag{50}$$

Problems

1. Show that the Green's function of the Neumann problem stated for an infinite region lying outside some sphere is expressed by

$$G(\xi, x) = \frac{1}{r} + \frac{a}{|x| r_1} + \frac{1}{a} \ln \frac{(1 - \cos\gamma)|x|\,|\xi|}{a^2 + r_1|x| - |x|\,|\xi| \cos\gamma}.$$

Method: Expand the function φ appearing in the relationship (40) in a series of Legendre polynomials:

$$\varphi = \sum_{k=0}^{\infty} a_k P_k(\cos\gamma) \left(\frac{a^2}{|x|\,|\xi|}\right)^{k+1},$$

and use the boundary conditions (44). In summing the series, use the formula obtained by integrating the equation

$$\frac{1}{\sqrt{1 + \rho^2 - 2\rho \cos\gamma}} = \sum_{k=0}^{\infty} P_k(\cos\gamma)\, \rho^k \qquad (|\rho| < 1).$$

2. Use the solution to problem 1 to show that the solution to the exterior Neumann problem for a sphere V is given by the following *Bjerknes formula:*

$$u(x) = \frac{1}{4\pi} \iint_{\mathcal{F}V} \left[\frac{2}{r} - \frac{1}{a} \ln \frac{a + r - |x| \cos\gamma}{(1 - \cos\gamma)|\xi|}\right] \psi \, dS.$$

Chapter XXII

SEVERAL QUESTIONS ON GRAVIMETRY AND THE THEORY OF THE SHAPE OF THE EARTH

1. *Equipotential distributions*

In this chapter, we shall consider the applications of the theory of a Newtonian potential to certain problems associated with the study of gravitational fields. In sections 1 and 2, we shall study the Newtonian potential

$$U(x) = \iiint\limits_V \tfrac{1}{2}\rho \, \mathrm{d}V \, , \tag{1}$$

where the density ρ is non-negative, and in sections 3-5, we shall turn to the study of gravitational fields.

We shall assume the mass distributions to be such that all the potentials in question exist and that closed surfaces on which the Newtonian potential (1) has a constant value also exist. Such surfaces are called equipotential surfaces. We shall denote by V_Σ the finite region whose boundary is the equipotential surface Σ, and we shall denote by $(R_E - V_\Sigma)$ the complement of the region V_Σ with respect to all space. We shall always assume below that $(R_E - V_\Sigma)$ is a region and that it contains no gravitational masses. We denote by $\mathrm{d}/\mathrm{d}n$ differentiation with respect to the outward normal n to the surface Σ, regarded as the boundary of the infinite region $R_E - V_\Sigma$ (fig. 56).

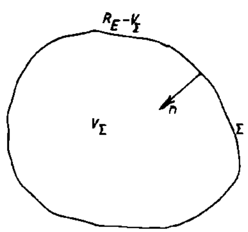

Fig. 56.

Suppose that U_0 is the value of the Newtonian potential $U(x)$ on an equipotential surface Σ. Since the Newtonian potential outside the region where the masses are located is harmonic (Chapter XIX, section 1), in the infinite region $R_E - V_\Sigma$ we may apply formula (45) of Chapter XVIII. By setting

$$u(x) = U(x)\ , \qquad L(\xi, x) = \frac{1}{4\pi}\,\frac{1}{r}$$

in it, we obtain

$$\frac{1}{4\pi}\iint\limits_{\Sigma}\left[\frac{1}{r}\frac{dU}{dn} - U_0\frac{d}{dn}\left(\frac{1}{r}\right)\right]dS = \begin{cases} U(x) & \text{when } x \in R_E - V_\Sigma\ , \\ \frac{1}{2}U_0 & \text{when } x \in \Sigma\ , \\ 0 & \text{when } x \in V_\Sigma - \Sigma\ . \end{cases} \tag{2}$$

On the other hand, the function $u(x) = U_0$ = constant is harmonic in the bounded region V_Σ; therefore, we may apply formula (44) of Chapter XVIII to it in that region. Since the normal n is *inward* with respect to the region V_Σ (so that we need to change the sign of the left side of formula (44)), we have

$$\frac{1}{4\pi}\iint\limits_{\Sigma} U_0\frac{d}{dn}\left(\frac{1}{r}\right)dS = \begin{cases} U_0 & \text{when } x \in V_\Sigma - \Sigma\ , \\ \frac{1}{2}U_0 & \text{when } x \in \Sigma\ , \\ 0 & \text{when } x \in R_E - V_\Sigma\ . \end{cases} \tag{3}$$

By combining formulae (2) and (3), we obtain

$$\frac{1}{4\pi}\iint\limits_{\Sigma}\frac{1}{r}\frac{dU}{dn}\,dS = \begin{cases} U(x) & \text{when } x \in R_E - V_\Sigma\ , \\ U_0 & \text{when } x \in V_\Sigma\ , \end{cases} \tag{4}$$

which gives an expression for the Newtonian potential in the region $R_E - V_\Sigma$ in terms of the values of its normal derivatives on the equipotential surface Σ. The expression on the left side of eq. (4) can be regarded as the single-layer potential of density

$$\rho = \frac{1}{4\pi}\,\frac{dU}{n}\,. \tag{5}$$

Let us show that, on the equipotential surface, dU/dn is positive. It follows from the definition (1) of a Newtonian potential that for positive ρ, the function $U(x)$ will also be positive. If dU/dn were zero or negative, the function U would assume values equal to or greater than U_0 in the region $R_E - V_\Sigma$. This is impossible because in this region the function U is harmonic and, consequently, it has its greatest value on the boundary Σ. Thus, the density ρ is positive; that is, the single layer (4) can be formed by *gravitational masses*.

Let us now use Green's theorem (7), Chapter XVII, in an arbitrary region V. Setting $u = U$ and $v = 1$ in Green's theorem, we obtain

$$-\iint\limits_{\mathcal{F}V}\frac{dU}{dn}\,ds = \iiint \Delta U\,dV\,. \tag{6}$$

But, as we know (Chapter XIX, section 1), the Newtonian potential satisfies Poisson's equation

$$\Delta U = -4\pi\rho\ ,$$

so that formula (6) can be transformed into the form

$$\frac{1}{4\pi}\iint\limits_{\mathcal{F}V}\frac{dU}{dn}\,dS = m\ , \tag{7}$$

where

$$m = \iiint_V \rho \, dV$$

is the gravitational mass concentrated in the region V. Eq. (7) is called *Gauss' formula*. Poincaré showed that Gauss' formula is valid even when the entire mass or a part of it is distributed on the surface $\mathcal{F}V$.

A single layer formed by a distribution of mass on some surface S in such a way that this surface S will be an equipotential surface is called an equipotential layer. The corresponding mass distribution is also called an equipotential distribution.

It follows from formulae (4) and (7) that it is always possible to distribute the mass contained within an equipotential surface on that surface itself in such a way as to form an equipotential layer, without causing any change in the Newtonian potential in the region $R_E - V_\Sigma$. For this, we need to determine the density by means of formula (5).

If the mass m is distributed over an arbitrary surface S in such a way that an equipotential layer is formed, this distribution is unique.

To show this, let us suppose that there are two distinct equipotential distributions of the mass m that are characterized by the densities $\bar{\rho}_1$ and $\bar{\rho}_2$. It follows from formulae (4) that in the finite region V_S whose boundary is the surface S, the potential of the equipotential layer will be constant. Consequently, the inward normal derivatives (Chapter XIX, section 8) of the potential of this layer will be equal to zero. Therefore, on the basis of formulae (38) of Chapter XIX, the outward normal derivatives of the potentials of layers of density $\bar{\rho}_1$ and $\bar{\rho}_2$ will be equal, respectively, to $4\pi\bar{\rho}_1$ and $4\pi\bar{\rho}_2$ and hence will be different. Since the solution of the exterior Neumann problem is unique (see Chapter XVIII, section 4), two distinct potentials U_1 and U_2, with values U_{10} and U_{20} on the surface S, correspond to these derivatives in the region $R_E - V_S$. These values must also be different because of the uniqueness of the solution to Dirichlet's problem.

Let us consider the function $U_1U_{20} - U_2U_{10}$. It is harmonic in the region $R_E - V_S$ and is equal to zero on the surface S. Therefore, in the region $R_E - V_S$,

$$U_1U_{20} - U_2U_{10} = 0 .$$

When the points of the layer are displaced to a great distance, the potentials U_1 and U_2 are equal to m/r with accuracy up to higher order terms (where r is the distance from the point of observation x to an arbitrary point of the layer) (see Chapter XIX, section 2). Therefore,

$$\frac{m}{r} U_{20} - \frac{m}{r} U_{10} = 0 ,$$

and hence

$$U_{10} = U_{20} ,$$

which contradicts the assumption that there are two distinct equipotential

distributions of the mass m. Thus, if there is an equipotential distribution on a surface S, it is unique.

The proof of the existence of equipotential distributions for arbitrary surfaces is much more complicated. The first to study the question was Gauss. Weierstrass showed that the proof that Gauss gave was not valid. Neumann first gave a rigorous proof of the existence of equipotential distributions for a sufficiently broad class of surfaces.

Problem

Suppose that Σ is an equipotential surface outside which there are no gravitational masses. Show that specifying the mass situated within Σ, the normal derivative of the potential on Σ, and the surface Σ itself uniquely determines the potential of the gravitational field outside Σ.

2. *The energy of a gravitational field. Gauss' problem*

We shall touch on the problem of the energy of a gravitational field in connection with the problem of the equilibrium distribution of masses (Gauss' problem).

As we know from physics, the energy of a system of distributed masses is, up to an additive constant, equal to

$$W = -\tfrac{1}{2}\kappa \iiint\limits_V U\rho \, dV , \tag{8}$$

where U is the potential of the gravitational field, ρ is the density of the substance, κ is the gravitational constant, and the integration is taken over an arbitrary region containing all the gravitational masses. Let us assume that among these regions there is one that is finite (that is, there are no masses at infinity). We shall not give a proof of formula (8). We note only that the minus sign in front of the integral is caused by the fact that the force acting between masses is one of attraction and not repulsion. For a system of electric charges of a single sign, the situation would be the reverse, with the result that the right side of eq. (8) would have the opposite sign.

Substituting into eq. (8) the value of ρ given by Poisson's equation and applying Green's theorem (7) of Chapter XVII for $u = v = U$, we obtain

$$W = -\frac{\kappa}{8\pi} \iiint\limits_V \left[\left(\frac{\partial U}{\partial x_1}\right)^2 + \left(\frac{\partial U}{\partial x_2}\right)^2 + \left(\frac{\partial U}{\partial x_3}\right)^2\right] dV + \frac{1}{8\pi} \iint\limits_{\mathcal{F}V} U \frac{\partial U}{\partial n} \, dS .$$

If the integration is taken over all space (which obviously does not change the value of the integral (8)), the surface integral over $\mathcal{F}V$ will vanish, so that

$$W = -\frac{\kappa}{8\pi}\iiint\limits_{R_E}\left[\left(\frac{\partial U}{\partial x_1}\right)^2 + \left(\frac{\partial U}{\partial x_2}\right)^2 + \left(\frac{\partial U}{\partial x_3}\right)^2\right]\mathrm{d}V\,. \tag{9}$$

The integral on the right is Dirichlet's integral, with which we are already familiar and which, as we see, characterizes the energy of the field.

The integrals (8) and (9) illustrate two possible representations of a field. The integral (8) connects the energy of the field with the distribution of charges, since the integrand vanishes in regions where there are no masses. This makes it possible for us to interpret the energy of the field as the energy of interaction of masses (thus associated with the masses themselves). The integral (9) expresses the energy only in terms of the field potential; here, to every element of volume of the field (including those portions of space *where there are no masses*), there corresponds a definite non-zero value of the integrand, as if some amount of energy were localized at every element of volume of the field. This allows us to interpret the energy W as the characteristic energy of a gravitational field, which it is then natural to regard as an independent physical object. In connection with this, the quantity

$$-\frac{\kappa}{8\pi}\left[\left(\frac{\partial U}{\partial x_1}\right)^2 + \left(\frac{\partial U}{\partial x_2}\right)^2 + \left(\frac{\partial U}{\partial x_3}\right)^2\right] \tag{10}$$

is called the *energy density* of the gravitational field.

Let us now turn to a famous problem of Gauss: With what distributions of mass within and on a given surface S will the potential energy of the field have an extremum?

This problem is closely related to the problem of the equilibrium of masses. A system of masses at rest is in a state of stable equilibrium if their energy of interaction has the minimum value compatible with the conditions limiting the possible configurations of the system (the principle of minimum potential energy). In the opposite case, either there is no equilibrium or the condition is one of unstable equilibrium.

Let us use variational methods. By varying the density ρ in some finite region V_ρ, we shall seek a mass distribution in that region such that the variation in the energy of the gravitational field will be

$$\delta W = 0\,. \tag{11}$$

This distribution will correspond to an extremum.

From eq. (8),

$$\delta W = \delta\iiint\limits_{V} U\rho\,\mathrm{d}V = \iiint\limits_{V}(U\delta\rho + \rho\,\delta U)\,\mathrm{d}V = 0\,, \tag{12}$$

where V is a region containing the region V_ρ (possibly coinciding with it), $\delta\rho$ is an arbitrary variation in the mass density, and δU is the variation in the potential caused by the variation in ρ. Here, the quantity $\delta\rho$ is arbitrary except that it obeys the condition of conservation of mass in the region V_ρ:

$$\iiint\limits_{V_\rho}\delta\rho\,\mathrm{d}V = 0\,. \tag{13}$$

In Green's theorem (7) of Chapter XVII, let us set $u = U$ and $v = \delta U$. This gives us

$$\iiint\limits_{V} (U\Delta\delta U - \delta U\Delta U)\, \mathrm{d}V = \iint\limits_{\mathcal{F}V} \left(U \frac{\partial \delta U}{\partial n} - \delta U \frac{\partial U}{\partial n}\right) \mathrm{d}S \,. \tag{14}$$

Let us have the volume V approach infinity. Outside the region occupied by the masses, the functions U and δU are harmonic. Therefore, on the basis of the lemma on the behaviour of a harmonic function at infinity (Chapter XVIII, section 3), we conclude that the integrand in the surface integral over $\mathcal{F}V$ approaches zero in proportion to $1/r^3$. Therefore this integral vanishes at infinity. Consequently,

$$\iiint\limits_{R_E} U\Delta\, \delta U\, \mathrm{d}V = \iiint\limits_{R_E} \delta U\, \Delta U\, \mathrm{d}V \,,$$

which, after we make the substitution

$$\Delta U = -4\pi\rho \;, \qquad \Delta\delta U = \delta\Delta U = -4\pi\delta\rho \;,$$

yields

$$\iiint\limits_{R_E} U\, \delta\rho\, \mathrm{d}V = \iiint\limits_{R_E} \rho\, \delta U\, \mathrm{d}V \,.$$

Since $\rho = \delta\rho = 0$ outside V_ρ, we have

$$\iiint\limits_{V_\rho} U\, \delta\rho\, \mathrm{d}V = \iiint\limits_{V_\rho} \rho\, \delta U\, \mathrm{d}V \tag{15}$$

and condition (12) is equivalent to

$$\iiint\limits_{V_\rho} U\, \delta\rho\, \mathrm{d}V = 0 \,. \tag{16}$$

But this equation can be satisfied for a variation $\delta\rho$, subject to condition (13) alone, only when the potential $U = U_0$ = constant in the region V. But if $U = U_0$ within the region, it must, because of the continuity of the potential, be equal to U_0 on the surface $\mathcal{F}V_\rho$ as well, and we know that this is possible only when the distribution of mass on $\mathcal{F}V_\rho$ has the form of an equipotential layer.

Thus, the energy of the gravitational field of the masses situated in the region V_ρ has an extremum when all the mass is distributed on the boundary $\mathcal{F}V_\rho$ of the region in the form of an equipotential layer. Hence it follows, in particular, that with this distribution there is an extremum to the Dirichlet integral.

It is easy to see that the extremum in the case of a level distribution of masses is a *maximum* in comparison with an arbitrary distribution of masses *within* the region V_ρ and a minimum in comparison with an arbitrary distribution of these masses *outside* or *on* the surface $\mathcal{F}V_\rho$. It then follows that if the masses from the surface $\mathcal{F}V_\rho$ can penetrate inside the region V_ρ, the equipotential distribution will be unstable; and in the opposite

case, it will be stable. Therefore, for example, a liquid "poured" onto a surface $\mathcal{F}V_\rho$ that is impervious to it will, under the action of gravity, be distributed on the surface in the form of an equipotential layer.

It is another matter when we consider distributions of electrical charges of the same sign. In this case, instead of a force of attraction, there is repulsion, and the energy of the field is determined by expressions of the forms (8) and (9), taken with opposite signs. Therefore, an equipotential distribution of charges which can be displaced in some finite volume V_ρ is stable. It then follows, as we have already had occasion to note in Chapter XVIII, that free charges that exist in a conductor are located on the surface of the conductor in a state of equilibrium. They thus form an equipotential layer, so that the potential of the conductor has the same value at all points on it. In particular, it follows from this last fact that an electric field does not penetrate inside a conductor. For if the potential of the field is constant, the intensity of the field is equal to zero.

Problems

1. Show that the energy of the gravitational field of a sphere of radius R with a uniformly distributed mass is equal to

$$W = -\frac{3\gamma}{5}\frac{m^2}{R}.$$

2. Show that the energy of a gravitational field formed by an equipotential distribution of a mass m on a spherical surface of radius R is equal to

$$W = -\frac{\gamma}{2}\frac{m^2}{R}.$$

3. Show that the energy of an electric field is equal to

$$W = \frac{1}{8\pi}\iiint_{R_E} E^2\, dV,$$

where E is the field intensity vector.

3. *Gravitational fields. Stokes' theorem*

Let us consider a massive body that is rotating at constant velocity ω around an axis, keeping its orientation in space. Such a body we shall call a *planetary* body.

The force of Newtonian gravitation (gravitational attraction) and the centrifugal force of inertia acts on every body that is at rest on the surface of a planet. We shall call their geometric sum the *weight* of the body.

Let us show that this weight has a potential. Since the gravitational attraction has a potential, we need only show the existence of a potential for the centrifugal force.

Let us introduce a rectangular Cartesian coordinate system with origin

at the center of inertia of the planet, and with axis 3 directed along the axis of rotation. The centrifugal force of inertia, whose components are equal to $x_1\omega^2$, $x_2\omega^2$, 0, acts on a body of unit mass that is associated with the planet. But these quantities are the partial derivatives with respect to x_1, x_2, and x_3 of the expression

$$\Omega = \tfrac{1}{2}\omega^2(x_1{}^2 + x_2{}^2) , \tag{17}$$

which therefore represents the potential of the centrifugal force.

Thus, the weight potential is equal to

$$W = \kappa U + \Omega = \kappa \iiint\limits_{V_\rho} \frac{\rho}{r}\,\mathrm{d}V + \tfrac{1}{2}\omega^2(x_1{}^2 + x_2{}^2) , \tag{18}$$

where U is the Newtonian potential, κ is the gravitational constant, V_ρ is the volume of the planet, and ρ is the density of the substance of which the planet is made.

It follows from eq. (18) that the weight potential satisfies Poisson's equation:

$$\Delta W = -4\pi\kappa\rho + 2\omega^2 . \tag{19}$$

We shall refer to the surfaces σ on which the weight potential has a constant value W_0 as the equipotential surfaces of the weight potential or simply as the equipotential surfaces σ. However, we should remember that the equipotential surfaces of the Newtonian potential and those of the weight potential do not coincide.

In analogy with the notations that we have used before, we shall denote by V_σ the finite volume bounded by the surface σ, and by $R_E - V_\sigma$ the complement of this region with respect to all space R_E. We denote by d/dn differentiation in the direction of the outward normal n to the surface σ, regarded as the boundary of the infinite region $R_E - V_\sigma$.

We shall call the derivative of the potential in the direction of the normal n to the equipotential surface σ

$$g = \mathrm{d}w/\mathrm{d}n \tag{20}$$

the acceleration due to gravity *. The acceleration due to gravity is the quantity that can be directly and most accurately measured on the surface of the earth in comparison with the other characteristics of the gravity field. The acceleration due to gravity is of great interest in various applications. Therefore, many gravimetric formulae have been derived with the purpose of expressing some quantity or other in terms of the acceleration due to gravity, or of determining this acceleration at all points of the surface of the earth from direct measurements at a finite number of points. We shall derive some of these gravimetric formulae later.

Suppose that σ is an equipotential surface lying entirely outside the region occupied by a planet. If we apply formula (43) of Chapter XVIII to the potential W and use the symbol d/dn defined above, we obtain

* By "acceleration due to gravity" the author means the acceleration resulting from the weight potential (and not from the Newtonian potential alone).

$$-\iint_{\sigma}\left[\frac{1}{r}\frac{dw}{dn} - W_0\frac{d}{dn}\left(\frac{1}{r}\right)\right]dS - \iiint_{V_\sigma}\frac{\Delta W}{r}\,dV = \begin{cases} 4\pi W(x) & \text{when } x \in V_\sigma - \sigma \\ 2\pi W_0 & \text{when } x \in \sigma \\ 0 & \text{when } x \in R_E - V_\sigma \end{cases} \tag{21}$$

It follows from eq. (19) that

$$\iiint_{V_\sigma}\frac{\Delta W}{r}\,dV = -4\pi\kappa\iiint_{V_\rho}\frac{\rho}{r}\,dV + 2\omega^2\iiint_{V_\sigma}\frac{1}{r}\,dV = -4\pi\kappa\; U(x) + 2\omega^2\iiint_{V_\sigma}\frac{dV}{r}\,.$$

Furthermore, if we set

$$u = W_0\,, \qquad L(\xi, x) = \frac{1}{4\pi}\,\frac{1}{r}$$

in formula (43) of Chapter XVIII, we obtain

$$\iint_{\sigma} W_0\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS = \begin{cases} 4\pi W_0 & \text{when } x \in V_\sigma - \sigma\,, \\ 2\pi W_0 & \text{when } x \in \sigma\,, \\ 0 & \text{when } x \in R_E - V_\sigma\,. \end{cases}$$

If we substitute the expression just obtained and also the expressions (17) and (20) into formula (21), we obtain

$$\frac{1}{4\pi}\iint_{\sigma}\frac{g}{r}\,dS + \frac{\omega^2}{2\pi}\iiint_{V_\sigma}\frac{dV}{r} + \tfrac{1}{2}\omega^2(x_1{}^2 + x_2{}^2) = W_0 \qquad (x \in V_\sigma)\,, \tag{22}$$

$$\frac{1}{4\pi}\iint_{\sigma}\frac{g}{r}\,dS + \frac{\omega^2}{2\pi}\iiint_{V_\sigma}\frac{dV}{r} = \kappa U(x) \qquad (x \in R_E - V_\sigma + \sigma)\,. \tag{23}$$

Formula (23) determines the potential $\kappa U(x)$ of the Newtonian force of gravitation outside the equipotential surface σ from the distribution of the acceleration due to gravity on σ. Thus, if we know the angular velocity of rotation ω of the planet, the equipotential surface σ, and the distribution of the acceleration due to gravity on the surface, then the gravitational field of the planet in the region $R_E - V_\sigma$ can be completely determined. However, a much stronger assertion, known as Stokes' theorem, is also valid: if (1) the equipotential surface σ enclosing the planet, (2) the mass of the planet, and (3) the angular velocity of the rotation of the planet are known, then the potential of the gravitational force outside σ and the acceleration due to gravity on σ are uniquely determined.

Thus, Stokes' theorem asserts that knowledge of just the mass of the planet, its angular velocity, and the equipotential surface of the weight potential makes it possible to solve the fundamental gravimetric problem, namely, the determination of the gravitational field of a planet and the distribution of the "weight" field on its surface. (This latter problem, of course, requires knowledge of an equipotential surface close to the surface of the planet.) It is not necessary to know the mass distribution within the planet.

Let us proceed to prove Stokes' theorem. From formula (23), it is sufficient to show that knowledge of the equipotential surface σ, the mass of the planet m, and its angular velocity ω determines uniquely the acceleration due to gravity g on σ. Let us suppose the opposite. Assume that g_1 and

g_2 are two distinct distributions of the acceleration due to gravity on the equipotential surface σ.

Integrating eq. (19) over the volume V_σ, we obtain

$$\iiint\limits_{V_\sigma} \Delta W \, dV = -4\pi\kappa m + 2\omega^2 V \, .$$

On the other hand, by applying the Ostrogradskii-Gauss formula and remembering the definition given above for the symbol d/dn, we see that

$$\iiint\limits_{V_\sigma} \Delta W \, dV = -\iint\limits_{\sigma} \frac{dW}{dn} \, dS = -\iint\limits_{\sigma} g \, dS \, .$$

Eliminating the volume integral from these two equations, we obtain

$$\iint\limits_{\sigma} g \, dS = 4\pi\kappa m + 2\omega^2 V \, . \tag{24}$$

If we now set $g = g_1$ in one case and $g = g_2$ in the other and if we take the difference of the resulting expressions, we see that

$$\iint\limits_{\sigma} (g_1 - g_2) \, dS = 0 \, . \tag{25}$$

On the other hand, if we make the substitution $g = g_1$ and $g = g_2$ in formula (22) and subtract the second of the resulting equations from the first, we see that

$$\iint\limits_{\sigma} \frac{g_1 - g_2}{r} \, dS = W_{01} - W_{02} = \text{constant} \quad \text{when} \quad x \in V_\sigma \, . \tag{26}$$

Here, the expressions W_{01} and W_{02} denote the weight potentials on the level surface σ for $g = g_1$ and $g = g_2$. The integral $\overline{U}$ on the left side of eq. (26) represents a single-layer potential. From formulae (38) of Chapter XIX, the difference between its external and internal normal derivatives on the surface σ is equal to

$$\frac{\partial \overline{U}}{\partial n_e} - \frac{\partial \overline{U}}{\partial n_i} = 4\pi(g_1 - g_2) \, .$$

Since the potential $\overline{U}$ is constant in the region V_σ, the derivative $\partial\overline{U}/\partial n_i = 0$ and the derivative $\partial\overline{U}/\partial n_e$ does not change sign. The first assertion is obvious. The second follows from the fact that in the region $R_E - V_\sigma$ the potential $\overline{U}$ is harmonic, so that the constant value that it has on σ represents either a maximum or a minimum. This would be impossible if the derivative $\partial\overline{U}/\partial n_e$ changes sign.

It follows from the retention of the sign of $\partial\overline{U}/\partial n_e$ that the difference $g_1 - g_2$ retains its sign on σ and it then follows from eq. (25) that $g_1 = g_2$. This proves Stokes' theorem.

Problems

1. Show that a change in the mass distribution in a planet can cause a change in the field of gravitation everywhere outside the planet. Explain how this does not contradict Stokes' theorem.

2. Show that a redistribution of masses in a planet may fail to cause a change in the equipotential surface of the weight potential.
Method: Examine a planet with a mass distribution that depends only on the radius.

3. Derive Pointcaré's formula

$$\rho_{\mathrm{av}} = \frac{\omega^2}{2\pi\kappa} + \frac{1}{4\pi\kappa V_S} \iint_S g \, dS ,$$

where ρ_{av} is the average density of the planet, S is an arbitrary closed surface within which the planet is located, and V_S is the volume of the finite region bounded by the surface S.
Method: Use formula (24), noting that it is valid even when σ is an arbitrary surface surrounding the planet.

4. Show that the velocity of rotation of a liquid planet obeys the inequality

$$\omega^2 \leqslant 2\pi\kappa\rho_{\mathrm{av}} ,$$

where ρ_{av} is the average density of the planet.
Hint: For a liquid planet, g must be positive.

4. *The basic gravimetric problem*

A figure bounded by an equipotential surface of the weight potential for the earth is called a geoid. The importance of determining experimentally the shape and position of the geoid containing all the earth's mass is clear from the preceding section. This would make it possible to determine completely its external gravitational field. Therefore, we concentrate our attention on the problem of finding the surface of the geoid from the observational data on the acceleration due to gravity.

Let us take some surface S, assumed to be close to the surface of the geoid, and let us call it the reference surface. The weight potential W calculated on the assumption that the reference surface is an equipotential surface we shall call *undisturbed*. As Stokes' theorem tells us, to compute the undisturbed potential, it is sufficient to know the mass and rotational velocity of the earth.

The difference

$$T = w - W \tag{27}$$

between the actual potential w and the undisturbed potential W of the field of gravity we shall call the *disturbing potential*. The existence of a disturbing potential is a consequence of the deviation of the reference surface from

the surface of the geoid. By assumption, this deviation is small. Therefore, the disturbing potential causes only small corrections in the values of the undisturbed potential. Since the potential of the centrifugal force obviously appears in the same way in the potentials w and W, the disturbing potential T depends only on the distribution of the earth's mass and not on the velocity of its rotation.

Let us now compare the reference surface S with that geoid on whose surface the weight potential is equal to the value W_0 of the undisturbed potential on the reference surface. Then, for points of the surface of the geoid,

$$W + T = W_0 \,. \tag{28}$$

Suppose that ξ is a point on the reference surface S, that n is the normal to S at the point ξ, that ζ is the point on the geoid in question that lies on the normal n, and that δn is the distance between the points ξ and ζ measured in the positive direction of the normal n. Then, the undisturbed potential at the point ξ of the geoid can be determined by the formula

$$W(\zeta) = W_0 + \frac{dW}{dn}\,\delta n = W_0 + g_0(\xi)\,\delta n \;,$$

where

$$g_0(\xi) = dW/dn \tag{29}$$

is the undisturbed acceleration due to gravity at the point ξ. But from formula (28),

$$W(\zeta) = W_0 - T(\zeta) \,.$$

By combining these expressions, we obtain

$$\delta n = -\frac{T(\zeta)}{g_0(\xi)} \,. \tag{30}$$

By carrying the analysis further, we can confirm the experimental observation that for every geoid there is a sphere (with center at the center of gravity of the earth) from which the geoid deviates only slightly (that is, in comparison with the radius of the sphere). We choose for our reference surface the surface S of the one of these spheres of radius R_e (that is, of radius close to the average value of the radius of the earth). Here, we are assuming that the earth's mass is enclosed by the reference surface and the geoids corresponding to it.

Let us introduce the spherical coordinates R, θ, and φ with origin at the center of gravity of the earth. Remembering that the undisturbed potential is a harmonic function with constant value W_0 on the spherical surface S, we easily find the analytic expression for it:

$$W = W_0 R_e / R \;, \tag{31}$$

so that

$$g_0 = \frac{dW}{dn} = -\frac{\partial W}{\partial R} = \frac{W_0 R_e}{R^2} \,. \tag{32}$$

The actual acceleration due to gravity at the point ζ of the surface of the geoid is, from formula (20), equal to

$$g(\zeta) = \frac{dw}{dn_g}\Big|_\zeta = \frac{dW}{dn_g}\Big|_\zeta + \frac{dT}{dn_g}\Big|_\zeta , \tag{33}$$

where d/dn_g denotes differentiation along the normal to the geoid and the subscript ζ means that the value of the derivative is taken at the point ζ. Since the surface of the geoid is, by assumption, close to the reference surface (a plumb line almost coincides with the direction towards the center of the earth), we may, with no significant error, take

$$\partial/\partial n_g = - \partial/\partial R ,$$

so that

$$\frac{\partial W}{\partial n_g}\Big|_\zeta \approx - \frac{\partial W}{\partial R}\Big|_\zeta = g_0(\xi) - \frac{\partial g_0}{\partial R}\Big|_\zeta \delta n .$$

Substituting the values for δn and g_0 given by formulae (30) and (32), we obtain

$$\frac{dW}{dn_g} \approx g(\xi) - \frac{2T(\zeta)}{R_e} ,$$

which, after substitution into formula (33), leads to the relationship

$$g(\zeta) - g_0(\xi) = \frac{\partial T}{\partial n}\Big|_\zeta - \frac{2T(\zeta)}{R_e} . \tag{34}$$

The difference

$$\psi(\xi) = g(\zeta) - g_0(\xi) \qquad (\xi \in S)$$

is called the *gravitational anomaly* of the geoid with respect to the reference surface. The gravitational anomaly can be considered as known (as a result of numerous measurements of the acceleration due to gravity that have been made on the surface of the earth).

Because of the assumption that the reference surface is close to the surface of the geoid, the disturbing potential on the reference surface must deviate only slightly from the values that it assumes at the corresponding points on the geoid. This makes it possible, without great error, to use the values of the disturbing potential on the right sides of formulae (30) and (34) for the reference surface, which gives us

$$\delta n = T(\xi)/g_0(\xi) , \tag{35}$$

$$\frac{\partial T}{\partial n} - \frac{2T}{R_e} = \psi(\xi) \qquad (\xi \in S) . \tag{36}$$

When the disturbing potential T is determined, the position of the geoid relative to the reference surface can be found from formula (35). Since the disturbing potential is obviously harmonic, in seeking to determine it we encounter the exterior mixed problem for Laplace's equation with the boundary condition (36). This problem is called the *basic problem of gravimetry*. In the following section, we shall solve it by Green's method.

5. *The solution of the basic problem of gravimetry by Green's method*

Before we set about solving the basic problem, let us establish certain properties of the disturbing potential. With this in mind, let us compare the expansions of a disturbed and an undisturbed potential in multipole series (Chapter XIX, sections 2 and 4), which we shall assume concentrated at the center of the reference surface. The terms that depend on the velocity of rotation of the earth coincide in the two expansions. The terms that represent a potential of zero order also coincide since they are determined only by the mass of the earth. Finally, the first-order terms are equal to zero in both expansions, since the multipoles are assumed to be concentrated at the center of the reference surface, which coincides with the center of gravity of the earth. Consequently, the expansion of the disturbing potential begins with second-order terms; that is, it is of the form

$$T = \sum_{\alpha=2}^{\infty} \frac{Y_\alpha}{R^{\alpha+1}} , \tag{37}$$

where the Y_α are spherical functions and R is the distance from the points of observation to the center of gravity of the earth. Because of the orthogonality of spherical functions of different orders, it follows that the disturbing potential and its derivative are orthogonal to all spherical functions of order lower than the second on an arbitrary spherical surface whose center is at the center of the reference surface. Therefore, by considering formula (36), we conclude that the gravitational anomaly $\psi(\theta,\varphi)$ satisfies the following orthogonality conditions:

$$\iint_S \psi(\xi)\, dS = 0 , \qquad \iint_S \psi(\xi)\, P_1(\cos\gamma)\, dS = 0 , \tag{38}$$

where $P_1(\cos\gamma)$ is the Legendre polynomial of the first-order and γ is the angle between an arbitrary fixed direction and the radius drawn from the center of the reference surface to the point ξ.

Let us now determine the Green's function $G(\xi,x)$ (the basic problem of gravimetry). From relationships (59) and (66) of Chapter XVIII, we have

$$G(\xi,x) = \frac{1}{4\pi}\left[\frac{1}{r} + \varphi(\xi,x)\right] , \tag{39}$$

where r is the distance between the two points ξ and x, and $\varphi(\xi,x)$ is the solution to the boundary problem

$$\Delta_\xi \varphi = 0 \quad \text{when} \quad \xi, x \in R_E - V_S , \tag{40}$$

$$\frac{d\varphi}{dn} - \frac{2}{R_e}\varphi = -\left(\frac{d}{dn} - \frac{2}{R_e}\right)\frac{1}{r} \quad \text{when} \quad \xi \in S ,\ x \in R_E - V_S . \tag{41}$$

We find the function φ by means of an expansion in Legendre polynomials. Denoting by γ the angle between the directions from the center of the reference surface S to the points x and ξ, respectively, and denoting by r the distance between the point ξ and the center of the reference surface (fig. 57), we find that

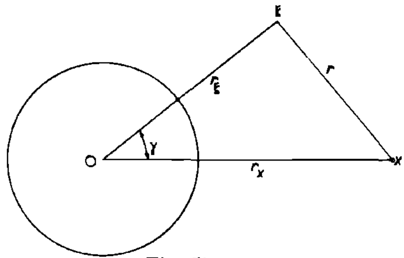

Fig. 57

$$\frac{1}{r} = \frac{1}{\sqrt{R^2 + r_\xi^2 - 2Rr_\xi \cos\gamma}} = \sum_{\alpha=0}^{\infty} P_\alpha(\cos\gamma) \frac{R^\alpha}{r_\xi^{\alpha+1}} . \tag{42}$$

Let us write the function φ in the form

$$\varphi = \varphi_1 - \frac{1}{R} - \frac{r_\xi \cos\gamma}{R^2} . \tag{43}$$

We expand the functions φ_1 in a series of Legendre polynomials:

$$\varphi_1 = \sum_{\alpha=0}^{\infty} a_\alpha P_\alpha(\cos\gamma) \left(\frac{R_e^2}{Rr_\xi}\right)^{\alpha+1} \qquad (R, r_\xi > R_e) , \tag{44}$$

where the a_α are as yet undetermined coefficients. To evaluate them, we use the boundary condition (41). Noting that

$$\frac{d}{dn} = -\frac{d}{dr_\xi} ,$$

we compute the derivatives

$$\frac{d\varphi}{dn} = -\frac{\cos\gamma}{R^2} + \sum_{\alpha=0}^{\infty} (\alpha+1)\, a_\alpha P_\alpha(\cos\gamma) \left(\frac{R_e^2}{Rr_\xi}\right)^{\alpha+1} ,$$

$$\frac{d}{dn}\left(\frac{1}{r}\right) = -\sum_{\alpha=0}^{\infty} \alpha P_\alpha(\cos\gamma) \frac{r_\xi^{\alpha-1}}{R^{\alpha+1}} .$$

Substituting these values into the boundary condition (41) and remembering that $r_\xi = R_e$ when ξ is a point on S, we obtain, after some simple manipulations,

$$-\frac{2}{RR_e} - \frac{3\cos\gamma}{R^2} + \sum_{\alpha=0}^{\infty} [(\alpha+2) - a_\alpha(\alpha-1) R_e] \frac{P_\alpha(\cos\gamma) R_e^{\alpha-1}}{R^{\alpha+1}} = 0 .$$

This equation will be identically satisfied if we set

$$a_0 = 0 , \qquad a_\alpha = \frac{\alpha+2}{\alpha-1}\frac{1}{R_e} = \left(1 + \frac{3}{\alpha-1}\right)\frac{1}{R_e} \qquad (\alpha \geqslant 2) ,$$

where the coefficient a_1 may be chosen arbitrarily. Substituting these values of the coefficients into the series (44) and substituting this series into eq. (43), we obtain

$$\varphi = -\frac{1}{R} - \frac{r_\xi \cos\gamma}{R^2} + a_1 \cos\gamma \frac{R_e^4}{R^2 r_\xi^2} + \frac{R_e}{r_\xi R} \sum_{\alpha=2}^{\infty} P_\alpha(\cos\gamma) \left(\frac{R_e^2}{Rr_\xi}\right)^\alpha + \frac{3R_e^3}{R^2 r_\xi^2} \sum_{\alpha=2}^{\infty} \frac{P_\alpha(\cos\gamma)}{\alpha - 1} \left(\frac{R_e^2}{Rr_\xi}\right)^{\alpha-1} . \quad (45)$$

These series are easily summed. By applying formula (15) of Chapter XV to the first of them, we easily see that

$$\frac{R_e}{Rr_\xi} \sum_{\alpha=2}^{\infty} P_\alpha(\cos\gamma) \left(\frac{R_e^2}{Rr_\xi}\right)^\alpha = -\frac{R_e}{Rr_\xi} - \frac{R_e^3 \cos\gamma}{R^2 r_\xi^2} + \frac{R_e}{\sqrt{R_e^4 + R^2 r_\xi^2 - 2R_e^2 Rr_\xi \cos\gamma}} .$$

To sum the second series, we need only integrate the equation

$$\sum_{\alpha=2}^{\infty} P_\alpha(\cos\gamma)\, s^{\alpha-2} = \frac{1}{s^2 \sqrt{1 + s^2 - 2s\cos\gamma}} - \frac{1}{s^2} - \frac{\cos\gamma}{s} .$$

We then obtain

$$\sum_{\alpha=2}^{\infty} \frac{P_\alpha(\cos\gamma)}{\alpha - 1} s^{\alpha-1} = \frac{1 - \sqrt{1 + s^2 - 2s\cos\gamma}}{s} - \cos\gamma \ln\left(1 - s\cos\gamma + \sqrt{1 + s^2 - 2s\cos\gamma}\right) + C ,$$

where C is a constant. Noting that, for $s = 0$, the left side of this equation is equal to zero and choosing C suitably, we obtain

$$\sum_{\alpha=2}^{\infty} \frac{P_\alpha(\cos\gamma)}{\alpha - 1} s^{\alpha-1} = \frac{1 - s\cos\gamma - \sqrt{1 + s^2 - 2s\cos\gamma}}{s} - \cos\gamma \ln \frac{1 - s\cos\gamma + \sqrt{1 + s^2 - 2s\cos\gamma}}{2} ,$$

on the basis of which,

$$\frac{3R_e^3}{R^2 r_\xi^2} \sum_{\alpha=2}^{\infty} \frac{P_\alpha(\cos\gamma)}{\alpha - 1} \left(\frac{R_e^2}{Rr_\xi}\right)^{\alpha-1} = \frac{3R_e}{Rr_\xi} - \frac{3R_e^3 \cos\gamma}{R^2 r_\xi^2} - \frac{3R_e}{R^2} \sqrt{R_e^4 + R^2 r_\xi^2 - 2R_e^2 Rr_\xi \cos\gamma} - \frac{3R_e^3}{R^2 r_\xi^2} \ln \frac{Rr_\xi - R_e^2 \cos\gamma + \sqrt{R_e^4 + R^2 r_\xi^2 - 2R_e^2 Rr_\xi \cos\gamma}}{2Rr_\xi} .$$

Substituting the expressions for the sums of the infinite series into eq. (45) and substituting the value of the function φ that we have found into formula (39), we get the following expression for the Green's function of the problem in question:

$$G(\xi,x) = \frac{1}{4\pi}\left\{\frac{1}{r} + \frac{2R_e}{Rr_\xi} - \frac{1}{R} - \frac{r\cos\gamma}{R^2} + \left(a_1 - \frac{4}{R_e}\right)\frac{R_e^4\cos\gamma}{R^2r_\xi^2}\right.$$
$$+ \frac{R_e}{\sqrt{R_e^2 + R^2r_\xi^2 - 2R_e^2Rr_\xi\cos\gamma}} - \frac{3R_e}{R^2r_\xi^2}\sqrt{R_e^4 + R^2r_\xi^2 - 2R_e^2Rr_\xi\cos\gamma}$$
$$\left. - \frac{3R_e^3}{R^2r_\xi^2}\ln\frac{Rr_\xi - R_e^2\cos\gamma + \sqrt{R_e^4 + R^2r_\xi^2 - 2R_e^2Rr_\xi\cos\gamma}}{2Rr_\xi}\right\}. \quad (46)$$

This expression can be considerably simplified by introducing the distance r^* from the point ξ to the point x^* (the harmonic conjugate of the point x with respect to the reference surface). This yields

$$G(\xi,x) = \frac{1}{4\pi}\left\{\frac{1}{r} + \frac{R_e}{Rr^*} - \frac{3R_er^*}{Rr_\xi^2} + \frac{2R_e}{Rr_\xi} - \frac{1}{R}\right.$$
$$\left. + \left(a_1 - \frac{4}{R_e}\right)\frac{R_e^4\cos\gamma}{R^2r_\xi^2} - \frac{r_\xi\cos\gamma}{R^2} - \frac{3R_e^3\cos\gamma}{R^2r_\xi^2}\ln\frac{Rr_\xi - R_e^2\cos\gamma + Rr^*}{2Rr_\xi}\right\} \quad (47)$$

On the reference surface S, where

$$r^* = rR_e/R\ ,$$

the Green's function is of the form

$$G(\xi,x)|_{\xi\in S} = \frac{1}{4\pi}\left\{\frac{2}{r} - \frac{3r}{R^2} + \frac{1}{R} + \left(a_1 - \frac{5}{R_e}\right)\frac{R_e^2\cos\gamma}{R^2}\right.$$
$$\left. - \frac{3R_e}{R^2}\cos\gamma\ln\frac{R - R_e\cos\gamma + r}{2R}\right\}. \quad (48)$$

Substituting this expression into formula (67) of Chapter XVIII and remembering the orthogonality relations (38), we obtain the Vening-Meinesz formula:

$$T = \frac{1}{4\pi}\iint_S \psi\left\{\frac{3r}{R^2} - \frac{2}{r} + \frac{3R_e}{R^2}\cos\gamma\ln\frac{R - 2r\cos\gamma + r}{2R}\right\}dS\ , \quad (49)$$

which gives the solution to the basic problem of gravimetry in closed form.

Chapter XXIII

APPLICATION OF THE THEORY OF SPHERICAL FUNCTIONS TO THE SOLUTION OF PROBLEMS IN MATHEMATICAL PHYSICS

1. *The electrostatic potential of a conducting sphere divided into two hemispheres by a dielectric layer*

Let us suppose that a conducting sphere is divided into two hemispheres by a layer of insulating material. Suppose that the upper hemisphere is charged to a potential U_1 and the lower to a potential U_2. Let us determine the potential of the electric field at an arbitrary point. We shall solve the problem in spherical coordinates (R, θ, φ) with the origin at the center of the sphere and the polar axis directed perpendicularly to the insulating layer.

This problem is obviously the Dirichlet problem for Laplace's equation with boundary condition

$$U|_{R=R_0} = \begin{cases} U_1 & \text{for } 0 \leqslant \theta < \frac{1}{2}\pi , \\ U_2 & \text{for } \frac{1}{2}\pi < \theta \leqslant \pi , \end{cases} \tag{1}$$

where $U = U(R, \theta)$ is the potential at the point (R, θ, φ) and R_0 is the radius of the sphere.

The solution to this problem is given for points inside and outside the sphere by expansions (32) and (33) of Chapter XXI:

$$U = \sum_{k=0}^{\infty} Y_k(\theta, \varphi) \left(\frac{R_0}{R}\right)^{k+1} \qquad (R_0 \leqslant R) , \tag{2}$$

$$U = \sum_{k=0}^{\infty} Y_k(\theta, \varphi) \left(\frac{R}{R_0}\right)^{k} \qquad (R_0 \geqslant R) \tag{3}$$

Since the field obviously does not depend on the coordinate φ, we set $a_{\beta k} = 0$ in formula (28) of Chapter XXI for $\beta \neq 0$. This gives us

$$Y_n(\theta, \varphi) = a_n P_n(\cos\theta) .$$

If we now compare the expansions

$$U = \sum_{k=0}^{\infty} a_k \left(\frac{R_0}{R}\right)^{k+1} P_k(\cos\theta) \qquad (R_0 \leqslant R) \tag{4}$$

and

$$U = \sum_{k=0}^{\infty} a_k \left(\frac{R}{R_0}\right)^{k} P_k(\cos\theta) \qquad (R_0 \geqslant R) \tag{5}$$

with expansion (7) of Chapter XV, we obtain, on the basis of formula (8) of Chapter XV, expressions for the coefficient a_n:

$$a_n = \frac{2n+1}{2} \int_0^{\pi} f(\theta)\ P_n(\cos\theta)\ \sin\theta\ d\theta\ , \tag{6}$$

where $f(\theta)$ is the value of the function U on the surface of the sphere. By substituting the boundary condition, we obtain

$$a_n = \frac{2n+1}{2} \left\{ \int_0^{\frac{1}{2}\pi} U_1 P_n(\cos\theta)\ \sin\theta\ d\theta + \int_{\frac{1}{2}\pi}^{\pi} U_2 P_n(\cos\theta)\ \sin\theta\ d\theta \right\}$$

$$= \frac{2n+1}{2} \left\{ U_1 \int_0^1 P_n(x)\ dx + U_2 \int_{-1}^0 P_n(x)\ dx \right\} .$$

Since the Legendre polynomial $P_n(x)$ is even or odd according as the subscript n is even or odd, that is, since

$$P_n(-x) = (-1)^n P_n(x)\ ,$$

it follows that

$$a_n = \frac{2n+1}{2} [U_1 + (-1)^n U_2] \int_0^1 P_n(x)\ dx\ .$$

We now use the formula (see problem 1 of Chapter XV)

$$\int_0^1 P_x(x)\ dx = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n = 2k,\ k > 0, \\ (-1)^k \dfrac{(2k)!}{2^{2k+1}\ k!\ (k+1)!} & \text{for } n = 2k+1. \end{cases}$$

It follows from this that

$$a_0 = \tfrac{1}{2}(U_1 + U_2)\ , \qquad a_{2k} = 0\ , \qquad a_{2k+1} = \frac{U_1 - U_2}{2^{2k+1}} (-1)^k \frac{(2k)!}{k!\,(k+1)!} (4k+3)\ .$$

If we substitute the values that we have found for the coefficient a_n in the expansions (4) and (5), we obtain the desired potential of the electrostatic field:

$$U = \tfrac{1}{2}(U_1 + U_2) \frac{R_0}{R} + \tfrac{1}{2}(U_1 - U_2) \left[\frac{3}{2} \left(\frac{R_0}{R}\right)^2 P_1(\cos\theta) \right.$$

$$\left. - \frac{7}{8} \left(\frac{R_0}{R}\right)^4 P_3(\cos\theta) + \ldots \right] \quad (R_0 \leqslant R)\ , \tag{7}$$

$$U = \tfrac{1}{2}(U_1 + U_2) + \tfrac{1}{2}(U_1 - U_2) \left[\frac{3}{2} \left(\frac{R}{R_0}\right) P_1(\cos\theta) \right.$$

$$\left. - \frac{7}{8} \left(\frac{R}{R_0}\right)^3 P_3(\cos\theta) + \ldots \right] \quad (R_0 \geqslant R)\ . \tag{8}$$

2. The problem of steady-state temperature in a sphere

Suppose that we have a metallic sphere with a black surface that is subject to the action of the sun's rays in the air. For simplicity, we assume its temperature to be equal to zero. Let us determine the steady-state temperature of the internal points of the sphere (fig. 58).

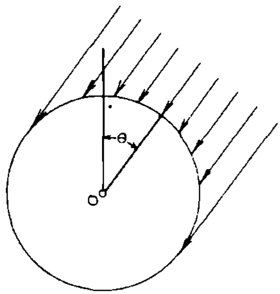

Fig. 58.

We know from section 1 of Chapter XVIII that this temperature must satisfy Laplace's equation. The boundary condition that must be satisfied on the surface of the sphere is

$$\frac{\partial u}{\partial R}\Big|_{R=R_0} = -p\,[u - f(\theta)]\,\Big|_{R=R_0}, \tag{9}$$

where R_0 is the radius of the sphere, $p = h/\gamma$ is the ratio of the coefficients of heat emission and internal thermal conductivity, and $f(\theta)$ is the temperature that would be observed on the surface of the sphere if there were no radiation from the surface into the surrounding air.

If we remember that the degree of heating is proportional to the sine of the angle of incidence of the rays hitting the surface, it is obvious that the function $f(\theta)$ is defined as follows

$$f(\theta) = \begin{cases} A\cos\theta & \text{for } 0 \leqslant \theta \leqslant \frac{1}{2}\pi\,, \\ 0 & \text{for } \frac{1}{2}\pi \leqslant \theta \leqslant \pi\,, \end{cases} \tag{10}$$

where A is a constant that depends on the intensity of the solar radiation.

Let us seek a solution to this problem in the form of an infinite series

$$u = \sum_{k=0}^{\infty} a_k \left(\frac{R}{R_0}\right)^k P_k(\cos\theta) \qquad (R_0 \geqslant R) \tag{11}$$

with as yet undetermined coefficients a_k.

If we expand the function $f(\theta)$ in a series of Legendre polynomials

$$f(\theta) = \sum_{k=0}^{\infty} b_k P_k(\cos\theta)\,, \tag{12}$$

where, from formula (8) of Chapter XV,

$$b_k = \frac{2k+1}{2} \int_0^{\pi} f(\theta)\, P_k(\cos\theta)\, \sin\theta\, d\theta \,, \tag{13}$$

and if we substitute eq. (11) into eq. (9), we obtain

$$\sum_{k=0}^{\infty} \left[a_k \left(\frac{k}{R_0} + p\right) - p b_k\right] P_k(\cos\theta) = 0 \,,$$

which is satisfied identically if we set

$$a_k = \frac{pR_0}{k + pR_0}\, b_k \,. \tag{14}$$

It remains now to calculate the numbers b_k, which is easily done by using formulae (10) and (13). It follows from these formulae that

$$b_k = \frac{2k+1}{2} \int_0^{\frac{1}{2}\pi} A \cos\theta\, P_n(\cos\theta)\, \sin\theta\, d\theta = \frac{2k+1}{2} A \int_0^1 P_k(x)\, x\, dx \,,$$

from which a direct calculation gives

$$b_0 = \tfrac{1}{2}A \int_0^1 x\, dx = \tfrac{1}{4}A \,, \qquad b_1 = \tfrac{3}{2}A \int_0^1 x^2\, dx = \tfrac{1}{2}A \,.$$

On the other hand, it was shown in Chapter XV in the discussion of Legendre polynomials that

$$\int_0^1 x\, P_k(x)\, dx = \begin{cases} 0 & \text{for } k = 2n+1 \quad (n > 0) \,, \\ \dfrac{(-1)^n\, (2n-2)!}{2^{2n}(n-1)!\,(n+1)!} & \text{for } k = 2n \quad (n > 0) \,, \end{cases}$$

from which it follows that

$$b_{2k+1} = 0 \qquad (k > 0) \,,$$

$$b_{2k} = (-1)^k A \frac{(2k-2)!\,(4k+1)}{2^{2k+1}\,(k-1)!\,(k+1)!} \qquad (k > 0) \,.$$

Consequently,

$$a_0 = \tfrac{1}{4}A \,, \qquad a_1 = \frac{A}{2}\, \frac{pR_0}{pR_0 + 1} \,,$$

$$a_{2k+1} = 0 \,, \qquad a_{2k} = (-1)^k A \frac{pR_0}{pR_0 + 1}\, \frac{(2k-2)!\,(4k+1)}{2^{2k+1}\,(k-1)!\,(k+1)!} \qquad (k > 0) \,.$$

Substituting the values that we have found for the coefficient a_k in the expansion (11), we obtain the temperature of the sphere in the form of the infinite series

$$u = pAR_0\left[\frac{1}{4pR_0} + \frac{1}{pR_0+1}\,\frac{1}{2}\,\frac{R}{R_0}\,P_1(\cos\theta) + \frac{1}{pR_0+2}\,\frac{5}{16}\left(\frac{R}{R_0}\right)^2 P_2(\cos\theta) + \ldots\right], \qquad (15)$$

where $R_0 \geqslant R$.

3. *The problem of charge distribution on an inductively charged sphere*

Let us apply the theory of spherical functions to the solution of the following electrostatic problem:

A body B is made of a dielectric material. The density of the electric charges distributed throughout the body is a known function

$$\rho = f(R_2, \theta_2, \varphi_2) \qquad (16)$$

of the coordinates of the point x (fig. 59).

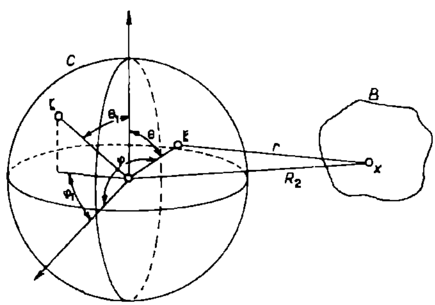

Fig. 59.

Let us suppose also that a spherical conductor C with a charge q is placed close to the body B. A continuously distributed layer of charge will be induced on this conductor. Let us determine the charge density ρ_1 of this layer at an arbitrary point $\zeta(R_0, \theta_1, \varphi_1)$, on the surface of the conductor. We denote by U the potential of the electrostatic field at an arbitrary point $\xi(R, \theta, \varphi)$ *within* the conductor. This potential can be represented as the sum

$$U = U_B + U_C\,, \qquad (17)$$

where U_B denotes the potential resulting from the presence of a charge in the body B and U_C denotes the potential caused by the charge that is induced on the surface of the conductor.

Let us first determine the potential U_B. We know from section 1 of Chapter XVIII that this potential is determined by the formula

$$U_B = \iiint\limits_{(B)} \frac{\rho}{r}\, dV ,$$

where r denotes the distance from the point $\xi(R, \theta, \varphi)$ to a variable point $x(R_2, \theta_2, \varphi_2)$ in the body B. It follows from this that

$$U_B = \iiint\limits_{(B)} \frac{f(R_2, \theta_2, \varphi_2)\, R_2^2 \sin\theta_2\, dR_2\, d\theta_2\, d\varphi_2}{\sqrt{R^2 + R_2^2 - 2RR_2 \cos\gamma}} ,$$

where

$$\cos\gamma = \cos\theta \cos\theta_2 + \sin\theta \sin\theta_2 \cos(\varphi - \varphi_2) .$$

By using the now familiar expansion (Chapter XV, section 5):

$$\frac{1}{\sqrt{R^2 + R_2^2 - 2RR_2 \cos\gamma}} = \sum_{k=0}^{\infty} P_k(\cos\gamma) \frac{R^k}{R_2^{k+1}} \quad (R_2 > R) ,$$

we can represent the potential U_B as the infinite series

$$U_B = \sum_{k=0}^{\infty} X_k(\theta, \varphi)\, R^k , \tag{18}$$

where

$$X_k(\theta, \varphi) = \iiint\limits_{(B)} f(R_2, \theta_2, \varphi_2)\, P_k(\cos\gamma) \sin\theta_2 \frac{dR_2\, d\theta_2\, d\varphi_2}{R_2^{k-1}} . \tag{19}$$

Since $f(R_2, \theta_2, \varphi_2)$ is assumed known over the entire volume B, the function $X_k(\theta, \varphi)$ is completely determined by formula (19).

Let us now find the potential U_C. We denote by

$$\rho_1 = f_1(\theta_1, \varphi_1) \tag{20}$$

the density of the layer of charge on the surface of the conductor. Then, the potential of the electric field at the point $\xi(R, \theta, \varphi)$ is determined by the formula

$$U_C = \int_0^{\pi} \int_0^{2\pi} \frac{f_1(\theta_1, \varphi_1)\, R_0^2 \sin\theta_1\, d\theta_1\, d\varphi_1}{\sqrt{R^2 + R_0^2 - 2RR_0 \cos\gamma_1}}$$

where

$$\cos\gamma_1 = \cos\theta \cos\theta_1 + \sin\theta \sin\theta_1 \cos(\varphi - \varphi_1) .$$

It then follows that the potential U_C can be represented as the infinite series

$$U_C = \sum_{k=0}^{\infty} \frac{R^k}{R_0^{k-1}} \int_0^{\pi} \int_0^{2\pi} f_1(\theta_1, \varphi_1)\, P_k(\cos\gamma_1) \sin\theta_1\, d\theta_1\, d\varphi_1 . \tag{21}$$

Let us now assume that the function $f_1(\theta_1,\varphi_1)$ is expanded in a series of spherical functions

$$f_1(\theta_1,\varphi_1) = \sum_{k=0}^{\infty} Y_k(\theta_1,\varphi_1) . \tag{22}$$

From formula (20), it is clear that when we determine the functions $Y_k(\theta_1,\varphi_1)$, we shall have the desired charge density. To find these functions, we substitute expansion (22) into formula (21). Then, by using the integral relationship (23) of Chapter XXI, we obtain the expansion

$$U_C = 4\pi \sum_{k=0}^{\infty} \frac{Y_k(\theta,\varphi)}{2k+1} \frac{R^k}{R_0{}^{k-1}} , \tag{23}$$

from which, because of formulae (17) and (18), it follows that

$$U = \sum_{k=0}^{\infty} \left\{ X_k(\theta,\varphi) + \frac{4\pi}{R_0{}^{k-1}} \frac{Y_k(\theta,\varphi)}{2k+1} \right\} R^k .$$

But we know that the potential U is *constant* within the conductor. Consequently, the right side of this equation must be independent of R, which is possible only if

$$X_k(\theta,\varphi) + \frac{4\pi}{R_0{}^{k-1}} \frac{Y_k(\theta,\varphi)}{2k+1} = 0 \qquad (k = 1,2,3,\ldots) .$$

These equations determine the functions Y_1, Y_2,... as follows:

$$Y_k(\theta_1,\varphi_1) = -\frac{2k+1}{4\pi} X_k(\theta_1,\varphi_1) R_0{}^{k-1} \qquad (k \geqslant 1) . \tag{24}$$

All we need to do now is determine the function $Y_0(\theta_1,\varphi_1)$, for which it is sufficient to equate the coefficients of R_0 in the right members of the expansions (21) and (23). We then obtain

$$Y_0(\theta,\varphi) = \frac{1}{4\pi} \int_0^{\pi}\int_0^{2\pi} f(\theta_1,\varphi_1) P_0(\cos\gamma_1) \sin\theta_1 \, d\theta_1 \, d\varphi_1 ,$$

from which it follows that

$$Y_0(\theta,\varphi) = \frac{1}{4\pi R_0{}^2} \iint_{\Sigma} \rho_1 \, d\Sigma , \tag{25}$$

where Σ is the surface of the sphere $R = R_0$. Remembering that

$$\iint_{\Sigma} \rho_1 \, d\Sigma = q ,$$

where q, as we stated above, denotes the total charge on the conductor, we find the desired function

$$Y_0(\theta_1,\varphi_1) = \frac{q}{4\pi R_0{}^2} . \tag{26}$$

If we substitute the values that we have found for the spherical functions into expansion (22), we obtain the following expression for the charge density

$$\rho_1 = \frac{q}{4\pi R_0{}^2} - \sum_{k=1}^{\infty} \frac{2k+1}{4\pi} X_k(\theta_1,\varphi_1) R_0{}^{k-1} , \tag{27}$$

where the functions $X_k(\theta_1,\varphi_1)$ are determined by formula (19).

Formula (27) shows that the density of the electric charge distributed on the surface of the spherical conductor consists of two parts: (1) the density $q/4\pi R_0{}^2$, representing the density of the charge q, uniformly distributed over the entire surface of the sphere (as if there were no external electric forces) and (2) the density

$$\sum_{k=1}^{\infty} \frac{2k+1}{4\pi} X_k(\theta_1,\varphi_1) R_0{}^{k-1} ,$$

which is induced by the charges in the body B.

Let us examine in greater detail a special case of formula (27) in which, instead of a charged body, we have a point charge q_0 that induces the electric charge on the surface C.

As before, we denote by $\xi(R,\theta,\varphi)$ some point within the sphere C and by R_2, θ_2, and φ_2 the spherical coordinates of the point x at which the charge q_0 is concentrated. We then have

$$U_{\rm B} = \frac{q_0}{\sqrt{R^2 + R_2{}^2 - 2RR_2\cos\gamma}} = \frac{q_0}{R_2}\sum_{k=0}^{\infty} P_k(\cos\gamma)\left(\frac{R}{R_2}\right)^k , \tag{28}$$

where

$$\cos\gamma = \cos\theta\cos\theta_2 + \sin\theta\sin\theta_2\cos(\varphi-\varphi_2)$$

Comparing this expansion with formula (18), we find that

$$X_k(\theta,\varphi) = q_0\frac{P_k(\cos\gamma)}{P_2{}^{k+1}}$$

If we now substitute this expression into formula (27), we obtain the following result:

$$\rho_1 = \frac{q}{4\pi R_0{}^2} - \frac{q_0}{4\pi R_2 R_0}\sum_{k=1}^{\infty}(2k+1)P_k(\cos\gamma_2)\left(\frac{R_0}{R_2}\right)^k ,$$

where

$$\cos\gamma_2 = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2) .$$

But, by differentiating the expansion (28) with respect to R, we can easily show the validity of an equation of the following kind:

$$\sum_{k=1}^{\infty} (2k+1)\, P_k(\cos\gamma_2) \left(\frac{R_0}{R_2}\right)^k = \frac{R_2(R_2^2 - R_0^2)}{(R_2^2 + R_0^2 - 2R_2R_0\cos\gamma_2)^{\frac{3}{2}}} - 1\,,$$

from which we finally get the following formula for the density ρ_1:

$$\rho_1 = \frac{q}{4\pi R_0^2} + \frac{q_0}{4\pi R_2 R_0}\left[1 - \frac{R_2(R_2^2 - R_0^2)}{(R_2^2 + R_0^2 - 2R_2R_0\cos\gamma_2)^{\frac{3}{2}}}\right], \tag{29}$$

where $\cos\gamma_2$ is defined by the formula given above.

4. *The flow of an incompressible liquid around a sphere*

Suppose that we are dealing with an irrotational flow of an incompressible liquid past a sphere of radius R_0. We assume the velocity of the liquid at an infinite distance from the sphere to have a constant value V_0. Close to the sphere, the liquid acquires some additional velocity, whose potential we denote by u. To determine this potential, we place the origin of a spherical coordinate system at the center of the sphere and direct the polar axis in the direction opposite to the motion of the liquid (fig. 60).

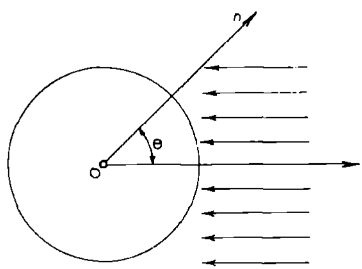

Fig. 60.

We know from Chapter XVIII that the velocity potential of an incompressible liquid satisfies Laplace's equation. Let us seek a solution to this equation in the form of an infinite series

$$u = \sum_{k=0}^{\infty} a_k P_k(\cos\theta) \left(\frac{R_0}{R}\right)^{k+1} \qquad (R_0 < R)\,, \tag{30}$$

where the $P_k(\cos\theta)$ are the Legendre polynomials. The coefficients a_k in this series can be determined from the conditions that exist on the boundary of the liquid surrounding the sphere. It is clear from fig. 60 that the normal component of the additional velocity of an element of the liquid at the surface of the sphere is given by the formula

$$AB = v_0 \cos(\pi - \theta) = - v_0 \cos\theta ,$$

from which it follows that

$$\frac{\partial u}{\partial R}\bigg|_{R=R_0} = - v_0 \cos\theta .$$

If we substitute the right side of expansion (30) for u, we find that

$$-\frac{1}{R_0}\sum_{k=0}^{\infty}(k+1)\, a_k P_k(\cos\theta) = - v_0 \cos\theta .$$

But this equation becomes an identity only when the coefficients a_k are chosen as follows:

$$a_0 = 0 , \quad a_1 = -\tfrac{1}{2}R_0 v_0 , \quad a_2 = a_3 = \ldots = 0 .$$

Substituting these values of the coefficients into expansion (30), we find the desired potential in Stokes' form, namely,

$$u = \frac{v_0 R_0^{\,3}}{2R^2}\cos\theta .$$

Problems

1. Find the Newtonian potential at a point $x(R, \theta, \varphi)$ of the field caused by attracting masses that are situated so as to form a disk of thickness δ and radius R_0.
   Answer: This potential is given by the formulae

$$U = \frac{2m}{R_0}\left[P_0(\cos\theta) - \frac{R}{R_0}P_1(\cos\theta) + \frac{1}{2}\left(\frac{R}{R_0}\right)^2 P_2(\cos\theta) - \frac{1\times 2}{2\times 4}\left(\frac{R}{R_0}\right)^4 P_4(\cos\theta) + \ldots\right] \quad (R < R_0,\ 0 < \theta < \tfrac{1}{2}\pi) ,$$

$$= \frac{2m}{R_0}\left[\frac{1}{2}\,\frac{R_0}{R}P_1(\cos\theta) - \frac{1\times 2}{2\times 4}\left(\frac{R_0}{R}\right)^3 P_2(\cos\theta) + \frac{1\times 2\times 3}{2\times 4\times 6}\left(\frac{R_0}{R}\right)^5 P_4(\cos\theta) - \ldots\right] \quad (R_0 > R) ,$$

   where m is the mass of the attracting layer.

2. A thin disk of radius R_0 is charged with q units of charge. Find the potential of the field at the point $x(R, \theta, \varphi)$ if the density of the charge at a varuable point ξ (fig. 61) varies according to

$$\rho = \frac{E}{4\pi R_0\sqrt{R_0^{\,2} - r^2}} .$$

   Answer: The potential at the point x is expressed by the formulae

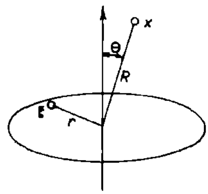

Fig. 61.

$$U = \frac{E}{R_0}\left[\frac{\pi}{2} - \frac{R}{R_0} P_1(\cos\theta) + \frac{1}{3}\left(\frac{R}{R_0}\right)^3 P_3(\cos\theta) - \frac{1}{5}\left(\frac{R}{R_0}\right)^5 P_5(\cos\theta) + \ldots\right] \quad (R < R_0,\ 0 < \theta < \tfrac{1}{2}\pi) ,$$

$$U = \frac{E}{R_0}\left[\frac{R_0}{R} - \frac{1}{3}\left(\frac{R_0}{R}\right)^3 P_2(\cos\theta) + \frac{1}{5}\left(\frac{R_0}{R}\right)^5 P_4(\cos\theta) - \frac{1}{7}\left(\frac{R_0}{R}\right)^7 P_6(\cos\theta) + \ldots\right] \quad (R > R_0) .$$

3. Find the steady-state temperature of points within a hemisphere if the surface of the hemisphere has a constant temperature T_0 and the base of the hemisphere has a temperature 0° at all times (fig. 62).

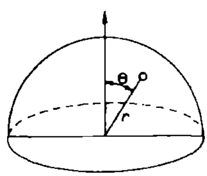

Fig. 62.

Method: Denote by $f(\theta)$ the temperature of the surface, so that

$$f(\theta) = \begin{cases} T_0 & \text{for } 0 < \theta < \frac{1}{2}\pi , \\ 0 & \text{for } \theta = \frac{1}{2}\pi . \end{cases}$$

Extend the function $f(\theta)$ to the interval $(\frac{1}{2}\pi, \pi)$ according to the definition

$$f(\pi - \theta) = -f(\theta) = -T_0 \qquad (\tfrac{1}{2}\pi < \theta < \pi) ,$$

and determine the coefficients a_{2k+1} in the expansion

$$T_0 = a_1 P_1(\cos\theta) + a_3 P_3(\cos\theta) + \ldots + a_{2k+1} P_{2k+1}(\cos\theta) + \ldots \qquad (0 < \theta < \tfrac{1}{2}\pi)$$

by using the formula

$$a_{2k+1} = \frac{4k+3}{2}\left[T_0 \int_0^{\frac{1}{2}\pi} P_{2k+1}(\cos\theta)\sin\theta\, d\theta - T_0 \int_{\frac{1}{2}\pi}^{\pi} P_{2k+1}(\cos\theta)\sin\theta\, d\theta\right].$$

Answer: The temperature is expressed by the expansion

$$T = T_0 \sum_{k=0}^{\infty} (-1)^k \frac{(4k+3)\ 1\times2\times3\times\ldots(2k-1)}{2\times4\times6\times\ldots(2k+2)} \left(\frac{R}{R_0}\right)^{2k+1} P_{2k+1}(\cos\theta)$$

$$(0 \leqslant \theta \leqslant \tfrac{1}{2}).$$

4. A conducting sphere of radius R_0 is grounded and placed in the electric field formed by a point charge q_0 located at a point x a distance $h > R_0$ from the center of the sphere (fig. 63). Determine the potential at the point $\xi(R, \theta, \varphi)$ from the charge induced on the surface of the sphere.

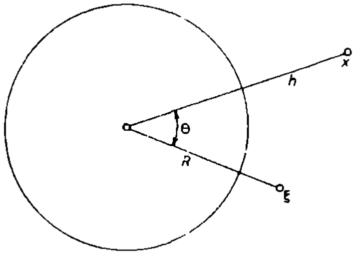

Fig. 63.

Method: Represent the potential in the form of the sum

$$U = U_B + U_C = \frac{q_0}{\sqrt{h^2 + R^2 - 2hR\cos\theta}} + U_C$$

where

$$U_C = \sum_{k=0}^{\infty} a_k \left(\frac{R}{R_0}\right)^k P_k(\cos\theta) \quad (R < R_0)$$

and

$$U_C = \sum_{k=0}^{\infty} a_k \left(\frac{R_0}{R}\right)^{k+1} P_k(\cos\theta) \quad (R > R_0),$$

and determine the coefficients a_k from the grounding condition $U_B + U_C = 0$ on the surface of the sphere.

Answer: The desired potential is given by the expression

$$U = U_B - \frac{q_0}{h}\sum_{k=0}^{\infty}\left(\frac{R}{h}\right)^k P_k(\cos\theta) \quad (R < R_0),$$

$$U = U_B - \frac{q_0 R_0}{hR}\sum_{k=0}^{\infty}\left(\frac{R_0^2}{hR}\right)^k P_k(\cos\theta) \quad (R > R_0).$$

Chapter XXIV

GRAVITY WAVES ON THE SURFACE OF A LIQUID

1. *Statement of the problem*

Let us examine the waves on the surface of an incompressible non-viscous liquid contained in a basin with solid walls.

The upper surface of the liquid, which does not come in contact with the walls, is said to be free. When there are no waves, its state is said to be *undisturbed*. In this state, we shall assume a free surface to be plane.

Let us set up a rectangular Cartesian coordinate system, for which, in this chapter, we shall use the old notation x, y, z, with the x- and y-axes situated on the undisturbed free surface. We direct the z-axis upward.

A state of a free surface that deviates from the undisturbed state is called disturbed (agitated).

We shall assume that the motion of the liquid was originally caused by a conservative system of forces and we shall consider the disturbance at the instant at which all these forces (except gravity) were removed. In this case, the waves on the surface of the liquid are called *gravity waves*. As we know from hydrodynamics, when only conservative forces act on a liquid, its motion will be non-turbulent and, therefore, there will be a velocity potential Φ; that is, the component v_l of the velocity of the liquid in the direction l can be represented in the form

$$v_l = -\partial\Phi/\partial l \ , \tag{1}$$

where $\Phi = \Phi(x, y, z; t)$ is some function of the coordinates and of time, that satisfies (in space coordinates x, y, z) Laplace's equation

$$\Delta\Phi = 0 \ . \tag{2}$$

Finally, we shall consider the disturbance of the liquid to be small; that is, we shall assume that all the derivatives of the velocity potential $\partial\Phi/\partial x$, $\partial\Phi/\partial y$, $\partial\Phi/\partial z$, $\partial\Phi/\partial t$ and also that the displacement of the free surface are sufficiently small that we may neglect the squares of these displacements and their products without introducing any significant error into the solution. Under these conditions, we reduce the problem of the disturbance on a free liquid surface to the boundary problem for Laplace's equation (2).

Let us first find the boundary conditions that the velocity potential Φ must satisfy.

On the motionless portion of the boundary (the walls and bottom of the basin), we have, on the basis of eq. (1),

$$\partial\Phi/\partial n = 0 \ , \tag{3}$$

since the liquid cannot penetrate the solid walls. On the free surface, we have

$$p = p_0 , \tag{4}$$

where p is the hydrodynamic pressure in the liquid and p_0 is the atmospheric pressure.

To transform this last condition into a more convenient form, we use Euler's equations (10a) of Chapter VI, which describe the motion of an ideal liquid. We take that equation in which the derivatives of the component v_z of the velocity appear. We now write that equation in the form

$$\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} = \bar{Z} - \frac{1}{\rho}\frac{\partial p}{\partial z} , \tag{5}$$

where $\bar{Z}$ is the external force acting on a unit of mass of the liquid along the z-axis and ρ is the density of the liquid. When only gravity acts on the liquid,

$$\bar{Z} = -g ,$$

where g is the acceleration due to gravity. Also, because of eq. (1), we have

$$\frac{\partial v_z}{\partial x} = \frac{\partial v_x}{\partial z} , \quad \frac{\partial v_z}{\partial y} = \frac{\partial v_y}{\partial z} , \quad \frac{\partial v_z}{\partial t} = -\frac{\partial^2 \Phi}{\partial z\, \partial t} ,$$

so that when there is a velocity potential, Euler's equation (5) can be written in the form

$$-\frac{\partial^2 \Phi}{\partial z\, \partial t} + v_x \frac{\partial v_x}{\partial z} + v_y \frac{\partial v_y}{\partial z} + v_z \frac{\partial v_z}{\partial z} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} ,$$

which can be immediately integrated with respect to z. We then obtain

$$\frac{p}{\rho} = \frac{\partial \Phi}{\partial t} - gz - \tfrac{1}{2}(v_x^2 + v_y^2 + v_z^2) + C , \tag{6}$$

where C is an arbitrary function of time. Since the addition of an arbitrary function of time to the potential does not violate eq. (1), we can choose $C = C(t)$ arbitrarily. Let us set

$$C = p_0/\rho .$$

As we stated above, we can neglect the squares of the velocities. Then, if we denote by ζ the coordinate z of the free surface, we obtain

$$\zeta = \frac{1}{g}\left(\frac{\partial \Phi}{\partial t}\right)_{z=\zeta} .$$

If we neglect terms of higher order, this expression can be written in the form

$$\zeta = \frac{1}{g}\left(\frac{\partial \Phi}{\partial t}\right)_{z=0} . \tag{7}$$

With this degree of approximation, if we remember that the normal to the

free surface makes a small angle with the z-axis, we can set the normal component of the velocity of the liquid on the free surface equal to

$$\frac{\partial \zeta}{\partial t} = -\left(\frac{\partial \Phi}{\partial z}\right)_{z=0} . \tag{8}$$

If we differentiate eq. (7) with respect to time and substitute the value that we get for $\partial\zeta/\partial t$ into eq. (8), we obtain

$$\left(\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z}\right)_{z=0} = 0 . \tag{9}$$

Eq. (9) gives the boundary condition for the free surface in a form that will be convenient for us in what follows. Eq. (7) makes possible the determination of the form of the free surface if the solution is known for Φ.

Let us find the solution to our problem that gives the purely periodic oscillation at every point of space occupied by the liquid. This oscillation will be of a single angular velocity ω but, generally speaking, will vary from point to point in amplitude and phase. Here, we set

$$\Phi = \mathrm{Re}\, u\, e^{-i\omega t} , \tag{10}$$

where u is a complex function of the coordinates. Since

$$\mathrm{Re}\, u\, e^{-i\omega t} = u' \cos \omega t + u'' \sin \omega t ,$$

where

$$u' = \mathrm{Re}\, u , \qquad u'' = \mathrm{Im}\, u ,$$

Eq. (10) does describe a harmonic oscillation with phase and amplitude dependent on these coordinates.

To find the equation and the boundary conditions that the function u must satisfy, let us replace Φ in eqs. (2), (3), and (9) by the product $u\, e^{-i\omega t}$. Then, at points within the liquid,

$$\Delta u = 0 , \tag{11}$$

on the wall of the basin,

$$du/dn = 0 , \tag{12}$$

and on the free surface,

$$g\frac{\partial u}{\partial z}\Big|_{z=0} - \omega^2 u\Big|_{z=0} = 0 . \tag{13}$$

Thus, we have the interior mixed problem for Laplace's equation. This problem is homogeneous. Therefore, is u is a solution, so is Au, where A is a quantity independent of the coordinates.

We note that after we find the function u that satisfies eq. (11) and the boundary conditions (12) - (13), we can find various solutions of a more general type by superposition of solutions of the form $A(\omega)\, u\, e^{-i\omega t}$. For example, the integral

$$\int_{-\infty}^{\infty} A(\omega)\; u\, e^{-i\omega t}\, d\omega \tag{14}$$

will give a very general solution. The quantity A, being a function of ω, can be chosen arbitrarily provided the integral (14) is meaningful.

2. *Two-dimensional waves in a basin of finite depth*

Waves that are independent of one of the coordinates are called two-dimensional. Let us direct the x-axis perpendicularly to the crest of the wave. Then, the picture of the disturbance will not depend on the y coordinate and eq. (11) will be of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} = 0 . \tag{15}$$

Let us consider the picture of two-dimensional disturbance in a basin of constant depth h. The dimension of the basin along the x-axis we shall consider infinite. The dimension along the y-axis either can be infinite or can have a finite value (a canal with vertical walls). Thus, of the boundary conditions (12) - (13), we retain the conditions on the free surface

$$g \frac{\partial u}{\partial z}\Big|_{z=0} - \omega^2 u\big|_{z=0} = 0 \tag{16}$$

and the condition at the bottom of the basin

$$\frac{\partial u}{\partial z}\Big|_{z=h} = 0 . \tag{17}$$

The condition on the walls of the canal is automatically satisfied since the function u does not depend on y.

Let us seek the solution to eq. (15) by the method of separation of variables. Setting $u = z(x)w(z)$, we obtain the two equations

$$\frac{\partial^2 v}{\partial x^2} + k^2 v = 0 , \qquad \frac{\partial^2 w}{\partial z^2} - k^2 w = 0 ,$$

where k^2 is an arbitrary number. The general solutions to these equations are

$$v = B_1\, e^{ikx} + B_2\, e^{-ikx} , \tag{18}$$

$$w = C_1\, e^{kz} + C_2\, e^{-kz} , \tag{19}$$

where B_1, B_2, C_1, and C_2 are arbitrary constants.

These constants and also k^2 must be chosen so that both the boundary conditions and the requirement of smallness (see section 1) will be satisfied.

We first consider eq. (18), which determines the dependence of the agitation on the coordinate x. If the number k^2 is complex or negative, eq. (18) indicates that along one of the branches of the x-axis the agitation not only will not be small, but will increase without bound. Therefore, the number k^2 must be chosen real and positive. We denote by k the positive square root of k^2.

We note, in passing, that for real k, eq. (18) immediately implies that

$$k = 2\pi/\lambda \ , \tag{20}$$

where λ is the wavelength. The number k is called the wave number.

By substituting the product vw in the boundary condition (17), we obtain

$$C_1 \, e^{kh} - C_2 \, e^{-kh} = 0 \ ,$$

from which it follows that, except for a constant factor,

$$C_1 = e^{-kh} \ , \qquad C_2 = e^{kh} \ ,$$

so that

$$w = e^{k(z-h)} + e^{-k(z-h)} \ . \tag{21}$$

By using boundary condition (16), we obtain

$$gk \sinh kh = \omega^2 \cosh kh$$

or

$$\omega^2 = gk \tanh kh \ ; \tag{22}$$

that is, the wave number k and the angular frequency of the oscillations are functionally related. Noting that the right side of eq. (22) monotonically increases without bound with increasing k, we conclude that to every value of ω there corresponds one and only one value of k satisfying this equation, and that k increases when ω increases.

We now return to eq. (18) for the function $v(x)$. Since all the conditions of the problem are satisfied by the choice of the constants C_1 and C_2 and by the limitations imposed on the values of k^2, the constants B_1 and B_2 are only limited by the requirement that the amplitude of the wave be small; otherwise they are arbitrary. This is natural since we have made no quantitative statements about the initial disturbance that caused the agitation. Therefore, the solution that we are looking for is not unique, and it only determines the class of possible motions of the liquid that satisfy the conditions stated.

Multiplying eq. (18) by $e^{-i\omega t}$, we obtain

$$v \, e^{-i\omega t} = B_1 \exp\left[ik\left(x - \frac{\omega}{k} t\right)\right] + B_2 \exp\left[-ik\left(x + \frac{\omega}{k} t\right)\right] \ , \tag{23}$$

from which it is clear that the first term on the right side of eq. (18) corresponds to a wave travelling with phase velocity

$$v_{ph} = \omega/k \tag{24}$$

in the direction of the x-axis and that the second term corresponds to a wave travelling with the same phase velocity in the opposite direction. If we square eq. (24) and substitute the value for ω^2 given by eq. (22), we find that

$$v_{ph}^2 = \frac{g}{k} \tanh kh = \frac{1}{2\pi} g\lambda \tanh 2\pi \, h\lambda \ ; \tag{25}$$

that is, the phase velocity of the waves depends on their length. This indi-

cates that if a composite wave is formed by the superposition of waves of different length, its form will, in general, change with the passage of time, since the individual waves composing it will be propagated with different velocities (wave dispersion). On the other hand, eq. (23) indicates that waves that are formed by the superposition of waves of a single wave length retain their shape with the passage of time. We note that these last waves are always unbounded in space (periodic).

If

$$kh = 2\pi \frac{h}{\lambda} \ll 1 \,,$$

that is, if a wavelength is very much greater than the depth of the basin, then tanh kh is approximately equal to kh and, from eq. (25), we obtain

$$v_{\mathrm{ph}} \approx \sqrt{gh} \tag{26}$$

This means that very long waves are propagated without dispersion.

To make further deductions from our solution, let us write the expression for the velocity potential Φ as applied to a wave travelling in the positive direction of the x-axis. From eq. (10), we have

$$\Phi = \mathrm{Re}\, A \cosh k(h-z) \exp\left[ik(x - \frac{\omega}{k}t)\right] . \tag{27}$$

A combination of solutions of the form (27) with different values of ω and the corresponding values of k will also satisfy the conditions of the problem. Therefore, if we consider A and ω as functions of k and if we integrate eq. (27) with respect to k, we obtain the more general solution

$$\Phi = \mathrm{Re} \int_{-\infty}^{\infty} A(k) \cosh k(h-z) \exp\left[ik(x - \frac{\omega}{k}t)\right] \mathrm{d}k . \tag{28}$$

The function $A(k)$ is restricted only by the requirement that the integral on the right side exists. (From a physical standpoint, the values of the number k that we are considering must be bounded above because, for very high frequencies, the viscosity and other characteristics that are not taken into account in the Euler equation cannot be neglected and our solution loses physical meaning. In other words, we must assume that, beginning with some sufficiently high values of k, the function $A(k)$ will vanish.)

Solution (28) represents the superposition of waves with infinitesimal amplitudes $A(k)\mathrm{d}k$. If to some frequency, or combination of frequencies, there correspond waves whose amplitudes are not infinitesimal (the amplitudes must still be small, as specified above), we need to add to eq. (28) the finite sum over the corresponding values of k:

$$\mathrm{Re} \sum_{(k)} A(k) \cosh k(h-z) \exp\left[ik(x - \frac{\omega}{k})t\right] . \tag{29}$$

This is the most general form of the solution to the problem that we are considering. However, we can deduce some very interesting conclusions from solution (28).

Differentiating the integrand in eq. (28) with respect to t and setting $z = 0$, we obtain, on the basis of (7), the following expression for the coordinate $z = \zeta$ of the free surface:

$$\zeta = \mathrm{Re}\left[-\mathrm{i}\int_{-\infty}^{\infty}\frac{\omega}{g}A(k)\cosh kh \exp\left[\mathrm{i}k(x-\frac{\omega}{k}t)\right]\mathrm{d}k\right].$$

Let us define

$$B(k) = -\frac{\mathrm{i}\omega}{g}A(k),$$

Eq. (28) then becomes

$$\zeta = \mathrm{Re}\int_{-\infty}^{\infty} B(k)\cosh kh \exp\left[\mathrm{i}k(x-\frac{\omega}{k}t)\right]\mathrm{d}k. \tag{30}$$

Let us assume that at some initial instant, the shape of the surface of the liquid is described by the function

$$\zeta|_{t=0} = \zeta_0(x).$$

Let us see how the surface will change at subsequent instants. From eq. (30),

$$\zeta_0(x) = \mathrm{Re}\int_{-\infty}^{\infty} B(k)\cosh kh\, \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k.$$

The functions $B(k)$, as we shall show, can be chosen in such a way that the integral on the right side of this equation will be real, that is, so that

$$\zeta_0(x) = \int_{-\infty}^{\infty} B(k)\cosh kh\, \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k. \tag{31}$$

In order to choose $B(k)$ in such a way that this will be true, we use the Fourier integral formula *

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k\int_{-\infty}^{\infty} f(\xi)\,\mathrm{e}^{-\mathrm{i}k\xi}\,\mathrm{d}\xi, \tag{32}$$

which will be valid if the function $f(x)$ is of bounded variation and continuous in an interval having x as an interior point and if, in addition, $f(x)$ is absolutely integrable over the entire real axis. Let us assume these conditions satisfied for the function $\zeta_0(x)$ for all x and let us set $f(x) = \zeta_0(x)$ in eq. (32). Then, it follows from a comparison of eqs. (31) and (32) that eq. (31) will be satisfied if we set

$$B(k) = \frac{1}{2\pi}\,\frac{\bar{\zeta}_0(k)}{\cosh kh}, \tag{33}$$

* See V.I.Smirnov [1]), Vol. 2, p. 160.

where

$$\bar{\zeta}_0(k) = \int_{-\infty}^{\infty} \zeta_0(x)\, e^{-ikx}\, dx\,. \tag{34}$$

Substituting this value of $B(k)$ into eq. (30), we obtain

$$\zeta(x,t) = \mathrm{Re}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\zeta}_0(k) \exp\left[ik\left(x - \frac{\omega}{k}\, t\right)\right] dk\,. \tag{35}$$

This formula makes it possible to determine the change in the form of the surface of the liquid with the passage of time.

Let us consider the particular case in which the initial disturbance is caused by extremely long waves ($\lambda \gg h$). Then, from eq. (26),

$$\omega/k = \sqrt{gh}$$

and eq. (35) takes the form

$$\zeta(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\zeta}_0(k) \exp\left[ik(x - t\sqrt{gh})\right] dk\,.$$

Noting that the expression $(x - t\sqrt{gh})$ is independent of k and plays the same role here as does x in eq. (32), we obtain

$$\zeta(x,t) = \zeta_0(x - t\sqrt{gh})\ ;$$

that is, the initial disturbance is propagated without distortion, as has been remarked.

We obtain another important special case if we assume that

$$\zeta_0(x) = \mathrm{Re}\ \psi(x)\, e^{ik_0 x}\,, \tag{36}$$

where $\psi(x)$ is a real function having the following properties:

(a) It changes only slightly over the extent of the wavelength $\lambda_0 = 2\pi/k_0$.

(b) It is different from zero only in a finite interval of variation in x. Such a disturbance is called a wave *group* or wave-*train* of length λ_0.

On the basis of property (a), it follows from the theory of Fourier integrals that the function $\bar{\zeta}_0(k)$, defined by formula (34), will be close to zero for all k with the exception of those values that are close to the value k_0. Therefore, in the expansion

$$\omega = \omega_0 + \frac{\partial \omega}{\partial k}\Big|_{k=k_0} (k - k_0) + \dots\,,$$

we may discard all but the first two terms without causing any significant error in the calculation of the integral (35). If we use the notation

$$k - k_0 = k'\,, \qquad \frac{\partial \omega}{\partial k}\Big|_{k=k_0} = v_g\,,$$

we obtain

$$\exp[i(kx-\omega t)] \approx \exp[i(k_0+k')x]\exp[-i(\omega_0+v_g k')t]$$
$$= \exp[i(k_0x-\omega_0 t)]\exp[ik'(x-v_g t)] ,$$

so that an approximation to the integral (35) can be written in the form

$$\zeta(x,t) = \mathrm{Re}\,\frac{1}{2\pi}\exp[i(k_0x-\omega_0 t)]\int_{-\infty}^{\infty}\bar{\zeta}_0(k')\exp[ik'(x-v_g t)]\,dk ,$$

where

$$\bar{\zeta}_0(k') = \int_{-\infty}^{\infty}\psi(x)\,e^{ik_0x}\,e^{-ikx}\,dx = \int_{-\infty}^{\infty}\psi(x)\,e^{-ik'x}\,dx .$$

By using the Fourier integral formula (32), we easily see that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{\zeta}_0(k')\exp[ik'(x-v_g t)]\,dk' = \psi(x-v_g t) ,$$

from which it follows that

$$\zeta(x,t) = \mathrm{Re}\,\psi(x-v_g t)\exp[i(k_0x-\omega_0 t)] . \tag{37}$$

Thus, as a first approximation, the wave train that we are considering is propagated *as a unit* with velocity

$$v_g = \frac{\partial\omega}{\partial k}\bigg|_{k=k_0}$$

This velocity is called the *group velocity*. However, the individual wave in the group (the crest and the trough) travels with velocity ω_0/k_0, representing the phase velocity of waves of corresponding wavelengths. These waves neither lag behind the train nor outstrip it, since their altitude at the boundart of the train, which is characterized by the factor $\psi(x-v_g t)$, vanishes. This, of course, is true only for that approximation which we are considering at the moment. Calculation of terms of higher order would show that, with the passage of time, the train, generally speaking, changes its shape and increases without bound as a result of wave dispersion.

We emphasize that *the concept of group velocity must not be introduced in the general case of an arbitrary disturbance*. It is applicable only in connection with wave trains whose spectrum, characterized by the function $\zeta_0(k)$, extends only over a sufficiently narrow interval of the values of the wave number k.

Problems

1. Show that when a two-dimensional wave passes, the particles of a liquid in a basin move in an elliptic orbit whose major axis a is directed along the direction of propagation of the wave and whose minor axis b is vertical and that

$$a = \frac{k}{\omega} A \cosh k(h-z) , \qquad b = \frac{k}{\omega} A \sinh k(h-z) .$$

Method: Use the expression for the velocity potential (27).
Note that differentiation of the potential with respect to the coordinates of the particle of the liquid when there is no disturbance (instead of differentiation with respect to the moving coordinates when the wave passes) leads only to a second-order error in the determination of the velocity of the particle. Then find the component of the velocity of a particle situated at depth z:

$$v_x = -\frac{\partial \Phi}{\partial x} = \mathrm{Re}\; ikA \cosh k(h-z) \exp\left[ik(x - \frac{\omega}{k}t)\right] ,$$

$$v_z = -\frac{\partial \Phi}{\partial z} = \mathrm{Re}\; kA \sinh k(h-z) \exp\left[ik(x - \frac{\omega}{k}t)\right] .$$

Integrate these expressions to determine the displacements x', z' of the particles from the equilibrium position as functions of time. Choose the constant of integration from the condition of periodicity of the motion. Finally, show that the quantities x' and z' satisfy the equation

$$(x'/a)^2 + (z'/b)^2 = 1 .$$

2. Examine the case of a basin of infinite depth.

3. *Annular waves*

A disturbance at any point of the surface of a liquid causes the appearance of annular waves with center at the point of the disturbance.

To study these waves, we use cylindrical coordinates r, φ, z with origin at the point of the disturbance and with the z-axis directed downward. On the basis of formula (3) of Chapter XVIII, Laplace's equation (2) for the velocity potential $\Phi(r, \varphi, z; t)$ then takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 . \tag{38}$$

Let us first consider oscillations that are purely periodic with respect to time. In connection with this, in accordance with section 1, we take

$$\Phi(r,\varphi,z;t) = \mathrm{Re}\; u(r,\varphi,z)\, e^{-i\omega t} , \tag{39}$$

where ω is the angular frequency of the oscillations and u is a complex function of the coordinates which, like Φ, satisfies eq. (38). Assuming that the waves in which we are interested are symmetric about the center, we obtain the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} = 0 . \tag{40}$$

If we now assume that the basin is infinite, we retain the initial conditions only for the free surface (13):

$$\text{for } z = 0 \qquad g\frac{\partial u}{\partial z} - \omega^2 u = 0 \tag{41}$$

and for the bottom of the basin (12),

$$\text{for } z = h \qquad \partial u/\partial z = 0 \, . \tag{42}$$

As in the preceding section, we shall seek a solution to eq. (40) by separating the variables. If we set $u(r, z) = v(r)w(z)$ and perform the obvious manipulations, we obtain the equations

$$\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} + k^2 v = 0 \, , \tag{43}$$

$$\frac{d^2 w}{dz^2} - k^2 w = 0 \, , \tag{44}$$

where k^2 is an arbitrary number.

Both boundary conditions (41) - (42) refer to eq. (44). Eq. (44) itself and these boundary conditions are the same as in section 2. Therefore, on the basis of formula (21), we can immediately assert that, except for a constant factor,

$$w(z) = \cosh k(h - z) \, , \tag{45}$$

where, because of (22), k is a real number dependent on the angular frequency ω of the oscillations as indicated by the equation

$$\omega^2 = gk \tanh kh \, . \tag{46}$$

Eq. (43) is Bessel's equation of zero order (see Chapter XII). Its solution, which is bounded and continuous for all r including $r = 0$, is Bessel's function of zero order $J_0(kr)$. Thus, by taking eqs. (39) and (45) into consideration, we conclude that all continuous solutions to eq. (40) with boundary conditions (41) - (42) are of the form

$$u(r, z) = AJ_0(kr) \cosh k(h - z) \, ,$$

where A is a quantity independent of both r and z. On the basis of eq. (39), it then follows that

$$\Phi(r, z; t) = \operatorname{Re} AJ_0(kr) \cosh k(h - z) \, e^{-i\omega t} \, . \tag{47}$$

By virtue of formula (7), the altitude of the free surface above the undisturbed level is determined by

$$\zeta(r, t) = - \operatorname{Re} \frac{i\omega}{g} AJ_0(kr) \cosh kh \, e^{-i\omega t} \, . \tag{48}$$

It is easy to see from the graph of the function $J_0(kr)$ that the distance between two adjacent crests of the periodic annular waves that we are considering (the analogue of wavelengths in the two-dimensional case) increases as the distance from the point $r = 0$ increases, while the altitude of the wave decreases as the distance from $r = 0$ increases.

Let us now examine the case of an arbitrary initial disturbance that is symmetric about the axes.

Suppose that, for $t = 0$,

$$\zeta = \zeta(r, 0) = \zeta_0(r) , \tag{49}$$

where $\zeta_0(r)$ is a given continuous function of r. Let us determine the motion of the liquid for subsequent values of t. We use the Fourier-Bessel integral *:

$$f(x) = \int_0^\infty kJ_0(kx)\,\mathrm{d}k \int_0^\infty \xi f(\xi)\, J_0(k\xi)\,\mathrm{d}\xi \qquad (0 < x < \infty) ,$$

where instead of the function $f(x)$ we have $\zeta_0(r)$. If we define

$$\bar{\zeta}_0(k) = \int_0^\infty \xi\zeta_0(\xi)\, J_0(k\xi)\,\mathrm{d}\xi , \tag{50}$$

we may write

$$\zeta_0(r) = \int_0^\infty \bar{\zeta}_0(k)\, J_0(kr)\, k\,\mathrm{d}k . \tag{51}$$

This equation gives the initial disturbance $\zeta_0(r)$ in the form of superposed annular waves of different frequencies. From this it is clear that the function

$$\Phi(r, z; t) = \mathrm{Re}\ \mathrm{i}g \int_0^\infty \bar{\zeta}_0(k)\, J_0(kr)\, \frac{\cosh k(h - z)}{\cosh kh}\, \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega}\, k\,\mathrm{d}k \tag{52}$$

gives the solution to our problem. This is true because (52) gives the superposition of functions (47) that satisfy the equation of the problem and the boundary conditions. Consequently, it possesses these properties itself. It then follows from (7) that

$$\zeta(r, t) = \mathrm{Re} \int_0^\infty \bar{\zeta}_0(k)\, J_0(kr)\, \mathrm{e}^{-\mathrm{i}\omega t}\, k\,\mathrm{d}k , \tag{53}$$

so that for $t = 0$, we obtain eq. (51). Thus, the function $\Phi(r, z; t)$ represents the velocity potential of the liquid for the boundary and initial conditions that are given and, therefore, it determines the motion of the liquid. Formula (53) determines the shape of the surface of the liquid at instants of time subsequent to the initial instant.

Problems

1. Show that, for a basin of infinite depth, formula (52) takes the form

* For the derivation of this formula see section 4, Chapter XXXI.

$$\Phi(r, z; t) = \operatorname{Re} ig \int_0^\infty \bar{\zeta}_0(k)\, J_0(kr)\, e^{-kz} \frac{e^{-i\omega t}}{\omega} k\, dk .$$

2. Suppose that

$$\zeta_0(r) = \begin{cases} 2\pi b & \text{for } 0 \leqslant r \leqslant r_0 , \\ 0 & \text{for } r > r_0 \end{cases} .$$

Suppose also that we let r_0 approach zero and let b become infinitely great in such a way that at all times

$$\pi b r_0^2 = 1 .$$

Show that then

$$\bar{\zeta}_0(k) \to 1 .$$

(A "unit disturbance" at the coordinate origin.)

3. Show that, in a basin of infinite depth, when there is a "unit disturbance" at the initial instant at the coordinate origin, the potential Φ may at subsequent instants be represented in the form of the series

$$\Phi(r, z; t) = \frac{gt}{2\pi} \left\{ \frac{P_1(\cos\theta)}{r_1^2} + \frac{gt^2}{3!} \frac{2!\, P_2(\cos\theta)}{r_1^3} + \frac{(gt^2)^2}{5!} \frac{3!\, P_3(\cos\theta)}{r_1^4} + \ldots \right\} ,$$

where $r_1 = \sqrt{(r^2 + z^2)}$, where θ is the angle between the z-axis and the ray directed from the coordinate origin to the point (r, z) and where $P_k(\cos\theta)$ (for $k = 1, 2, 3, \ldots$) are Legendre polynomials.
Method: Expand the ratio $e^{-i\omega t}/\omega$ in a series of powers of ω, introduce the quantity k instead of ω in accordance with (46), and use the formula

$$\frac{1}{n!} \int_0^\infty e^{-kz} J_0(kr)\, k^n\, dk = \frac{P_n(\cos\theta)}{r_1^{n+1}} .$$

4. Show that under the conditions of the preceding problem the altitude of the surface of the liquid can be represented in the form of the series

$$\zeta(r, t) = \frac{1}{2\pi r^2} \left[\frac{1^2}{2!} \frac{gt^2}{r} - \frac{1^2 \times 3^2}{6!} \left(\frac{gt^2}{r}\right)^3 + \frac{1^2 \times 3^2 \times 5^2}{10!} \left(\frac{gt^2}{r}\right)^5 - \ldots \right] .$$

5. Show that under the conditions stated for problem 3, each phase of the motion is propagated from the coordinate origin along the radii with a constant acceleration.
Method: Start with the solution to problem 4, according to which

$$\zeta(r, t) = \zeta(gt^2/r) .$$

4. *Stationary phase method*

Stokes (1850) and Kelvin (1887) developed the method of approximate calculation of integrals of the type obtained above, known as the stationary

phase method. This method has numerous applications and generalizations; we shall present it without being rigorous.

Consider the integral

$$J = \int_a^b \psi(\xi)\, e^{\mathrm{i}f(\xi)}\, d\xi \;, \tag{54}$$

where $\psi(\xi)$ and $f(\xi)$ are real functions such that for most of the interval of integration (that is, except in certain neighbourhoods of points at which $f'(\xi) = 0$) the function $\psi(\xi)$ changes by only a small amount of its original value as the function $f(\xi)$ changes by an amount 2π. The interval of integration is assumed to be large enough to contain a large number of oscillations of the function $e^{\mathrm{i}f(\xi)}$.

When we integrate over the segment in which the function $f(\xi)$ changes rapidly, the real and imaginary parts of the integrand change sign frequently, with the result that the value of the integral barely changes. The integral can have significant increases only in the neighbourhood of points at which $f'(\xi) = 0$.

The stationary phase method consists in carrying out the integration only over the neighbourhoods of those points at which the derivative of $f(\xi)$ is equal to zero and neglecting all other segments of the interval of integration – which, because of the considerations explained above, we are justified in doing.

Suppose that $\alpha_1, \alpha_2, \ldots$ are roots of the equation

$$f'(\xi) = 0 \;,$$

such that for $\xi = \alpha_1, \alpha_2, \ldots$, the second derivative $f''(\xi) \neq 0$. We denote by ξ_k small deviations of the argument ξ from α_k; that is, we set

$$\xi_k = \xi - \alpha_k \;.$$

As a first approximation, we have, in a small neighbourhood of the point $\xi = \alpha_k$,

$$f(\xi) = f(\alpha_k) + \tfrac{1}{2}\xi_k^2 f''(\alpha_k) \;. \tag{55}$$

Substituting this equation into eq. (54) and remembering what has just been said, we can write

$$J \approx \sum_{(k)} \psi(\alpha_k) \exp\,[\mathrm{i}f(\alpha_k)] \int_{(\alpha_k)} \exp[\tfrac{1}{2}\mathrm{i}f''(\alpha_k)\xi_k^2]\, d\xi_k \;,$$

where the summation is taken over all values of k and the integration is taken over a small neighbourhood of the point $\xi = \alpha_k$. Because of the oscillation of the integrand, we may, without causing any great error, set

$$\int_{(\alpha_k)} \exp\,[\tfrac{1}{2}\mathrm{i}f''(\alpha_k)\xi_k^2]\, d\xi_k \approx \int_{-\infty}^{\infty} \exp\,[\tfrac{1}{2}\mathrm{i}f''(\alpha_k)\xi_k^2]\, d\xi_k$$

$$= \left(\frac{2\pi}{|f''(\alpha_k)|}\right)^{\frac{1}{2}} \exp\,[\tfrac{1}{4}\mathrm{i}\pi s_k] \;,$$

where s_k is equal to 1 or -1, depending on whether the second derivative $f''(\xi)$ is greater or less than zero at $\xi = \alpha_k$. Finally, as a first approximation, we obtain

$$J = \sum_{(k)} \psi(\alpha_k) \left(\frac{2\pi}{|f''(\alpha_k)|}\right)^{\frac{1}{2}} \exp\left[i(f(\alpha_k) + \tfrac{1}{4}\pi s_k)\right] . \tag{56}$$

Generally speaking, the approximation (56) gives good results only if the terms of the series

$$f(\xi) = f(\alpha_k) + \tfrac{1}{2}\xi_k{}^2 f''(\alpha_k) + \tfrac{1}{6}\xi_k{}^3 f'''(\alpha_k) + \dots$$

decrease sufficiently rapidly in a neighbourhood of $\xi = \alpha_k$.

Let us apply the stationary phase method to the problem of annular waves considered in the preceding section.

Let us assume for simplicity that the basin is of infinite depth and that at the initial instant a "unit disturbance" (see problem 2 of the preceding section) is acting at the coordinate origin. The unit disturbance represents that case in which the surface of the liquid is originally disturbed only in a very small neighbourhood of the coordinate origin, so that this disturbance represents a sink of unit volume, symmetric about the origin. It is easy to see that, in the case of a unit disturbance, $\bar{\zeta}_0(k) = 1$ (see, for example, the problem referred to). Therefore, formula (53) takes the form

$$\zeta(r, t) = \mathrm{Re} \int_0^\infty J_0(kr)\, e^{-i\omega t}\, k\, dk . \tag{57}$$

Let us seek an approximate value of $\zeta(r, t)$ for sufficiently large values of r and t.

For small values of k, the integrand is close to zero; consequently, the segments of the interval of integration which determine the value of the integral must not be sought for small values of k. Therefore, the values of kr that correspond to these segments can be considered large and, recalling the asymptotic representation (29) of Chapter XII, we set

$$J_0(kr) \approx \sqrt{\frac{2}{\pi kr}} \sin(kr + \tfrac{1}{4}\pi) = \frac{1}{2i}\sqrt{\frac{2}{\pi kr}} \{\exp[i(kr + \tfrac{1}{4}\pi)] - \exp[-i(kr + \tfrac{1}{4}\pi)]\} .$$

Furthermore, from (46), we have, for $h = \infty$,

$$\omega = \sqrt{gk} .$$

Substituting these expressions into (57), we obtain

$$\zeta(r, t) = \mathrm{Re}\, \frac{1}{2i}\sqrt{\frac{2}{\pi r}} \int_0^\infty \{\exp[i(kr - t\sqrt{gk} + \tfrac{1}{4}\pi)] - \exp[-i(kr + t\sqrt{gk} + \tfrac{1}{4}\pi)]\} \sqrt{k}\, dk .$$

In accordance with the stationary phase method, the integral of the second term on the right side can be set equal to zero. The expression $(kr + t\sqrt{gk})$ has neither maxima nor minima if r, t, and k are all positive. Consequently, its derivative with respect to k does not vanish in the region of integration and the sum (56) is equal to zero.

The equation

$$\frac{\partial}{\partial k}(kr - t\sqrt{gk}) = r - \tfrac{1}{2}t\sqrt{g/k} = 0$$

has a unique root

$$k_1 = gt^2/4r^2 . \tag{58}$$

Furthermore, we find that

$$\frac{\partial^2}{\partial k^2}(kr - t\sqrt{gk})\,|_{k=k_1} = \frac{1}{4}\frac{g^{\frac{1}{2}}t}{k_1^{\frac{1}{2}}} > 0$$

and, from (56), we obtain

$$\zeta(r,t) = \mathrm{Re}\,\frac{1}{2\mathrm{i}}\sqrt{\frac{2}{\pi r}}\sqrt{k_1}\sqrt{\frac{8\pi k_1^{\frac{3}{2}}}{g^{\frac{1}{2}}t}}\,\exp\left[\mathrm{i}(k_1 r - t\sqrt{gk_1} + \tfrac{1}{2}\pi)\right] .$$

Substitution into this equation of the value of k_1 given by eq. (58) leads, after some simple manipulations, to the final relationship, which is valid for sufficiently large values of r and t:

$$\zeta(r,t) \approx \frac{gt^2}{2^{\frac{3}{2}}r^3}\cos\frac{gt^2}{4r} . \tag{59}$$

Let us analyze the formula that we have obtained.

If we fix t, we see that, with increasing r, the profile of the surface of the liquid is made up of longer and longer waves with progressively smaller height. They finally become an infinitely long "hump", whose height vanishes only at infinity. On the other hand, if we fix r, we see that the oscillations at every given point first take place slowly and have a small amplitude, but with the passage of time, they speed up without bound and increase in amplitude.

Without going into a detailed investigation, we shall confine ourselves to some brief remarks on these results. First of all, we note that the velocity of propagation of the disturbance is infinitely great. For, at any instant subsequent to the initial instant, there is no undisturbed region. It is easy to show that this fact is a direct consequence of the assumption that the liquid is incompressible. In practice, every liquid is compressible, which is the reason there is no infinitely-fast-moving component of the disturbance.

Furthermore, we conclude that the amplitudes of the oscillations increase without bound with the passage of time. It is easy to show that this conclusion is due to the supposition that the area of the original sink is infinitesimal (since its volume is assumed finite). Therefore, the amplitude of the initial disturbance is infinitely great. The propagation of this "infinite amplitude" also implies an unbounded increase in the amplitude of the high-frequency component. A detailed investigation shows that, in the case of finite area of the region of the initial sink, when waves of length considerably less than the cross section move out of the different points of the sink, they absorb each other as a result of interference. Therefore, in practice, the high-frequency component, containing waves of length consid-

erably less than the cross section of the region of the initial disturbance is, as it were, "trimmed" and there is no infinite increase in the amplitude.

In view of the remarks that we have made (about the trimming of the instantaneously propagating component and of the short waves), formula (59) gives an accurate picture of the propagation of the disturbance. Initially, the long waves with a greater speed of propagation leave an originally localized "wave packet" and reach an arbitrary point of observation. Then, shorter waves of greater amplitude reach that point. As the crests and troughs get farther from the center of the disturbance, they become longer (which reflects the wave dispersion that takes place). If we observe the propagation of an individual crest, it seemingly belongs to a wave of ever greater length, so that the crests (generally, points of a single phase in the motion) are accelerated. Their magnitude rapidly drops. All this describes what is observed when a stone is thrown into the water.

Chapter XXV

THE HELMHOLTZ EQUATION

1. *The connection between the Helmholtz equation and certain hyperbolic and parabolic equations*

Consider the equation

$$\Delta w = a_0 \frac{\partial^2 w}{\partial t^2} + 2a_1 \frac{\partial w}{\partial t} a_2 w \ , \tag{1}$$

where a_0, a_1, and a_2 are constants. If all these constants are positive, eq. (1) is the telegraph equation. We already had occasion to speak of its one-dimensional analogue in Chapter I. If a_1 and a_2 are both zero and if a_0 is positive, it becomes the wave equation (Chapter I); if a_0 and a_2 are both zero and a_1 is positive, it becomes the thermal-conductivity and diffusion equation; if $a_0 = 0$, $a_1 > 0$, $a_2 \neq 0$, it becomes the equation for diffusion in a medium in which chemical or chain reactions take place. We encountered the last two equations in Chapter III.

Following the method of separation of variables, let us seek solutions to eq. (1) of the form

$$w(x, t) = u(x)v(t) \ , \tag{2}$$

where $u(x)$ is a function only of the space coordinates and $v(t)$ is a function only of time. Substituting eq. (2) into eq. (1), we obtain

$$\frac{1}{u}\,\Delta v = \frac{1}{v}\left(a_0 \frac{d^2 v}{dt^2} + 2a_1 \frac{dv}{dt} + a_2 v\right) .$$

Since the left side of this equation does not depend on t and the right side does not depend on the coordinates of the point x, we have

$$\Delta u + k^2 u = 0 \ , \tag{3}$$

$$a_0 \frac{d^2 v}{dt^2} + 2a_1 \frac{dv}{dt} + (a_2 - k^2)v = 0 \ , \tag{4}$$

where k^2 is some constant.

The elliptic equation (3) is called the *Helmholtz equation*. It plays an important role in mathematical physics because of its simplicity and because of the significance of the problems that lead to it (wave processes, heat flow, diffusion, etc.).

It follows from formula (2) that the Helmholtz equation immediately determines the intensities, which obey a single time law and vary from point to point, for processes occurring at all points of a region under scru-

tiny. In the special case in which the function $v(t)$ is constant, it determines the steady state. By a combination of solutions of the form (2), one can obtain almost any space-time dependence.

The possibility of constructing an arbitrary time dependence (satisfying only certain general requirements) by means of a combination of solutions of the form (2) is retained if, instead of arbitrary functions $v(t)$, we consider only functions that together form a complete system. Therefore, with no loss of generality, we may examine instead of the substitution (2), the substitution corresponding to harmonic oscillations with amplitude and phase that vary from point to point. In this case, according to Fourier's theorem, an arbitrary space-time relationship can be obtained by a combination of oscillations of different frequencies.

It is convenient to describe harmonic oscillations in terms of complex functions of the form

$$u(x)\ e^{-i\omega t} \tag{5}$$

or

$$u(x)\ e^{i\omega t}\ , \tag{6}$$

where ω is the angular frequency of the oscillations and $u(x)$ is a complex function of the coordinates of the points x. The real parts of the expressions (5) and (6) determine at every point x the same harmonic oscillation

$$\operatorname{Re}\left[u(x)\ e^{\mp i\omega t}\right] = |u(x)|\ \cos\ (\omega t + \theta) \tag{7}$$

with amplitude $|u(x)|$ and phase θ, the latter being a root of the equations

$$\sin\theta = \frac{\mp \operatorname{Im} u}{|u|}\ , \qquad \cos\theta = \frac{\operatorname{Re} u}{|u|}\ .$$

The symbols Re and Im denote respectively the real and imaginary parts of the functions following them. The plus-or-minus sign in front of Im in the expression for the sine is chosen according to whether expression (5) or (6) is used. If we substitute expression (5) into eq. (1) and divide by $e^{-i\omega t}$ we obtain the Helmholtz equation (3) with parameter k^2, which will, in the general case be complex valued:

$$k^2 = \omega^2 a_0 - a_2 + 2a_1 i\ . \tag{8}$$

If instead we substitute expression (6), we obtain the Helmholtz equation that is the complex conjugate of the equation obtained when we substitute (5). Its solutions, $u^*(x)$, and the solutions obtained in the first case will be complex conjugates. However, the real function Re $u^*(x)e^{i\omega t}$ which is a solution to the original equation (1) will be the same in both cases because the real parts of complex conjugate numbers are equal. Therefore, the two substitutions (5) and (6) are equivalent, and we may use only one of them. We shall use the substitution (5).

Besides the homogeneous equation (1), we shall also examine the nonhomogeneous Helmholtz equation

$$\Delta u + k^2 u = -4\pi\rho\ . \tag{9}$$

The function ρ, as we shall see in section 4 of Chapter XXVI, can be thought of as the density of the distribution of the sources of the wave.

The theory of the Helmholtz equation is in many ways like the theory of the Laplace and Poisson equations. In particular, for the Helmholtz equation the same characteristic statement of the boundary problems (Dirichlet, Neumann, mixed) is used. These problems can be either external or internal. Formulation of the internal problems coincides with what was said in section 2 of Chapter XVIII. In the formulation of the external problems, it is necessary to introduce an additional condition relating to the behaviour of the solution at an infinitely distant point. We shall examine this condition below.

As in the study of the Laplace and Poisson equations, it is the regular solutions to the boundary problem which will interest us. In using the integral formulae, and without making special assumptions, we shall assume the continuity of the first derivative of the solutions in the region in question up to its boundary.

In addition to the boundary problems, an entirely new type of problem, having to do with the natural oscillations, arises in the case of the Helmholtz equation. We shall encounter problems of this type in the following section.

2. *Spherically symmetric solutions to the Helmholtz equation in a bounded region*

Let us first acquaint ourselves with the characteristic features of the solutions to the boundary problem for the Helmholtz equation in the case of real spherically symmetric solutions in a bounded region.

By using the expression (4) of Chapter XVIII for Laplace's operator in spherical coordinates r, θ, and φ, the origin of which we place at the center of symmetry of the solutions sought, we easily see that these satisfy the equation

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} + k^2u = 0 . \tag{10}$$

Multiplying this equation by r and performing the obvious transformation, we obtain

$$\frac{d^2ru}{dr^2} + k^2ru = 0 . \tag{11}$$

From this it is clear that all spherically symmetric solutions to the Helmholtz equation are included in the general solution

$$u(r) = A\,\frac{e^{ikr}}{r} + B\,\frac{e^{-ikr}}{r} , \tag{12}$$

where A and B are arbitrary constants.

Let us examine the boundary problem for a sphere $r \leqslant r_0$:

$$\frac{d^2ru}{dr^2} + k^2ru = 0 \quad \text{when} \quad r < r_0 ; \qquad \alpha\frac{du}{dr} + \beta u = C \quad \text{when} \quad r = r_0 , \tag{13}$$

where r_0, α, β, and C are real constants and r_0 is positive.

To solve the problem, we need to determine the constants A and B in the general solution (12) so that it will be regular for $r \leqslant r_0$ and will satisfy the given boundary conditions.

To meet the first requirement, it is necessary to set $B = -A$. This condition is also sufficient because if we differentiate and use l'Hôpital's rule we easily see that the solution (12) is regular throughout all space. Thus, the solutions to the problem (13) are of the form

$$u(r) = A\,\frac{1}{r}\,(e^{ikr} - e^{-ikr}) \ , \tag{14}$$

where the constant A must be determined from the boundary conditions. As we know, the constant k^2 can be an arbitrary complex number.

In this section, we shall consider solutions to the problem (13) for real values of k^2. We first assume that k^2 is positive. Then, the solution (14) can be written in the form

$$u(r) = A\,\frac{\sin kr}{r} \ . \tag{15}$$

Substituting this expression into the boundary condition for problem (13), we find that the constant A must satisfy the equation

$$A\left[(\beta r_0 - \alpha)\sin kr_0 - \alpha kr_0 \cos kr_0\right] = Cr_0^2 \ . \tag{16}$$

This is possible for values of k that are not roots of the equation

$$\tan kr_0 = \frac{\alpha kr_0}{\beta r_0 - \alpha} . \tag{17}$$

However, there exists a set of positive values $k_1 < k_2 < \ldots < k_i < \ldots$ of the constant k at which eq. (17) is satisfied and, consequently, for $C \neq 0$ there are no numbers A satisfying eq. (16). The values $k_1, k_2, \ldots$ are those values of k for which

$$\left(\alpha\frac{d}{dr} + \beta\right)\frac{\sin kr}{r} = 0 \quad \text{when} \quad r = r_0 \ . \tag{18}$$

Together with the problem (13), let us examine the corresponding homogeneous problem

$$\frac{d^2 ru}{dr^2} + k^2 ru = 0 \quad \text{when} \quad r < r_0 \ ; \qquad \alpha\frac{du}{dr} + \beta u = 0 \quad \text{when} \quad r = r_0 \ . \tag{19}$$

For values of k that satisfy the condition (18), (19) has the solutions $\sin k_1 r/r, \sin k_2 r/r, \ldots$ which are not identically equal to zero. For all other values of k, it follows from formula (16) that $A = 0$, that is, that the homogeneous problem (19) has no solutions other than the trivial solutions $u \equiv 0$.

Thus, we have the following alternatives: Either the homogeneous problem (19) does not have solutions other than the trivial solution $u \equiv 0$ for a given value of k^2 (and then the non-homogeneous problem (13) has a unique solution given by formulae (15) - (17)) or the homogeneous problem (19) has a non-trivial solution (and then the non-homogeneous problem (13) is insoluble).

The functions

$$u_m = \frac{\sin k_m r}{r} \quad (m = 1, 2, 3, \ldots) \ , \tag{20}$$

which are non-trivial solutions to the problem (19) are called the eigenfunctions of the problem (13) and the numbers ${k_m}^2$ at which the homogeneous problem (19) has non-trivial solutions are called the eigenvalues of the problem (13).

In studying the boundary problems for the Laplace and Poisson equations, we dealt only with the first of the alternatives developed above. The corresponding homogeneous boundary-value problems did not have continuous solutions other than the trivial one: $u = 0$, and the solution to the non-homogeneous problem was unique. This means that the boundary-value problems for the Laplace and Poisson equations do not have eigenfunctions.

In order to understand in greater detail the physical significance of the results that we have put forward, let us examine the corresponding solutions $w(r,t)$ of eq. (1), which was the starting point for our discussions in section 1. It follows from formula (8) that for positive values of k^2, we may choose as a special case of eq. (1) the wave equation

$$\Delta w = \frac{1}{c^2} \frac{\partial^2 w}{\partial t^2} \, , \tag{21}$$

where $1/c^2 \equiv a_0$. Then, the constant k^2 is determined by

$$k^2 = \omega^2/c^2 \ . \tag{22}$$

The general expressions for spherically symmetric solutions to the wave equation (21) will be obtained if we multiply the general solution (14) by $e^{-i\omega t} = e^{-ickt}$ and take the real parts of the results. This gives us

$$w(r, t) = A \frac{\sin kr}{r} \cos (kct + \theta) \ . \tag{23}$$

This solution represents a system of standing spherical waves. (We remember that a system of waves in which the geometrical position of points of the same amplitude of oscillation does not change is called a system of standing waves.) The solution (15) represents the amplitude of these oscillations.

The oscillations can be more clearly visualized if they are thought of as being purely mechanical. As an example, let us assume that k^2 is not an eigenvalue and that the boundary condition of the problem (13) is of the form

$$u = C \, , \qquad \text{on the boundary } r = r_0 \ . \tag{24}$$

This means that on the boundary of a sphere of radius $r \leqslant r_0$ oscillations of amplitude

$$A = \frac{C r_0}{\sin k r_0}$$

take place. Oscillations then take place within the sphere, $r \leqslant r_0$, determined by formula (23). On the spherical surfaces

$$r = r_n \equiv n\frac{\pi}{k} \quad (n = 1, 2, 3, \dots), \tag{25}$$

for which $\sin kr = 0$, the amplitude of the oscillations is equal to zero. These are called nodal surfaces. If $r_1 \equiv \pi/k > r_0$, there will be no nodal surfaces. Noting that $k = \omega/c$, we conclude that this will be the case if the frequency of the oscillations in question $\omega < \omega_1 = \pi c/r_0$. The amplitude of the oscillations at the centre of the sphere is equal to $Ak = Cr_0k/\sin kr_0$; that is, it is always non-zero.

Suppose now that $k^2 = k_m{}^2$; that is, k^2 is equal to one of the eigenvalues of the problem (13) with the boundary conditions (24). It follows from formula (25) that this will be the case for a frequency of the excited oscillations

$$\omega_m = m\frac{\pi}{cr_0}, \tag{26}$$

and that the amplitude of the oscillations on the boundary $r = r_0$ is equal to zero. It is clear that oscillations of this kind cannot be caused by oscillations that originate on the boundary, which explains the insolubility of problem (13) for values of k^2 that are eigenvalues. From the point of view of a mechanical interpretation, this solution, which vanishes on the boundary, represents oscillations of a medium in a spherical container with solid walls.

Thus, for values of k^2 that are not eigenvalues, the oscillations on the boundary cause oscillations within the sphere $r \leqslant r_0$. Oscillations for which the boundary $r = r_0$ of the sphere is a nodal surface, cannot then arise. On the other hand, when k^2 is equal to an eigenvalue of the problem, the oscillations on the boundary cannot cause oscillations in the space $r \leqslant r_0$. However, oscillations that have the boundary $r = r_0$ as their nodal surface can exist *. Oscillations of the first kind are called *forced* and oscillations of the second kind are *free* or *natural*.

Consequently, the solutions to the problem (13) determine the amplitude of the forced oscillations, and the solutions to the homogeneous problem corresponding to it (19) determine the distribution of the amplitudes of the free oscillations. The absolute value of the amplitudes of the free oscillations is not determined by the homogeneous problem. The set of frequencies (26) with $m = 1, 2, 3, \dots$ or the eigenvalues $k_m{}^2$ corresponding to them constitute the spectrum of free oscillations.

We should arrive at exactly analogous results if we examined the boundary conditions

$$\mathrm{d}u/\mathrm{d}r = C \quad \text{when} \quad r = r_0 .$$

From the standpoint of mechanical interpretation, the derivative $\mathrm{d}u/\mathrm{d}r$ corresponds to the stresses of the vibrating medium. The homogeneous problem then determines the distribution of the amplitude in a vibrating spherical body with a free surface (that is, the stresses on the boundary are equal

* The method by which these are brought about does not interest us at the moment.

to zero). The general boundary condition of problem (13) gives the different intermediary cases, the mechanical interpretation of which is more complicated, but the general interpretation of the solution as the amplitude of free and forced vibrations still remains valid.

The significance of these results consists in the fact that a pair of alternatives similar to those shown above arises in the case of the general internal boundary-value problem for the Helmholtz equation. However, in the processes that lead to the boundary-value problem of the general form, the amplitude of the oscillations depends on several coordinates and therefore, the forced oscillations can exist along with the free. For example, in a sphere with solid walls, not only radial but also angular oscillations, which may be caused without radial displacement of the boundary, are possible. Therefore, the general boundary-value problem leads to the following alternatives: Either the homogeneous problem corresponding to the given non-homogeneous problem has no solutions other than the trivial solution identically equal to zero (and then the non-homogeneous problem has a unique solution) or the homogeneous problem has non-trivial solutions (and then the non-homogeneous problem can have a number of solutions differing from each other by the solution to the homogeneous problem).

On the basis of the material of this chapter, the reader should not find it difficult to construct solutions for a number of particular boundary-value problems (for example, for a sphere) that confirm the validity of the general statements made above or to see that, when there are non-trivial solutions to the homogeneous problem, the boundary condition for the non-homogeneous problem cannot be completely arbitrary. We shall return to the general formulation of this pair of alternatives in Chapter XXVII, section 6.

Let us suppose now that k^2 is a *negative* real number. The square root of k^2 will then be purely imaginary so that the general regular spherically symmetric solution to eq. (14) will be of the form

$$u(r) = A\,\frac{1}{r}\,(e^{k''r} - e^{-k''r}) \;, \tag{27}$$

where k'' is a positive real number. For the boundary condition of the problem (13) to be satisfied, the constant A must satisfy the equation

$$A(\beta K_1 + \alpha K_2) = C r_0^2 \;, \tag{28}$$

where

$$K_1 \equiv r_0(e^{k''r_0} - e^{-k''r_0}) \;, \qquad K_2 \equiv (k''r_0 - 1)\,e^{k''r_0} + (k''r_0 + 1)\,e^{-k''r_0} \;.$$

The function K_1 is always positive if r_0 and k'' are positive. Let us show that the function K_2 is also positive. The derivative

$$\frac{\partial K_2}{\partial k''} = k''r_0^2(e^{k''r_0} - e^{-k''r_0}) > 0 \quad \text{when} \quad k'' > 0 \;;$$

that is, the function K_2 increases with increasing k''. But the function $K_2 = 0$ when $k'' = 0$. Our assertion follows from this.

Consequently, when α and β have the same sign or when one of them is equal to zero, the coefficient of A in eq. (28) does not vanish and, there-

fore, the problem (13) has a unique solution and the homogeneous problem (19) corresponding to it has no solution other than the trivial one of $u \equiv 0$. In other words, under these conditions, undamped free vibrations are impossible. As we shall see later, this is related to the fact that when the imaginary part k'' of k is different from zero, energy is dispersed in the vibrating medium, as a result of which undamped vibrations cannot arrive without a supply of energy from the outside.

Let us take a close look at the picture of the oscillations that take place when the parameters α and β are of different signs, so that they may be chosen in such a way that the expression $\beta K_1 + \alpha K_2$ will vanish in eq. (28). The non-homogeneous problem (13) will then be insoluble, but, on the other hand, the expression (27) will satisfy the homogeneous problem (19) for arbitrary values of the constant A; that is, free oscillations will be possible in spite of the dispersion of energy in the medium.

Let us consider the oscillations on the boundary when β is positive and $\alpha = |\alpha| e^{i \arg \alpha}$ has an arbitrary *complex* value. We therefore turn from the amplitudes u to the functions of time $w = \operatorname{Re} u e^{-i\omega t}$. We then obtain

$$\operatorname{Re} \beta u \, e^{-i\omega t} = \frac{A}{r_0^2} \beta K_1 \cos \omega t \, .$$

$$\operatorname{Re} \alpha \frac{du}{dn} e^{-i\omega t} = \frac{A}{r_0^2} \operatorname{Re} |\alpha| \, e^{i \arg \alpha} K_2 \, e^{-i\omega t} = \frac{A}{r_0^2} K_2 \, |\alpha| \cos (\omega t - \arg \alpha) \, .$$

The first of these expressions describes the oscillations of the medium on the boundary. With regard to the second, as we have mentioned, the values of the derivative du/dn characterize the stresses in the medium when there are mechanical vibrations. Consequently, these expressions show that when $\arg \alpha \neq 0$, the phase of the vibrations of the stress of the medium is displaced relative to the phase of the vibrations of the medium. In particular, when $\arg \alpha = \pi$, that is, when the parameter α represents a negative number, the phases of the vibrations of the stress and of the medium are opposite. This means, for example, that the compression of the medium coincides with the action of forces corresponding to its stretching; that is, the forces strive not to return the elements of the medium to the equilibrium position but to move them yet farther away from it, so that the energy of the vibrations on the boundary is not dispersed but increases. There exists then the possibility of undamped vibrations arising despite the dispersion in the medium.

Such self-excitation of vibrations on the boundary without an influx of energy from without would, for ordinary physical media, be a violation of the law of conservation of energy. Therefore, in boundary-value problems connected with technical problems, the coefficients α and β cannot be of opposite sign.

Problem

Show that in a plane the solutions to the Helmholtz equation that are circularly symmetric satisfy the zero order Bessel equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + k^2u = 0 .$$

3. *Eigenvalues and eigenfunctions of a general boundary-value problem. Expansions in eigenfunctions*

In section 2, we studied the boundary-value problem

$$\Delta u + k^2u = f \quad \text{when} \quad x \in V - \mathcal{F}V , \qquad \alpha\frac{du}{dn} + \beta u = \psi \quad \text{when} \quad x \in \mathcal{F}V , \tag{29}$$

in the special case in which the region V was a sphere. We saw that the homogeneous problem corresponding to it

$$\Delta u + k^2u = 0 \quad \text{when} \quad x \in V - \mathcal{F}V , \qquad \alpha\frac{du}{dn} + \beta u = 0 \quad \text{when} \quad x \in \mathcal{F}V , \tag{30}$$

had non-trivial solutions for real positive values of k^2 and that these values formed an unbounded increasing sequence of numbers

$$k_1{}^2, k_2{}^2, \ldots, k_m{}^2, \ldots . \tag{31}$$

To each value $k_m{}^2$ corresponded one non-trivial solution u_m.

It is shown in the theory of integral equations that these results carry over completely to the problem stated for an arbitrary bounded region V, and also to the corresponding problem in an arbitrary plane region S with this difference that to a single number $k_m{}^2$ there may correspond not one but several linearly independent non-trivial solutions to the problem.

The numbers $k_m{}^2$ (where $m = 1, 2, 3, \ldots$) are determined by the region V (or S) and the boundary condition of problem (29). As before, we shall call them eigenvalues of the problem (29) and we shall call the solutions u_m the eigenfunctions of this problem corresponding to the eigenvalues $k_m{}^2$.

The coefficients α and β in the boundary condition to problem (29) can be functions of a point on the boundary. We shall assume that the boundary can be partitioned into a finite number of continuously smooth portions. On each of these portions we shall assume the coefficients α and β to have constant values which are non-negative and not simultaneously equal to zero, and that the coefficient α is either identically equal to zero or is everywhere positive. In the latter case, we may divide eq. (29) by it. Therefore, we may assume with no loss of generality that the coefficient α is equal either to zero or to one. Corresponding to these values of α, we may assume that on each of these continuously smooth portions of the boundary the coefficient β is either equal (when $\alpha \equiv 0$) to one or (if $\alpha = 1$) it can be different from one.

If to every eigenvalue $k_m{}^2$ correspond several linearly independent

eigenfunctions, for example, the two functions u_m and u_{m+1}, then these functions can always be chosen such that they are normalized and orthogonal in the region V, that is, so that

$$\iiint\limits_V u_m\, u_{m+1}\, \mathrm{d}V = 0\ , \qquad \iiint\limits_V u_m^2\, \mathrm{d}V = 1\ , \qquad \iiint\limits_V u_{m+1}^2\, \mathrm{d}V = 1\ . \tag{32}$$

To show this, suppose that v_m and v_{m+1} are eigenfunctions that do not have this property. Let us set

$$u_m = av_m\ , \qquad u_{m+1} = a_1 v_m + bv_{m+1}\ ,$$

where a, a_1, and b are constants. The functions u_m and u_{m+1} are also eigenfunctions and, for $b \neq 0$, they are linearly independent since the functions v_m and v_{m+1} are linearly independent. Let us now find constants a, a_1, and b that will satisfy the first of eqs. (32). To determine these constants, we take the equation

$$aa_1(v_m, v_m) + ab(v_m, v_{m+1}) = 0\ ,$$

where

$$(v_m, v_m) \equiv \iiint\limits_V v_m^2\, \mathrm{d}V\ , \qquad (v_m, v_{m+1}) \iiint\limits_V v_m\, v_{m+1}\, \mathrm{d}V\ .$$

We conclude from this that a can be chosen arbitrarily and that

$$a_1 = b\kappa\ ,$$

where

$$\kappa = -\frac{(v_m, v_{m+1})}{(v_m, v_m)}\ .$$

We can then write

$$u_m = av_m\ , \qquad u_{m+1} = b(\kappa v_m + v_{m+1})\ .$$

The constants a and b can always be chosen in such a way that the last two of eqs. (32) are satisfied. This completes the proof.

If there is a third eigenfunction v_{m+2} such that it, v_m, and v_{m+1} are linearly independent, this process of orthogonalization can be continued by setting

$$u_{m+2} = a_2 v_m + b_1 v_{m+1} + cv_{m+2}\ ,$$

and choosing the constants a_2, b_1, and $c \neq 0$ in such a way that the orthogonality and normalization conditions will be satisfied.

The number of linearly independent eigenfunctions corresponding to the eigenvalue k_m^2 we shall call the dimensionality of this eigenvalue and, in numbering these eigenvalues, we shall count each of them as many times as it has dimensions. For example, if the dimensionality of the number k_m^2 is 2 in the sequence of eigenvalues $k_1^2, k_2^2, k_3^2, \ldots$, we shall write not k_m^2 but

$$k_m^2 = k_{m+1}^2, k_{m+2}^2,$$

etc. We then have the eigenfunctions numbered in the same way as the eigenvalues.

Let us now show that the eigenfunctions corresponding to the different eigenvalues are mutually orthogonal. For this, we use Green's theorem (7) of Chapter XVII. Assuming $\alpha = 1$ and noting that

$$v\left(\frac{du}{dn} + \beta u\right) - u\left(\frac{dv}{dn} + \beta v\right) = v\frac{du}{dn} - u\frac{dv}{dn},$$

we transform Green's theorem into the form

$$\iiint_V (v\Delta u - u\Delta v)\, dV = \iint_{\mathcal{F}V}\left[v\left(\frac{du}{dn} + \beta u\right) - u\left(\frac{dv}{dn} + \beta v\right)\right] dS .$$

If we make the substitutions $v = u_m$ and $u = u_l$ and if we substitute the values of the quantities given in the problem (30), we obtain

$$(k_l^2 - k_m^2)\iiint_V u_m u_l \, dV = 0 .$$

Since, by hypothesis,

$$k_l^2 \neq k_m^2 ,$$

we have

$$\iiint_V u_m u_l \, dV = 0 \qquad (l \neq m) , \tag{33}$$

as was stated. If $\alpha = 0$, the proof is analogous; in such a case we should use Green's theorem in its original form.

Since all the eigenfunctions are determined up to a constant factor, it is always possible to normalize them by dividing by

$$\iiint_V u_m^2 \, dV .$$

Consequently, when we orthogonalize the linearly independent eigenfunctions corresponding to the multiple eigenvalues, we can write the general relationship:

$$\iiint_V u_m u_l \, dV = \begin{cases} 1 & \text{when } l = m , \\ 0 & \text{when } l \neq m , \end{cases} \tag{34}$$

which we shall, throughout the present chapter, always assume to be satisfied.

The system of eigenfunctions is complete. The most general result pertaining to the completeness of the system of eigenfunctions of problem (29) belongs to V. A. Il'in *. We shall give some of them.

Suppose that f is an arbitrary continuous function defined throughout the region V and that the integrals

* Uspekhi Matematicheskikh Nauk 13 (1958) 89.

$$\iiint\limits_V \left(\frac{\partial f}{\partial x_m}\right)^\lambda \mathrm{d}V \quad (m = 1, 2, 3, \ldots) \tag{35}$$

are defined for some value of $\lambda > 3$. Then,
(1) if $\alpha \neq 0$, the function f can be expanded in the series

$$f = \sum_{\nu=1}^{\infty} f_\nu \, u_\nu \tag{36}$$

of eigenfunctions u_ν of the problem (29). This series converges uniformly when the summation is taken in order of increase in the eigenfunctions at all points $x \in V - \mathcal{F}V$. The coefficients in the series can be calculated from the formula

$$f_\alpha = \iiint\limits_V f \, u_\alpha \, \mathrm{d}V \,. \tag{37}$$

(2) If $\alpha = 0$, the same results are valid under the additional condition that

$$f = 0 \quad \text{when} \quad x \in \mathcal{F}V \,. \tag{38}$$

If we impose more stringent restrictions on the function f by assuming that it and its first derivatives are continuous throughout the region V and that the integrals

$$\iiint\limits_V \left(\frac{\partial^2 u}{\partial x_m \, \partial x_l}\right)^2 \mathrm{d}V \quad (m, l = 1, 2, 3, \ldots)$$

exist, then the series (36) will converge absolutely and uniformly.

An analogous expansion theorem is valid for problems in two dimensions. (Here, λ must be taken greater than 2 in formula (35).)

These expansion theorems can be applied immediately to the solution of the non-homogeneous boundary-value problem:

$$\Delta u + k^2 u = f \quad \text{when} \quad x \in V - \mathcal{F}V \,, \qquad \alpha \frac{\mathrm{d}u}{\mathrm{d}n} + \beta u = 0 \quad \text{when} \quad x \in \mathcal{F}V \,. \tag{39}$$

Let us expand the function u in the series

$$u = \sum_{\nu=1}^{\infty} a_\nu \, u_\nu \tag{40}$$

of eigenfunctions u_ν of the problem (39). By substituting this series in the equation for the problem (39) and by formally carrying out termwise differentiation, we obtain

$$\Delta u + k^2 u = \sum_{\nu=1}^{\infty} a_\nu (\Delta u_\nu + k^2 u_\nu) = \sum_{\nu=1}^{\infty} a_\nu (k_\nu{}^2 - k^2) u_\nu \,.$$

By comparing the series on the right side with the expansion (36) of the function f, we see that the conditions of the problem (39) will be satisfied if we set

$$a_\nu = \frac{f_\nu}{k_\nu^2 - k^2} \equiv \frac{1}{k_\alpha^2 - k^2} \iiint_V f\, u_\nu \,\mathrm{d}V \,. \tag{41}$$

This equation shows that the method that we have been examining for solving the problem (39) can be applied if the parameter k^2 is not equal to any of the eigenvalues of the problem (39). As we let the parameter k^2 approach any one of the eigenvalues k_m^2, then under the condition that $f_m \neq 0$, the corresponding coefficient a_m will increase without bound.

The physical interpretation of this is that when $k^2 = k_m^2$, in the region V natural vibrations take place whose amplitudes increase in time without bound, due to the action of the disturbance f which is in *resonance* with them. As a consequence, there will be no steady-state vibrations. This is shown by the insolubility of the problem (39). In actuality, when there is resonance, the vibrations are bounded either because of the presence of damping ($\operatorname{Im} k^2 > 0$) or because of non-linear phenomena that were not taken into consideration in the mathematical formulation of the problem.

When $k^2 = k_m^2$, it is easy to find the non-resonant part of the vibrations by means of the method that we have been examining. For example, let us suppose that the function f satisfies the condition

$$\iiint_V f\, u_m \,\mathrm{d}V = 0 \ ,$$

with the consequence that free vibrations do not arise. With this assumption, we may set $a_m = 0$ and we can compute the remaining coefficients from formula (41). To the solutions that we have found by this method, we may add any of the eigenfunctions of the problem that correspond to the eigenvalue k_m^2; when we do, we obtain the same solution. Thus, we are dealing now with the second general alternative described in section 2.

We may also use the method of expansion in eigenfunctions for solving the general boundary-value problem (29). Here, we need to find the function u_1 that is continuous, that has continuous first and second derivatives, and that satisfies the boundary condition of the problem in question. By setting u identically equal to $v + u_1$, we get the problem

$$\Delta v + k^2 v = f - \Delta u_1 - k^2 u_1 \quad \text{when} \quad x \in V - \mathcal{F}V \ ,$$

$$\alpha \frac{\mathrm{d}v}{\mathrm{d}n} + \beta v = 0 \quad \text{when} \quad x \in \mathcal{F}V \ ,$$

which is analogous to the problem (39).

The method of solving boundary-value problems by means of expansion in eigenfunctions of homogeneous problems is not always successful because of the difficulty in finding the eigenfunctions. However, when it is possible to separate the variables, the eigenfunctions can frequently be expressed in terms of familiar functions. We shall give examples of this in the following section.

In the above, we have not gone into the question of the admissibility of the termwise differentiation of the series (40). We shall discuss this somewhat in Chapter XXXIX. Here, we note that the series (40), with coeffi-

cients computed from the formulae (41), converges to the solution of the problem (39) under extremely general conditions that do not require termwise differentiability.

Problem

Find the solution to the Dirichlet problem for the Poisson equation

$$\Delta u = f \quad \text{when} \quad x \in V - \mathcal{F}V\,, \qquad u = 0 \quad \text{when} \quad x \in \mathcal{F}V$$

by means of an expansion in eigenfunctions.
Method: Expand in terms of the eigenfunctions of the homogeneous problem for the Helmholtz equation:

$$\Delta u + k^2 u = 0 \quad \text{when} \quad x \in V - \mathcal{F}V\,, \qquad u = 0 \quad \text{when} \quad x \in \mathcal{F}V\,.$$

4. *The separation of variables in the Helmholtz equation in cylindrical and spherical coordinates*

We find from formulae (3) and (4) of Chapter XVIII that the Helmholtz equation has the forms:
in cylindrical coordinates r, φ, z,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0\,, \tag{42}$$

and in spherical coordinates r, θ, φ,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial u}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} + k^2 u = 0\,. \tag{43}$$

Eqs. (42) and (43) allow separation of variables. In the case of cylindrical coordinates, if we set

$$u = u_1(r)u_2(\varphi)u_3(z) \tag{44}$$

and substitute this expression into eq. (42), we obtain

$$\frac{1}{u_1}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_1}{\mathrm{d}r}\right) + \frac{1}{u_2}\frac{1}{r^2}\frac{\mathrm{d}^2 u_2}{\mathrm{d}\varphi^2} + \frac{1}{u_3}\frac{\mathrm{d}^2 u_3}{\mathrm{d}z^2} + k^2 = 0\,.$$

Since the term $(1/u_3)(\mathrm{d}^2u_3/\mathrm{d}z^2)$ depends only on z and the remaining terms do not depend on z, it follows that

$$\begin{gathered}\frac{1}{u_3}\frac{\mathrm{d}^2 u_3}{\mathrm{d}z^2} + k^2 = \mu^2\,,\\ \frac{1}{u_1}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_1}{r}\right) + \frac{1}{r^2}\frac{1}{u_2}\frac{\mathrm{d}^2 u_2}{\mathrm{d}\varphi^2} = -\mu^2\,,\end{gathered} \tag{45}$$

where μ^2 is a constant. If we multiply the second of eqs. (45) by r^2, we also conclude that

$$\frac{1}{u_2}\frac{d^2u_2}{d\varphi^2} = -\lambda^2 , \qquad \frac{r}{u_1}\frac{d}{dr}\left(r\frac{du_1}{dr}\right) + r^2\mu^2 = \lambda^2 ,$$

where λ^2 is also a constant. The numbers μ^2 and λ^2 are called the constants of separation.

Let us rewrite these equations in the form

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du_1}{dr}\right) + \left(\mu^2 \frac{\lambda^2}{r^2}\right)\mu_1 = 0 , \tag{46}$$

$$\frac{d^2u_2}{d\varphi^2} + \lambda^2 u_2 = 0 , \tag{47}$$

$$\frac{d^2u_3}{dz^2} - (\mu^2 - k^2)u_3 = 0 . \tag{48}$$

Eigenfunctions can be defined by means of the solutions to eqs. (46)-(48) for regions that have the shape of a cylinder, a hollow cylinder, or either of these with a sector removed by cutting along radial planes.

For example, let us construct eigenfunctions of the boundary-value problem for eq. (42) with the boundary condition

$$\alpha\frac{du}{dn} + \beta u = 0 \quad \text{when} \quad r = r_0 , \qquad u = 0 \quad \text{when} \quad z = \pm z_0 , \tag{49}$$

where β, r_0, and z_0 are positive constants. The region V then is a cylinder

$$r \leqslant r_0 , \qquad |z| \leqslant z_0 .$$

Since the eigenfunctions are single-valued and continuous in the region that we are studying, it follows that they must have period 2π with respect to the coordinate φ. Therefore, we need to set $\lambda^2 = n^2$ (where $n = 1, 2, 3, \ldots$) in eq. (47). Then, the general solution of eq. (47) will be of the form $u_2 = \cos(n\varphi + \psi_n)$, where ψ_n is an arbitrary constant.

The solutions to Bessel's equation (46), which are regular at $r = 0$, are the Bessel functions $J_0(\mu r)$. Therefore, let us set $u_1(r) = J_n(\mu r)$. Since

$$d/dn = d/dr ,$$

on the surface $r = r_0$, if we substitute the expression found for $u_1(r)$ in the boundary condition (49), we obtain the equations

$$\alpha\mu J_n'(\mu r_0) + \beta J_n(\mu r_0) = 0 ,$$

which determine the admissible values of the constant of separation μ^2. As we know from the theory of Bessel functions, this equation has an infinite set of distinct roots $\mu_{1n}, \mu_{2n}, \ldots, \mu_{mn}$, which we shall assume numbered in increasing order of magnitude.

Let us now turn to eq. (48). For $\mu^2 = \mu_{mn}{}^2$, its general solution is of the form

$$u_3(z) = A \sin \nu z + B \cos \nu z ,$$

where

$$\nu = \sqrt{k^2 - \mu_{mn}^2} \ .$$

Substituting the expression for the function u_3 in the boundary condition (49), we obtain the system of homogeneous equations for determining the values of the constants A and B:

$$A \sin \nu z_0 + B \cos \nu z_0 = 0 \ ,$$

$$-A \sin \nu z_0 + B \cos \nu z_0 = 0 \ .$$

Non-trivial solutions to this system will exist if its determinant is equal to zero, that is, if

$$\sin 2\nu z_0 = 0 \ .$$

The roots of this equation, numbered in increasing order of magnitude, we denote by ν_l. The constants A and B can then be represented by the expression

$$A_l = \tfrac{1}{2} \cos \nu_l z_0 \ , \qquad B_l = \tfrac{1}{2} \sin \nu_l z_0 \ ,$$

and the general expression for the function $u_3(z)$ takes the form

$$u_3(z) = \sin \nu_l (z + z_0) \ .$$

When we multiply the expressions that we have found for the functions u_1, u_2, and u_3, we get the general expression for the eigenfunctions of the problem in question:

$$u_{lmn}(r, \varphi, z) = C_{lmn} J_n(\mu_{mn} r) \cos(n\varphi + \psi_n) \sin \nu_l (z + z_0) \ ,$$

where C_{lmn} are arbitrary constants. The eigenvalues of the problem are determined by the relationship

$$k_{lmn}^2 = \nu_l^2 + \mu_{mn}^2 \ .$$

We are also interested in writing the general expression for the solutions to the Helmholtz equation that are of the form $u_1(r)u_2(\varphi)u_3(z)$ for variations of φ in the interval $(0, 2\pi)$. In this case, $\lambda^2 = n^2$, where n is an integer and the solution to eq. (47), as before, is of the form $u_2 = \cos(n\varphi + \psi_n)$. The general solution u_1 to Bessel's equation (46) for $\lambda^2 = n^2$ we denote by $Z_n(\mu r)$. Finally, we write the general solution to eq. (48) in the form $u_3 = \cos(\nu z + \psi_\nu)$, where $\nu = \sqrt{(k^2 - \mu^2)}$, and ψ_ν is an arbitrary constant. Thus, the general expression for the solution of the form (44) that we are seeking can be written in the form

$$u_n = A Z_n(\mu r) \cos(n\varphi + \psi_n) \cos(\nu z + \psi_\nu) \qquad (\nu \equiv \sqrt{k^2 - \mu^2}) \ , \tag{50}$$

where A is an arbitrary constant.

Let us turn to the question of separating the variables in the Helmholtz equation written in spherical coordinates. When we substitute the expression $u = v_1(r)v_2(\theta)v_3(\varphi)$ into eq. (43) and separate the variables, we obtain the equations

$$\frac{d}{dr}\left(r^2\frac{dv_1}{dr}\right) + (k^2r^2 - \lambda)v_1 = 0\ , \tag{51}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dv_2}{d\theta}\right) + \left(\lambda - \frac{m^2}{\sin\theta}\right)v_2 = 0\ , \tag{52}$$

$$\frac{d^2v_3}{d\varphi^2} + m^2v_3 = 0\ , \tag{53}$$

where λ and m^2 are constants of separation.

When we find the corresponding solutions to these equations, we can construct the eigenfunctions of the region having the form of a sphere, a hollow sphere, or either of these with a deletion in the form of a circular cone having vertex at the center of the sphere or of a lune formed by two half-planes which meet along a diameter of the sphere.

If the region in question encompasses the entire interval of variation of φ, the function $v_3(\varphi)$ must be of period 2π. The general solution to eq. (53) satisfying this requirement is of the form

$$v_3(\varphi) = \cos(m\varphi + \psi_m)\ , \tag{54}$$

where $m = 1, 2, 3, \ldots$ and ψ_m is an arbitrary constant.

Eq. (52), for $\lambda = n(n+1)$, where n is an integer, is the equation for associated Legendre polynomials $P_{nm}(\cos\theta)$. As we know (Chapter XXI), the product formed by multiplying the associated Legendre polynomials by functions of the form (54) constitute a complete system of spherical functions in the intervals $0 \leqslant \varphi \leqslant 2\pi$ and $0 \leqslant \theta \leqslant \pi$ of variation of the variables φ and θ. Therefore, for given intervals of variation of the variables φ and θ, no other products that are linearly independent with these can go into the composition of the eigenfunctions. Consequently, if the region in question encompasses the entire interval of variation of θ, we need to set $\lambda = n(n+1)$.

Let us turn to eq. (51). For $\lambda = n(n+1)$, we can, by means of the substitution $w = \sqrt{r}v_1(r)$, transform it into Bessel's equation of half-integral order:

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \left[k^2 - \frac{(n+\frac{1}{2})^2}{r^2}\right] = 0\ , \tag{55}$$

the solution of which we denote by $Z_{n+\frac{1}{2}}(kr)$. Then,

$$v_1(r) = \frac{1}{\sqrt{r}}Z_{n+\frac{1}{2}}(kr)\ .$$

If we multiply the functions v_1, v_2, and v_3, we get the following general expression for the eigenfunctions for regions in which the coordinates φ and θ vary in the intervals $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \varphi \leqslant 2\pi$ (sphere and hollow sphere):

$$u_{mn}(r,\theta,\varphi) = \frac{1}{\sqrt{r}}Z_{n+\frac{1}{2}}(kr)\ P_{nm}(\cos\theta)\cos(m\varphi + \psi_m)\ . \tag{56}$$

The particular choice of the solution $Z_{n+\frac{1}{2}}(kz)$ and the eigenvalues k_l^2 are determined by the given boundary conditions.

For example, let us examine the boundary conditions

$$u = 0 \quad \text{when} \quad r = r_0 \quad \text{and} \quad r = r_1 \quad (r_1 < r_0) ,$$

corresponding to the Dirichlet problem for a hollow spherical shell with internal radius r_1 and external radius r_0. The general solution of eq. (55) is of the form

$$Z_{n+\frac{1}{2}}(kr) = AJ_{n+\frac{1}{2}}(kr) + BY_{n+\frac{1}{2}}(kr) ,$$

from which,

$$v_3(r) = A\frac{1}{\sqrt{r}}J_{n+\frac{1}{2}}(kr) + B\frac{1}{\sqrt{r}}\,Y_{n+\frac{1}{2}}(kr) .$$

Substituting this expression into the given boundary condition, we obtain the system of equations for determining A and B:

$$AJ_{n+\frac{1}{2}}(kr_0) + BY_{n+\frac{1}{2}}(kr_0) = 0 , \qquad AJ_{n+\frac{1}{2}}(kr_1) + BY_{n+\frac{1}{2}}(kr_1) = 0 .$$

There will be non-trivial solutions to this system if its determinant is equal to zero, that is, for values of the parameter k that satisfy the condition

$$J_{n+\frac{1}{2}}(kr_0)\; Y_{n+\frac{1}{2}}(kr_1) - J_{n+\frac{1}{2}}(kr_1)\, Y_{n+\frac{1}{2}}(kr_0) = 0 .$$

The roots of this equation, numbered in increasing order of magnitude, we denote by k_{ln}. For $k = k_{ln}$,

$$\frac{A}{B} = -\frac{Y_{n+\frac{1}{2}}(k_{ln}r_0)}{J_{n+\frac{1}{2}}(k_{ln}r_0)} ,$$

so that we may set

$$v_3(r) = \frac{1}{\sqrt{r}}\left[Y_{n+\frac{1}{2}}(k_{ln}r_0)\; J_{n+\frac{1}{2}}(k_{ln}r) - J_{n+\frac{1}{2}}(k_{ln}r_0)\; Y_{n+\frac{1}{2}}(k_{ln}r)\right] .$$

From this, we obtain the expression for the eigenfunctions of the Dirichlet problem stated for a region having the form of a hollow sphere:

$$u_{lmn} = A_{lmn}\frac{1}{\sqrt{r}}\left[Y_{n+\frac{1}{2}}(k_{ln}r_0)\; J_{n+\frac{1}{2}}(k_{ln}r) - J_{n+\frac{1}{2}}(k_{ln}r_0)\; Y_{n+\frac{1}{2}}(k_{ln}r)\right]$$
$$\times\, P_{nm}(\cos\theta)\, \cos(m\varphi + \psi_m) , \qquad (57)$$

where the A_{lmn} are arbitrary constants.

Problems

1. Show that the eigenfunctions of the boundary problem for eq. (42) under the boundary condition

$$\alpha\frac{\mathrm{d}u}{\mathrm{d}n} + \beta u = 0 \quad \text{when} \quad r = r_0 \quad \text{or} \quad z = \pm z_0 ,$$

which corresponds to the mixed boundary problem in a cylindrical region, are of the form

$$u_{lmn}(r, \varphi, z) = A_{lmn} J_n(\mu_{mn} r) \cos(n\varphi + \psi_n) \times [\beta \sin \nu_l (z + z_0) + \alpha \nu_l \cos \nu_l (z + z_0)] ,$$

where ν_l is a root of the equation

$$\tan 2\nu z_0 = \frac{2\alpha\beta\nu}{\alpha^2\nu^2 - \beta^2} .$$

2. Show that the eigenfunctions of the boundary-value problem for eq. (43) with the boundary condition

$$u = 0 \quad \text{when} \quad r = r_0 ,$$

corresponding to the Dirichlet problem for a sphere, are of the form

$$u_{lmn}(r, \theta, \varphi) = A_{lmn} \frac{1}{\sqrt{r}} J_{n+\frac{1}{2}}(k_{ln} r)\, P_{nm}(\cos\theta) \cos(m\varphi + \psi_m) ,$$

where the k_{ln} are roots of the equation

$$J_{n+\frac{1}{2}}(kr_0) = 0 .$$

5. *Spherically symmetric solutions of the Helmholtz equation in an infinite region*

We begin our study of the *external* boundary-value problem for the Helmholtz equation with problems characterized by spherical symmetry.

Let us examine the external problem

$$\frac{d^2 ru}{dr^2} + k^2 ru = 0 \quad \text{when} \quad r > r_0 , \qquad \alpha \frac{du}{dn} + \beta u = C \quad \text{when} \quad r = r_0 , \tag{58}$$

where r_0 is positive and α, β, and C are real constants. We shall return to the condition at an infinitely distant point later.

Let us first assume that the parameter k^2 is a real positive number and let us seek the real solution to the problem (58). From the expression (12) for the general spherically symmetric solution to the Helmholtz equation, we find that the general expression for the real solution is of the form

$$u(r) = a \frac{\cos kr}{r} + b \frac{\sin kr}{r} , \tag{59}$$

where a and b are arbitrary constants. This solution is regular throughout the entire infinite region $r \geqslant r_0$ and it vanishes at infinity. Values for the constants a and b can in the general case be chosen without difficulty and in an infinite number of ways so that the boundary condition of problem (58) will be satisfied. Consequently, the problem (58) has an infinite number of regular solutions that vanish at infinity. To understand the meaning of these solutions, let us, as in section 2, examine the solutions $w(r, t)$ of the wave equation (21) corresponding to them. We multiply the general solution (12) by $e^{-i\omega t}$. This gives us

$$A\,\frac{e^{i(kr-\omega t)}}{r} + B\,\frac{e^{-i(kr+\omega t)}}{r}\,. \tag{60}$$

If, in accordance with eq. (22), we make the substitution $\omega = kc$ in the above expression and if we remember that the numbers A and B are complex, we easily see that, for real values of k, the real part of this expression can be represented in the form

$$A_1\,\frac{1}{r}\cos[k(r-ct)+\theta] + B_1\,\frac{1}{r}\cos[k(r+ct)+\theta]\,, \tag{61}$$

where A_1, B_1, and θ are constants.

As we shall see right away, this last expression corresponds to two systems of spherical waves. Let us examine the first term of the expression (61). We can represent the argument of the cosine function in the form of a sum $2n\pi + \theta_0$, where n is an integer and θ_0 is a positive number not exceeding 2π. To every value of the difference $r - ct$ there correspond values of n and θ_0. We shall call the number θ_0 the phase of the wave. The position of points of like phase represents for the first term of (61) a system of spherical surfaces concentric with the surface $r = r_0$. The expression $k(r-ct)+\theta$ has a constant value on each of these surfaces. Consequently, with the passage of time, their radius r increases with the velocity c. Thus, the first term corresponds to a system of spherical waves leaving the surface $r = r_0$ with phase velocity c. In the same way, we can show that the second term corresponds to a system of spherical waves *converging* on the surface $r = r_0$ from infinity.

If we regard the surface $r = r_0$ as a source of waves, we must ascribe a physical meaning only to the first term of the sum (61), considering the term corresponding to the system of waves that originate at infinity as having no physical meaning. Consequently, in the sum (60), and hence in the general expression (12) for a spherically symmetric solution to the Helmholtz equation, we must keep one term, namely, the first.

Let us seek an analytic criterion that will allow us to pick out those solutions of the Helmholtz equation that correspond to the diverging waves. Consider the functions

$$u = e^{ikr}/r \quad \text{and} \quad v = e^{-ikr}/r\,,$$

that appear in the general solution (12). When we differentiate these functions with respect to r, we find that they satisfy the differential relationships:

$$r\left(\frac{\partial u}{\partial r} - iku\right) = -u\,, \qquad r\left(\frac{\partial v}{\partial r} + ikv\right) = -v\,.$$

Letting r go to approach infinity and noting that

$$\lim_{r\to\infty} u = 0\,, \qquad \lim_{r\to\infty} v = 0\,,$$

we obtain the limiting relationships

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0\ , \qquad \lim_{r\to\infty} r\left(\frac{\partial v}{\partial r} + ikv\right) = 0\ ,$$

which provide a general analytic criterion that will allow us to distinguish the functions u and v by their behaviour at infinity. It is easy to see that, if we reverse the positions of u and v, these limiting relationships will not be satisfied.

This criterion becomes trivial when the spherically symmetric solutions to the Helmholtz equation are considered. However, it is of great heuristic value in setting up the general external boundary-value problem for this equation.

First of all, it is natural to try to extend it to an arbitrary system of spherical waves with an angular distribution of amplitudes since this distribution should not change the general rule of decrease of amplitudes along the radii. Also, it is natural to assume that at a sufficiently great distance from a bounded region in which a source of waves is located, waves of an arbitrary system will be very nearly spherical. Therefore, at a great distance from their source an arbitrary system of diverging waves must display the same features as does a system of diverging spherical waves; that is, the limiting relations

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0 \quad \text{and} \quad \lim_{r\to\infty} u = 0 \tag{62}$$

will be observed. This convergence to zero must be uniform as the distance increases along an arbitrary radius beginning in the bounded region. These relationships, which were first shown by Sommerfeld, are called the radiation condition. As we shall prove rigorously, the radiation condition ensures a unique solution to the external problem for the Helmholtz equation; that is, it plays the same role as does the condition of vanishing at infinity in the problems for the Laplace equation.

Let us return to problem (58). We seek a solution to it that will satisfy the radiation condition. As we have shown, the general spherically symmetric solution that satisfies the radiation condition is of the form

$$A(e^{ikr}/r)\ . \tag{63}$$

Substituting this expression into the boundary condition of the problem (58), we obtain the following equation for determining the value of the constant A:

$$A\ e^{ikr_0}(i\alpha k + \beta) = C\ . \tag{64}$$

If we equate separately the real and imaginary parts of this equation, we obtain

$$A'(\beta \cos kr_0 - \alpha k \sin kr_0) - A''(\beta \sin kr_0 + \alpha k \cos kr_0) = C\ ,$$

$$A'(\beta \sin kr_0 + \alpha k \cos kr_0) + A''(\beta \cos kr_0 - \alpha k \sin kr_0) = 0\ ,$$

where A and A'' denote the real and imaginary parts of the complex number A. The determinant of this system

$$\Delta = \beta^2 + \alpha^2 k^2$$

does not vanish for positive values of k^2. Consequently, its solution always exists and is unique. In particular, for $C = 0$, it has only the trivial solution $A' = A'' = 0$ so that the homogeneous problem

$$\frac{d^2 ru}{dr^2} + k^2 ru = 0 \quad \text{when} \quad r > r_0 , \qquad \alpha \frac{du}{dn} + \beta n = 0 \quad \text{when} \quad r = r_0 , \tag{65}$$

has no solutions that are not identically equal to zero. This means that in an infinite region free vibrations are impossible and only forced wave processes can arise.

After some simple manipulations, we see that

$$A' = \frac{C}{\sqrt{\beta^2 + \alpha^2 k^2}} \cos k(r_0 + r_k) , \qquad A'' = - \frac{C}{\sqrt{\beta^2 + \alpha^2 k^2}} \sin k(r_0 + r_k) ,$$

where the constant r_k is determined by the relationship

$$\cos kr_k = \frac{\beta}{\sqrt{\beta^2 + \alpha^2 k^2}} , \qquad \sin kr_k = \frac{\alpha k}{\sqrt{\beta^2 + \alpha^2 k^2}} .$$

From this, we see that

$$A = A' + iA'' = \frac{C}{\sqrt{\beta^2 + \alpha^2 k^2}} e^{-ik(r_0 + r_k)} \tag{66}$$

Substituting this value of A into the expression (63), we find the solutions that we have been seeking:

$$u = \frac{C}{\sqrt{\beta^2 + \alpha^2 k^2}} \frac{e^{ik(r - r_0 - r_k)}}{r} \tag{67}$$

For real values of k, this solution is always complex.

Let us now examine the general case, assuming that k is an arbitrary complex number. If we denote by $k \equiv k' + ik''$ the square root of k^2, whose real part k' is positive, we can transform the general expression (12) for a spherically symmetric solution into the form

$$A\, e^{-k''r} \frac{e^{ik'r}}{r} + B\, e^{k''r} \frac{e^{-ik'r}}{r} . \tag{68}$$

It is easy to show that, as before, the first term of this sum

$$A\, e^{-ik''r} \frac{e^{ik'r}}{r} . \tag{69}$$

corresponds to a system of diverging waves. However, depending on the sign of k'', it may either become infinitesimally small or increase without bound.

To grasp the meaning of this result, we use a well-known proposition in the theory of oscillations, according to which the flow of energy transmitted by a wave through an area dS normal to the direction of its propagation, is equal to $q|u|^2 dS$, where $|u|$ is the modulus of the amplitude of the wave and q is a constant depending on the choice of the unit.

For $k'' = 0$, we found the solution (67) such that

$$|u|^2 = \frac{C^2}{r^2(\beta^2 + \alpha^2 k^2)}.$$

For this value of $|u|^2$, the energy flow through an arbitrary spherical surface r = constant, of constant phase of the wave is equal to

$$4\pi \frac{qC^2}{\beta^2 + \alpha^2 k^2};$$

that is, it does not depend on r. In other words, the energy of the wave is conserved as it is propagated. In contrast, for $k'' \neq 0$, we obtain from the expression (69)

$$|u|^2 = \frac{|A|^2}{r^2} e^{-2k'' r}.$$

If k'' is positive, the value of $|u|^2$ will decrease exponentially as r increases, that is, much faster than the area $4\pi r^2$ of the spherical surface of constant phase increases. Consequently, the energy of the wave will be dispersed exponentially. However, if k'' is negative the energy of the wave will increase exponentially as it gets farther from the region of its formation. This process has no physical meaning for an infinite space and therefore must be excluded from our examination.

Thus, the quantity k'' characterizes the dispersion of energy in a medium that transmits a wave process and is called the complex absorption of the medium.

It is useful to show the connection between the sign of the imaginary part of the parameter k^2 and the sign of the complex absorption. Let us represent k^2 in trigonometric form:

$$k^2 = |k^2| (\cos 2\theta + i \sin 2\theta) \qquad (-\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi).$$

The two values of the root of this expression are

$$\pm\sqrt{|k^2|} (\cos\theta + i \sin\theta).$$

Above, we denoted by k the root with positive real parts. Since

$$\cos\theta \geqslant 0,$$

we have

$$k = \sqrt{|k^2|} (\cos\theta + i \sin\theta).$$

For $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, the functions $\sin\theta$ and $\sin 2\theta$ have the same sign. Consequently, the sign of the complex absorption coincides with the sign of the imaginary part $\operatorname{Im} k^2$ of the parameter k^2. Therefore, in the case of actual media, $\operatorname{Im} k^2$ is non-negative.

The condition of radiation for media with positive complex absorption obviously can be defined as the condition of exponential convergence of the solution to zero at an infinite distance from the wave source. The condition of radiation can also be formulated in the form of the equations

$$\lim_{r\to\infty} e^{|k''|} r\left(\frac{\partial u}{\partial r} - iku\right) = 0 , \quad \lim_{r\to\infty} e^{|k''|} u = 0 , \tag{70}$$

which become eqs. (62) when $k'' = 0$. The verification of this formulation is left to the reader.

Let us now find an expression for the solution to problem (58) when $k'' \neq 0$ that satisfies the boundary condition of the problem. If we substitute the expression (69) into the boundary condition, we obtain the following equation for determining the constant A:

$$A\ e^{-k''r_0}\ e^{ik'r_0}\ (i\alpha k' + \beta - \alpha k'') = C . \tag{71}$$

This equation can be obtained from eq. (64) by the substitutions

$$A \to A\ e^{-k''r_0} , \quad kr_0 \to k'r_0 , \quad \alpha k \to \alpha k' , \quad \beta \to \beta - \alpha k'' .$$

Consequently, the solution to the problem (58) for the case in question can be obtained by making the analogous substitutions in the expression (67) This gives us

$$u = \frac{C}{\sqrt{(\beta - \alpha k'')^2 + \alpha^2 k'^2}}\ e^{-k''r_0}\ \frac{e^{ik'(r-r_0-r_k)}}{r} , \tag{72}$$

where the constant r_k satisfies the relations

$$\cos k'r_k = \frac{\beta - \alpha k''}{\sqrt{(\beta - \alpha k'')^2 + \alpha^2 k'^2}} , \quad \sin k'r_k = \frac{\alpha k'}{\sqrt{(\beta - \alpha k'')^2 - \alpha^2 k'^2}} .$$

The expression $(\beta - \alpha k'')^2 + \alpha^2 k'^2$ in the denominator of the solution (72) is the determinant of the system of linear equations determining the complex constant A. For $k'' \neq 0$ and for arbitrary real values of α and β, this determinant does not vanish. Consequently, the problem (58) has a unique solution, and the homogeneous problem (65) corresponding to it has no solution not identically equal to zero.

If $k' = 0$ (that is, if the parameter k^2 is a negative real number), the expression (72) will be of the form

$$u = \frac{C}{\beta - \alpha k''}\ \frac{e^{-k''r}}{r} .$$

It follows from the expression (71) that in the given case, for $\beta - \alpha k'' = 0$, the problem (57) has no solution since the homogeneous problem corresponding to it has solutions of the form $A(e^{-k''r}/r)$ for an arbitrary value of the constant A. These solutions correspond to self-excitation on the boundary, discussed in section 2.

Let us briefly pause for the case of two dimensions. The solutions to the Helmholtz equation that are circularly symmetric satisfy the zero order Bessel equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + k^2u = 0 . \tag{73}$$

The general solution of this equation can be represented in the form

$$u(r) = AH_0^{(1)}(kr) + BH_0^{(2)}(kr) \ , \tag{74}$$

where A and B are arbitrary constants and $H_0^{(1)}(kr)$ and $H_0^{(2)}(kr)$ are Hankel functions of the first and second kind (see Chapter XII, section 6). This form of the solution corresponds to the solution, expressed in terms of the exponential functions e^{ikr} and e^{-ikr}, for space. In real form (for real k), the solution (74) is expressed in Bessel and Weber functions:

$$u(r) = aJ_0(kr) + bY_0(kr) \ ,$$

where a and b are real constants.

The particular solution $J_0(kr)$ (like $r^{-1} \sin kr$ in the three-dimensional case) is regular over the entire plane but the solution $Y_0(kr)$ becomes infinite, of the order of $\ln(1/r)$ at the point $r = 0$. Using the asymptotic expansions of Hankel functions (64) of Chapter XII

$$H_0^{(1)}(kr) \sim \sqrt{\frac{2}{\pi}}\ \frac{e^{i(kr-\frac{1}{4}\pi)}}{\sqrt{kr}} \ , \qquad H_0^{(2)}(kr) \sim \sqrt{\frac{2}{\pi}}\ \frac{e^{i(kr-\frac{1}{4}\pi)}}{\sqrt{kr}} \ , \tag{75}$$

we find that the solutions in question approach zero at infinity.

By using formulae (74) and (75), it is easy to construct systems of solutions that correspond to the divergent and convergent waves. Then, the system of waves that diverge at infinity can, for a real value of the parameter k, be singled out by means of the radiation condition:

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0 \ , \qquad \lim_{r\to\infty} u = 0 \ . \tag{76}$$

It is suggested that the reader derive the radiation condition on a plane for an arbitrary complex value of the parameter k.

Problems

1. Assume that the coefficient β in the boundary condition of the problem (58) can have complex values. Show that a condition of self-excitation on the boundary will be attained for $\beta = (k'' \pm ik')\alpha$.

2. Show that if we take the time factor not in the form $e^{-i\omega t}$ but in the form $e^{i\omega t}$, the minus sign in front of the term iku must be changed to plus in the radiation conditions (62) and (76).

3. Show that the Hankel function $H_0^{(1)}(kr)$ of the first kind satisfies the radiation condition (76) and that the Hankel function $H_0^{(2)}(kr)$ of the second kind does not satisfy it.

6. *Integral formulae*

We saw in sections 2 and 5 that the function

$$e^{ikr}/r \ , \tag{77}$$

where r is the radius vector of a spherical system of coordinates, is a solution to the Helmholtz equation for $r \neq 0$. In the expression (77), the letter r obviously may be understood as the distance from a variable point ξ to an arbitrary fixed point x. The expression (77) is then a solution to the Helmholtz equation $\Delta_\xi u + k^2 u = 0$, stated in arbitrary coordinates, for all $\xi \neq x$. Because of the symmetry of the expression (77) with respect to the coordinates of the point ξ and x, it will also satisfy the equation $\Delta_x u + k^2 u = 0$, in which the differentiation is with respect to the coordinates of the point x.

Consider the function

$$L(\xi, x) \equiv \frac{1}{4\pi}\left[\frac{e^{ikr}}{r} + \varphi(\xi, x)\right], \tag{78}$$

where $\varphi(\xi, x)$ is a solution to the Helmholtz equation $\Delta_\xi \varphi + k^2\varphi = 0$, which is regular in some region V. We shall call the function $L(\xi, x)$ the *fundamental solution* of this equation in the region V.

If $\xi = x$, the fundamental solutions of the Helmholtz and Laplace equations approach infinity as functions of the same order. Therefore, the fundamental solutions of the Helmholtz equation satisfy a number of the same integral relationships as do those of the Laplace equation. We shall give these relationships without proof. The proofs are almost word for word the same as those in Chapters XVIII and XIX. By using Green's theorem (7) of Chapter XVII and passing to the limit, we obtain the relationship

$$\iint_{\mathcal{F}V}\left(L\frac{du}{dn} - u\frac{dL}{dn}\right) dS_\xi = \begin{cases} 0 & \text{when } x \in R_E - V, \\ u(x) & \text{when } x \in V - \mathcal{F}V, \end{cases} \tag{79}$$

where R_E is 3-dimensional space, V is a bounded region, and $u(x)$ is a regular solution of the Helmholtz equation that, along with its first derivative, is continuous in the region V. This formula is analogous to formula (44) of Chapter XVIII.

Let us suppose now that the function $u(x)$ is a regular solution of the Helmholtz equation in an *infinite* region V and that it satisfies the radiation condition (62) at infinity. Suppose that V_σ is a finite subregion of the region V and that it is contained within some sphere σ. Let us apply formula (79) to the region V_σ, making the substitution $L(\xi, x) = (1/4\pi)(e^{ikr}/r)$. This gives us

$$\frac{1}{4\pi}\iint_{\mathcal{F}V}\left[\frac{e^{ikr}}{r}\frac{du}{dn} - u\frac{d}{dn}\left(\frac{e^{ikr}}{r}\right)\right] dS + \frac{1}{4\pi}\iint_{\mathcal{F}\sigma}\left[\frac{e^{ikr}}{r}\frac{du}{dn} - u\frac{d}{dn}\left(\frac{e^{ikr}}{r}\right)\right] dS$$

$$= \begin{cases} 0 & \text{when } x \in R_E - V_\sigma, \\ u(x) & \text{when } x \in V_\sigma - \mathcal{F}V_\sigma. \end{cases} \tag{80}$$

The integral over $\mathcal{F}\sigma$ can be written in the form of the sum

$$\iint_{\mathcal{F}\sigma} u\frac{e^{ikr}}{r^2} dS + \iint_{\mathcal{F}\sigma}\frac{e^{ikr}}{r}\left(\frac{\partial u}{\partial r} - iku\right) dS.$$

Since the function u satisfies the radiation condition, as the radius of the surface $\mathcal{F}\sigma$ increases without bound, both terms of this sum approach zero.

Let us consider, for example, the second term. Remembering that, because of the radiation condition, the imaginary part of k is non-negative, we see that

$$\left|\iint_{\mathcal{F}\sigma} \frac{e^{ikr}}{r}\left(\frac{\partial u}{\partial r} - iku\right) dS\right| \leq \left| r\left(\frac{\partial u}{\partial r} - iku\right)\right|_H \iint_{\mathcal{F}\sigma} \frac{dS}{r^2},$$

where

$$\left| r\left(\frac{\partial u}{\partial r} - iku\right)\right|_H$$

is the greatest absolute value of

$$r\left(\frac{\partial u}{\partial r} - iku\right)$$

on $\mathcal{F}\sigma$. Since the integral

$$\iint_{\mathcal{F}\sigma} \frac{dS}{r^2}$$

is bounded, when the radius of the surface $\mathcal{F}\sigma$ increases without bound, the right side of this inequality approaches zero in accordance with the condition of radiation. It can be proved in an analogous way that the first term of the sum approaches zero. Thus, if we take the limit in the relationship (80) as the radius of the surface $\mathcal{F}\sigma$ increases without bound, we obtain

$$\frac{1}{4\pi}\iint_{\mathcal{F}V}\left[\frac{e^{ikr}}{r}\frac{du}{dn} - u\frac{d}{dn}\left(\frac{e^{ikr}}{r}\right)\right] dS = \begin{cases} 0 & \text{when } x \in R_E - V, \\ u(x) & \text{when } x \in V - \mathcal{F}V. \end{cases} \tag{81}$$

By using this formula, let us show that the regular solutions of the Helmholtz equation that satisfy the radiation condition decrease at least as fast as $1/r$ as r approaches infinity. We take for $\mathcal{F}V$ a spherical surface with center at an arbitrarily chosen point ζ. We choose the radius of the surface $\mathcal{F}V$ sufficiently large that in an infinite region V with surface $\mathcal{F}V$ as its boundary, the function u is everywhere defined and is a regular solution of the Helmholtz equation. Suppose that R is the distance between the points ζ and x, that r_1 is the distance between the point ζ and a variable point ξ on the surface $\mathcal{F}V$, and that ψ is the angle between the segment $\overline{\zeta x}$ and $\overline{\zeta\xi}$. Then, as R becomes infinitely great,

$$r \equiv \sqrt{R^2 + r_1{}^2 - 2Rr_1\cos\psi} = R\sqrt{1 + \frac{r_1{}^2}{R^2} - 2\frac{r_1}{R}\cos\psi} = R[1 + o(1)],$$

where o(1) denotes a set of infinitesimal terms. Using this expression, let us write formula (81) in the form

$$u(x) = \frac{e^{ikr}}{R}\left[\iint_{\mathcal{F}V}\left(\exp[ikr_1\cos\psi]\frac{du}{dn} - u\frac{d}{dn}\exp[ikr_1\cos\psi]\right) dS\right][1 + o(1)]$$

$$(x \in V - \mathcal{F}V).$$

If we note that the integral on the right hand side, which is taken over the

bounded surface $\mathcal{F}V$, is known to be bounded and that the quantities R and r are of the same order, we obtain the following evaluation, valid for large values of r:

$$u(x) = e^{-k''r}\, O(1/r)\ , \tag{82}$$

where $O(1/r)$ is a factor of the same order as $1/r$ and k'' is the imaginary part of k. This formula proves the assertion made.

Now, formula (81) can be generalized to the case of an arbitrary fundamental solution $L(\xi, x)$. For in the derivation of this formula, we used the fact that the fundamental solution $r^{-1}\, e^{ikr}$ decreases as $1/r$ at infinity. But, from what has been said, an arbitrary fundamental solution must have this property. Therefore, for an infinite region, we obtain a formula coinciding with formula (79).

Finally, let us suppose that u is a solution to the Helmholtz equation that is regular throughout all space and that satisfies the radiation condition at infinity. Suppose that x is an arbitrary point and that σ is a sphere of radius r with center at the point x. Let us use formula (79) to express the value of the function u at the point x in terms of its values on the surface of the sphere, $d/dn = \partial/\partial r$, we obtain

$$u(x) = \frac{1}{4\pi} \iint_{\mathcal{F}\sigma} \left[\frac{e^{ikr}}{r}\,\frac{\partial u}{\partial r} - u\,\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\right)\right] dS = \frac{1}{4\pi} \iint_{\mathcal{F}\sigma} \frac{e^{ikr}}{r^2}\left[u + r\left(\frac{\partial u}{\partial r} - iku\right)\right] dS\ .$$

As r increases without bound, the integral on the right hand side approaches zero and since the value of $u(x)$ does not depend on r, it must be identically equal to zero.

It follows from the fact that the point x is chosen arbitrarily that the solution to the Helmholtz equation which is regular throughout all space and which satisfies the radiation condition is identically equal to zero. In physical terms, this means that when there are no sources of radiation (points at which the solution is not regular), a steady-state system of waves that diverge at infinity is impossible.

By analogy with Newtonian potentials, let us define the functions

$$v(x) \equiv \iiint_V \rho\, \frac{e^{ikr}}{r}\, dV\ , \tag{83}$$

$$\bar{v}(x) \equiv \iint_S \bar{\rho}\, \frac{e^{ikr}}{r}\, dS\ , \tag{84}$$

$$\bar{\bar{v}}(x) \equiv \iint_S \bar{\bar{\rho}}\, \frac{d}{dn}\left(\frac{e^{ikr}}{r}\right) dS\ , \tag{85}$$

which we shall call the vibrational potentials of a volume, of a single layer, and of a double layer, respectively. Since the factors e^{ikr} do not change the conditions of convergence of the integrals at the point $r = 0$, all the analytic properties of Newtonian potentials are carried over to the vibrational potentials. Let us enumarete the basic properties of these.

A volume vibrational potential (83) with a density which is integrable

and whose integral is bounded is continuous throughout all space. Its first derivatives can be computed by differentiation under the integral sign, and if the density ρ is uniformly bounded, the first derivative will be continuous throughout all space. At points at which the density ρ is differentiable, the second derivatives of the potential exist and satisfy the inhomogeneous Helmholtz equation

$$\Delta \bar{v} + k^2 v = -4\pi\rho \ .$$

The vibrational potential of a single layer (84) is continuous throughout all space. At points not belonging to the layer, it is infinitely many times differentiable and it satisfies the homogeneous Helmholtz equation $\Delta \bar{v} + k^2 \bar{v} = 0$; at points ζ of the layer, its normal derivatives satisfy the equation

$$\frac{d\bar{v}}{dn_0} = \frac{d\bar{v}}{dn_0} + 2\pi\rho(\zeta) \ , \qquad \frac{d\bar{v}}{dn_i} = \frac{d\bar{v}}{dn_0} - 2\pi\rho(\zeta) \ , \tag{86}$$

where $d\bar{v}/dn_e$ and $d\bar{v}/dn_i$ are the limiting values of the normal derivatives as the point ζ is approached from without and within the surface S, respectively, and

$$\frac{d\bar{v}}{dn_0} = \left[\iint_S \rho \frac{d}{dn_x}\left(\frac{e^{ikr}}{r}\right) dS\right]_{x=\zeta}$$

which we call the direct value of the normal derivatives at the point ζ of the layer.

The vibrational potential of a double layer (85) at points outside the layer is infinitely many times differentiable and satisfies the Helmholtz equation $\Delta \bar{\bar{v}} + k^2 \bar{\bar{v}} = 0$, and at points ζ of the layer it satisfies the equations

$$\bar{\bar{v}}_e(\zeta) = \bar{\bar{v}}_0(\zeta) - 2\pi\rho(\zeta) \ , \qquad \bar{\bar{v}}_i(\zeta) = \bar{\bar{v}}_0(\zeta) + 2\pi\rho(\zeta) \ , \tag{87}$$

where $\bar{\bar{v}}_e(\zeta)$ and $\bar{\bar{v}}_i(\zeta)$ are the limiting values of the potential as the point ζ is approached from without and within S, respectively, and $\bar{\bar{v}}_0(\zeta)$ is the direct value of the potential at the point ζ.

The boundary-value problems for the Helmholtz equation, like those for the Laplace and Poisson equations, admit the construction of Green's functions, by means of which the solution of the problem can be written in integral form. These Green's functions can describe fields arising from point sources.

For example, let us consider the relationship determining the Green's function of Dirichlet's problem:

$$\Delta u + k^2 u = 0 \quad \text{when} \quad x \in V - \mathcal{F}V \ , \qquad u = \psi \quad \text{when} \quad x \in \mathcal{F}V \tag{88}$$

We shall define the function $\varphi(\xi, x)$ that appears in the general expression for the fundamental solution (78) as a solution of the boundary problem

$$\Delta\varphi + k^2\varphi = 0 \quad \text{when} \quad x, \xi \in V - \mathcal{F}V \ ,$$

$$\varphi = -\,e^{ikr}/r \quad \text{when} \quad \xi \in \mathcal{F}V \ , \ x \in V - \mathcal{F}V \ .$$

In this case, the fundamental solution (78) vanishes on the boundary of the

region in question and formula (79), in view of the boundary condition of the problem (88), gives

$$u(x) = -\iint_{\mathcal{F}V} \psi \frac{dG}{dn} dS_\xi , \tag{89}$$

where

$$G = G(\xi, x) \equiv \frac{1}{4\pi}\left[\frac{e^{ikr}}{r} + \varphi(\xi, x)\right].$$

The fundamental solution $G(\xi, x)$ represents Green's function of the problem (88). It can be shown that Green's function $G(\xi, x)$ exists if the solution to the problem (88) is unique. If it is not, we can still construct a function $G(\xi, x)$, known as the generalized Green's function, such that formula (89) will hold. We shall speak in greater detail of this in Chapter XXVII.

Problems

1. Show by direct differentiation that the function $r^{-1} e^{ikr}$, where

$$r = \sqrt{\sum_{\alpha=1}^{3} (x_\alpha - \xi_\alpha)^2}$$

satisfies the Helmholtz equation.

2. Show that, in contrast with harmonic functions, the derivatives of the regular solutions of the Helmholtz equation need not satisfy the condition

$$\lim_{r\to\infty} r^2 \left|\frac{\partial u}{\partial x_i}\right| < \infty \qquad (i = 1, 2, 3) .$$

3. Show that the boundary-value problem of the non-homogeneous equation $\Delta u + k^2 u = f$ can be reduced to the boundary-value problem for the homogeneous equation $\Delta u_1 + k^2 u_1 = 0$.
Method: Set $u = u_1 + v$, where v is the volume vibrational potential with density $\rho = -(1/4\pi) f$.

4. Show that Green's function of the boundary-value problem (88) is symmetric with respect to both its arguments, that is, that $G(\xi, x) = G(x, \xi)$.

5. Consider Dirichlet's boundary-value problem $\Delta u + k^2 u = 0$ when x belongs to $V - \mathcal{F}V$ and $u = \psi$ when x belongs to $\mathcal{F}V$. Assume that the solution to this boundary-value problem can be represented in the form of the vibrational potential of a double layer:

$$u = \iint_{\mathcal{F}V} \bar{\bar{\rho}} \frac{d}{dn}\left(\frac{e^{ikr}}{r}\right) dS .$$

Show that the density $\bar{\bar{\rho}}$ satisfies the Fredholm integral equation of the second kind:

$$2\pi\bar{\bar{\rho}} + \iint_{\mathcal{F}V} \bar{\bar{\rho}} \frac{d}{dn}\left(\frac{e^{ikr}}{r}\right) dS = \psi .$$

Method: Use formula (79).

6. Suppose that $G(\xi, x)$ is Green's function of the Laplacian operator (that is, that $\Delta_\xi G(\xi, x) = 0$ when $\xi \neq x$) that satisfies the homogeneous boundary condition

$$\alpha \frac{dG}{dn_\xi} + \beta G = 0 \quad \text{when} \quad \xi \in \mathcal{F}V , \ x \in V - \mathcal{F}V .$$

Show that the solution u of the homogeneous boundary-value problem for the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{when} \quad x \in V - \mathcal{F}V , \qquad \alpha \frac{du}{dn} + \beta u = 0 \quad \text{when} \quad x \in \mathcal{F}V ,$$

satisfies the homogeneous Fredholm integral equation of the second kind:

$$u - k^2 \iiint_V Gu \, dV_\xi = 0 .$$

Method: Write the Helmholtz equation in the form $\Delta u = -k^2 u$ and use formula (67) of Chapter XVIII.

7. Assume that u is a regular solution of the Helmholtz equation in a bounded region V and that it has continuous first derivatives in that region. Derive the formula

$$\iiint_V \left[\sum_{\alpha=1}^{3} \left(\frac{\partial u}{\partial x_\alpha}\right)^2 - k^2 u^2\right] dV = \iint_{\mathcal{F}V} u \frac{du}{dn} dS .$$

Then use this formula to show the uniqueness of the solutions of the Dirichlet and Neumann problems for the Helmholtz equation with negative k^2 in the class of functions that are continuous and that have continuous first derivatives in the region V.
Method: Make the substitutions of u^2 for u and 1 for v in Green's theorem (7) of Chapter XVII.

7. *Series expansions in particular solutions of the Helmholtz equation in an infinite region*

We introduce the spherical coordinates r, θ, and φ. Suppose that $u(r, \theta, \varphi)$ is a solution to the Helmholtz equation which is regular for $r \geqslant r_0$ and which satisfies the radiation condition. Within the region of regularity, the function u can be expanded in a series of spherical functions (Chapter XXI, section 3):

$$u = \sum_{\alpha=0}^{\infty} \left(a_{\alpha 0}(r) P_\alpha(\cos\theta) + \sum_{\beta=1}^{\alpha} [a_{\alpha\beta}(r)\cos\beta\varphi + b_{\alpha\beta}(r)\sin\beta\varphi] P_{\alpha\beta}(\cos\theta)\right) . \quad (90)$$

Let us show that, up to a factor that is independent of the coordinate, the coefficients $a_{\alpha 0}$, $a_{\alpha\beta}$ and $b_{\alpha\beta}$ in this series are, for every fixed $\alpha = n$, equal to the functions

$$h_n(kr) = \sqrt{\pi/2kr}\, H_{n+\frac{1}{2}}{}^{(1)}(kr) \qquad (n = 0, 1, 2, \ldots,\ 0 \leqslant \beta \leqslant n)\ , \tag{91}$$

where $H_{n+\frac{1}{2}}{}^{(1)}(kr)$ represents a Hankel function of the first kind and of half-integral order.

We use the particular solution (56) of the Helmholtz equation in spherical coordinates and we set $Z_{n+\frac{1}{2}}(kr) = H_{n+\frac{1}{2}}{}^{(1)}(kr)$. With the notation (91), it takes the form

$$u_{nm} = h_n(kr)\, P_{nm}(\cos\theta)\, \cos(m\varphi + \psi_m) \qquad (0 \leqslant m \leqslant n)\ , \tag{92}$$

where we set $P_{n0}(\cos\theta) \equiv P_n(\cos\theta)$. It is easy to show that the function u_{nm} is a solution to the Helmholtz equation in spherical coordinates that is regular in an arbitrary region not containing the point $r = 0$ and that satisfies the radiation condition.

Suppose that Σ_0 and Σ are two spherical surfaces with radii r_0 and $r > r_0$ with center at the point $r = 0$. Let us apply Green's theorem (7) of Chapter XVII to the functions u and u_{nm} in the region V between the surfaces Σ_0 and Σ. Since the functions u and u_{nm} by hypothesis satisfy the Helmholtz equation, we have

$$\iiint_V (u\, \Delta u_{nm} - u_{nm}\, \Delta u)\, dV = 0\ ,$$

and therefore,

$$\iint_{\Sigma_0} \left(u \frac{du_{nm}}{dn} - u_{nm} \frac{du}{dn}\right) dS = \iint_{\Sigma} \left(u_{nm} \frac{\partial u}{\partial r} - u \frac{\partial u_{nm}}{\partial r}\right) dS\ . \tag{93}$$

For large values of r, it follows from the radiation condition and from the approximation (82) that

$$\partial u/\partial r = iku + O(1/r^2)\ , \qquad \partial u_{nm}/\partial r = iku_{nm} + O(1/r^2)\ ,$$
$$u = O(1/r)\ , \qquad u_{nm} = O(1/r)\ ,$$

from which

$$u_{nm} \frac{\partial u}{\partial r} - u \frac{\partial u_{nm}}{\partial r} = O(1/r^3)\ .$$

Therefore, the integral on the right hand side of eq. (93) approaches zero as r increases without bound and since the integral on the left hand side of this equation is independent of r, we have

$$\iint_{\Sigma_0} \left(u_{nm} \frac{\partial u}{\partial r} - u \frac{\partial u_{nm}}{\partial r}\right) dS \equiv 0\ .$$

Let us substitute in this equation the expressions for the functions u and u_{nm} given by eqs. (90) and (92). Recalling the orthogonality relationship for spherical functions (Chapter XXI, section 2), we obtain, after integrating over Σ,

$$h_n(kr)\,\frac{\partial}{\partial r}\,[C_{nm}a_{nm}(r) + D_{nm}b_{nm}(r)] = [C_{nm}a_{nm}(r) + D_{nm}b_{nm}(r)]\,\frac{\partial}{\partial r}\,h_n(kr)$$

$$(0 \leqslant m \leqslant n)\ ,$$

or, because of the arbitrariness of the values of C_{nm} and D_{nm},

$$h_n(kr)\,\frac{\partial a_{nm}(r)}{\partial r} = a_{nm}\,\frac{\partial h_n(kr)}{\partial r}\ , \qquad h_n(kr)\,\frac{\partial b_{nm}(r)}{\partial r} = b_{nm}(r)\,\frac{\partial h_n(kr)}{\partial r}.$$

It then follows that $h_n(kr)$, $a_{nm}(r)$, and $b_{nm}(r)$ differ from each other only by a factor independent of r since otherwise the above equations could not hold. Thus, our assertion is proven.

If we set

$$a_{nm}(r) = A_{nm}h_n(kr)\ , \qquad b_{nm}(r) = B_{nm}h_n(kr)\ ,$$

where A_{nm} and B_{nm} are independent of the coordinates and if we substitute these expressions into the series (90), we obtain

$$u = \sum_{\alpha=0}^{\infty} h_\alpha(kr) \sum_{\beta=0}^{\alpha} P_{\alpha\beta}(\cos\theta)\,[A_{\alpha\beta}\cos\beta\varphi + B_{\alpha\beta}\sin\beta\varphi]\ . \tag{94}$$

This proves the possibility of expanding a solution to the Helmholtz equation in a series of particular solutions of the form (56).

If we multiply the series (94) by the complex conjugate series and if we take into consideration the orthogonality relationship for spherical functions (Chapter XXI, section 3), we can compute the integral of this product over the surface $\mathcal{F}\sigma$ of some sphere σ. The result is

$$\iint\limits_{\mathcal{F}\sigma} |u|^2\,\mathrm{d}S = \sum_{\alpha=0}^{\infty} \frac{4\pi}{2\alpha+1}\,r^2|h_\alpha(kr)|^2$$

$$\times\left\{|A_{\alpha 0}|^2 + \tfrac{1}{2}\sum_{\beta=1}^{\alpha}\frac{(\alpha+\beta)!}{(\alpha-\beta)!}\,(|A_{\alpha\beta}|^2 + |B_{\alpha\beta}|^2)\right\}. \tag{95}$$

The following assertion, known as the fundamental lemma of the theory of the Helmholtz equation, follows from this relationship: a solution to the Helmholtz equation that is regular in an infinite region, that has k real, and that satisfies at infinity both the radiation condition and the condition

$$\lim_{r\to\infty}\iint\limits_{\mathcal{F}\sigma} |u|^2\,\mathrm{d}S = 0\ , \tag{96}$$

where σ is a sphere of radius r, is identically equal to zero. For, if k is real, the expression

$$r^2|h_n(kr)|^2 = \frac{\pi}{2k}\,r\left|H_{n+\frac{1}{2}}^{(1)}(kr)\right|^2 > 0$$

does not approach zero as r increases without bound. Therefore, eq. (95) and the condition (96) can both be satisfied only when

$$A_{\alpha\beta} = B_{\alpha\beta} = 0\ , \qquad 0 \leqslant \alpha \leqslant \infty\ , \qquad 0 \leqslant \beta \leqslant \alpha\ .$$

But since all the coefficients of the series (94) are equal to zero, u must also be equal to zero. This proves the lemma.

8*. *Questions concerning the uniqueness of solutions to the external boundary-value problems for the Helmholtz equation*

We have touched on the problem of the uniqueness of the solution to the internal boundary-value problem for the Helmholtz equation and have formulated a pair of alternatives (see section 2) according to which the solutions of the internal problem are always unique up to additive factors corresponding to the free vibrations.

Let us turn now to the external boundary value problem. We first prove the following uniqueness theorem:

THEOREM. *The solution of the external boundary-value problems of Dirichlet and Neumann for the Helmholtz equation*

$$\Delta u + k^2 u = 0 \ , \qquad k = k' + \mathrm{i}k'' \ ,$$

in the class of regular functions that satisfy the radiation condition

$$\lim_{r\to\infty} \mathrm{e}^{|k''|r} r\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) = 0 \ , \qquad \lim_{r\to\infty} u = 0 \ , \tag{97}$$

is unique.

To prove this theorem, it is sufficient to show that the solutions of the external homogeneous problems of Dirichlet and Neumann that satisfy the radiation condition are identically equal to zero.

Suppose that V is an infinite region in which we are seeking a solution to the problem, that $\mathcal{F}V$ is its boundary, and that Σ is a spherical surface of radius r containing the surface $\mathcal{F}V$ inside it.

Let us consider the positive-definite form

$$\mathcal{E} = \tfrac{1}{2}\left[\sum_{\alpha=1}^{3} \left(\frac{\partial v}{\partial x_\alpha}\right)^2 + \frac{1}{c^2}\left(\frac{\partial v}{\partial t}\right)^2\right] , \tag{98}$$

where

$$v = \mathrm{Re}\ \mathrm{e}^{-\mathrm{i}ckt}\, u \ . \tag{99}$$

We note that the function v satisfies the wave equation

$$\Delta v = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} \ . \tag{100}$$

This can be seen to follow also from the fact that the function u satisfies the Helmholtz equation, so that

$$\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} = \mathrm{Re}\ (-\,k^2\, \mathrm{e}^{-\mathrm{i}ckt}\, u) = \mathrm{Re}\ \mathrm{e}^{-\mathrm{i}ckt}\, \Delta u = \Delta\ \mathrm{Re}\ \mathrm{e}^{-\mathrm{i}ckt}\, u = \Delta v \ ,$$

If we differentiate $\mathcal{E}$ with respect to t, we get

$$\frac{\partial \mathcal{E}}{\partial t} = \sum_{\alpha=1}^{3} \frac{\partial v}{\partial x_\alpha} \frac{\partial^2 v}{\partial x_\alpha \partial t} + \frac{1}{c^2} \frac{\partial v}{\partial t} \frac{\partial^2 v}{\partial t^2} .$$

Let us substitute in the term on the right the expression for $(1/c^2)(\partial^2 v/\partial t^2)$ given by eq. (100). After certain uncomplicated transformations, we obtain

$$\frac{\partial \mathcal{E}}{\partial t} = \sum_{\alpha=1}^{3} \frac{\partial}{\partial x_\alpha} \frac{\partial v}{\partial t} \frac{\partial v}{\partial x_\alpha} .$$

If we now apply Gauss's theorem to $\partial\mathcal{E}/\partial t$ in the region V^* between the surfaces $\mathcal{F}V$ and Σ, we obtain

$$\frac{\partial}{\partial t} \iiint_{V^*} \mathcal{E} \, \mathrm{d}V = \iint_{\mathcal{F}V} T_\mathrm{n} \, \mathrm{d}S + \iint_{\Sigma} T_\mathrm{n} \, \mathrm{d}S , \tag{101}$$

where T_n is the component of the vector $\boldsymbol{T}$ that is normal to the boundaries $\mathcal{F}V$ and Σ, the components of $\boldsymbol{T}$ along the *axes* being determined by

$$T_j \equiv \frac{\partial v}{\partial t} \frac{\partial v}{\partial x_j} \quad (j = 1, 2, 3) .$$

It is clear from the expressions for T_j that

$$T_\mathrm{n} = \frac{\partial v}{\partial t} \frac{\mathrm{d}v}{\mathrm{d}n} .$$

Let us use eq. (99) to express v in terms of u. Using the notation

$$k = k' + \mathrm{i}k'' , \qquad u = u' + \mathrm{i}u'' , \tag{102}$$

where k' and k'' on the one hand and u' and u'' on the other are the real and imaginary parts of k and u, respectively, we then rewrite eq. (99) in the form

$$v = \mathrm{e}^{ck''t} (u' \cos ck't + u'' \sin ck't) . \tag{103}$$

Hence

$$T_\mathrm{n} \equiv \frac{\partial v}{\partial t} \frac{\mathrm{d}v}{\mathrm{d}n} = c \, \mathrm{e}^{2ck''t} \left(\frac{\mathrm{d}u'}{\mathrm{d}n} \cos ck't + \frac{\mathrm{d}u''}{\mathrm{d}n} \sin ck't\right)$$
$$\times \left[(k''u' + k'u'') \cos ck't + (k''u'' - k'u') \sin ck't\right] . \tag{104}$$

The normal component of T, T_n, is zero on $\mathcal{F}V$ for either of the boundary conditions $u = 0$ or $\mathrm{d}u/\mathrm{d}n = 0$, x belongs to $\mathcal{F}V$, so that the integral over $\mathcal{F}V$ in formula (101) is equal to zero.

To evaluate the integral over Σ, we use the radiation condition (70) and the evaluation (82):

$$\text{as } r \to \infty \qquad u = \mathrm{e}^{-|k''| r} \, \mathrm{O}(1/r) , \tag{105}$$

which, in the notations (102), we represent in the form

$$u' = e^{-|k''|r}\,O(1/r)\,, \qquad u'' = e^{-|k''|r}\,O(1/r)\,,$$

$$-\frac{\partial u'}{\partial r} = k''u' + k'u'' + o\Big(\frac{e^{-|k''|r}}{r}\Big)\,, \qquad -\frac{\partial u''}{\partial r} = k''u'' - k'u' + o\Big(\frac{e^{-|k''|r}}{r}\Big)\,.$$

Here, the symbol $o(\zeta)$ denotes the set of terms of higher order of smallness than ζ. Substituting these evaluations into formula (104) and remembering that $d/dn = \partial/\partial r$, on Σ, we obtain

$$T_n = -c\,e^{2ck''t}\,[(k''u' + k'u'')\cos ck't + (k''u'' - k'u')\sin ck't]^2 + o\Big(\frac{e^{-2|k''|r}}{r^2}\Big)\,,$$

from which it is clear that

$$\lim_{r\to\infty}\frac{\partial}{\partial t}\iiint_V \mathcal{E}\,dV = -c\,e^{2ck''t}\lim_{r\to\infty}\iint_\Sigma [(k''u' + k'u'')\cos ck't + (k''u'' - k'u')\sin ck't]^2\,dS\,. \tag{106}$$

If $k'' \neq 0$, the integrand of the integral over Σ represents an infinitesimal quantity in comparison with $1/r^2$. Therefore, this integral approaches zero as r increases without bound, with the result that

$$\frac{\partial}{\partial t}\iiint_V \mathcal{E}\,dV = 0\,, \qquad \iiint_V \mathcal{E}\,dV = \text{constant}\,. \tag{107}$$

We now substitute in these equations the value of $\mathcal{E}$ given by eq. (98), keeping formula (103) in mind and setting $t = 0$ once and $t = t$ once. We then conclude, on the basis of the relationship (107), that, for all t,

$$\begin{aligned}\iiint_V \Big[\sum_{\alpha=1}^{3}\Big(\frac{\partial u'}{\partial x_\alpha}\Big)^2 + c^2(k''u' + k'u'')^2\Big]\,dV \\ = e^{2ck''t}\cos ck't\iiint_V \Big[\sum_{\alpha=1}^{3}\Big(\frac{\partial u'}{\partial x_\alpha}\Big)^2 + c^2(k''u' + k'u'')^2\Big]\,dV \\ + e^{2ck''t}\sin ck't\iiint_V \Big[\sum_{\alpha=1}^{3}\Big(\frac{\partial u''}{\partial x_\alpha}\Big)^2 + c^2(k''u'' - k'u')^2\Big]\,dV \\ + 2e^{2ck''t}\cos ck't\sin ck't\iiint_V \Big[\sum_{\alpha=1}^{3}\frac{\partial u'}{\partial x_\alpha}\frac{\partial u''}{\partial x_\alpha} \\ + c^2(k''u' + k'u'')(k''u'' - k'u')\Big]\,dV\,.\end{aligned}$$

This is possible only if all the integrals appearing here are equal to zero. Since the integrands of the integrals that are coefficients of $\cos ck't$ and $\sin ck't$ are non-negative, it follows that, in the region V, $k''u' + k'u'' = 0$ and $k''u'' - k'u' = 0$, which implies that in this region $u' = u'' = 0$ and u is identically equal to zero. Thus, the theorem is proven for $k'' \neq 0$.

If $k'' = 0$, the relations (98) and (103) imply that

$$\iiint_V \mathcal{E}\, dV$$

is a periodic function of time and hence that its derivative with respect to t changes sign infinitely many times. But the right side of eq. (106) has a definite sign. Therefore, eq. (106) is possible only if both sides are equal to zero; hence,

$$\lim_{r\to\infty} \iint_\Sigma (u'' \cos ck't + u' \sin ck't)^2\, dS = 0 \ .$$

For arbitrary t, this is possible only if

$$\lim_{r\to\infty} \iint_\Sigma u'^2\, dS = \lim_{r\to\infty} \iint_\Sigma u''^2\, dS = 0 \ ,$$

from which it follows that

$$\lim_{r\to\infty} \iint_\Sigma |u|^2\, dS = 0 \ .$$

It then follows on the basis of the fundamental lemma of the theory of the Helmholtz equation (see section 7) that u is identically equal to zero in the region V. This completes the proof of the theorem.

It is easy to show that the theorem can also be proven if, for $k'' \neq 0$, the radiation condition is taken not in the form (70) but in the form

$$\lim_{r\to\infty} u = 0 \ , \qquad k'' > 0 \ ,$$

since the vanishing of the integral

$$\iint_\Sigma T_n\, dS$$

as r increases without bound is ensured by the exponential decrease, shown by (82), of the functions u' and u''.

The reader will find it interesting to attempt to extend this proof to the mixed external problem.

Problems

Show that the considerations which, for $k'' \neq 0$, lead to the proof of the uniqueness theorem after eq. (93) is established do not lead to this result when $k'' = 0$.

Chapter XXVI

THE EMISSION AND SCATTERING OF SOUND

1. *The fundamental relationships for sound fields*

In this chapter, we shall examine a number of problems relating to the emission of sound waves, that is, waves representing *small* oscillations of some elastic medium. In connection with this, we recall the basic relationship regarding sound waves (see Chapter VI, section 2) and we shall transform them to a form that is more suitable for our purposes.

The velocity potential of particles of a non-viscous sound transmitting medium obeys the wave equation (27) of Chapter VI, which, in the case of steady-state sound vibrations that take place with angular frequency ω, can be replaced by the Helmholtz equation

$$\Delta u + k^2 u = 0 \, , \tag{1}$$

where u is the amplitude of the oscillations of the potential and where $k^2 = \omega^2/a^2$; a is the velocity of sound in the medium.

We note that we are using for the amplitude of the potential the same notation u that we used in Chapter VI for the potential itself. The potential in our present notation is represented by the expression $u\,e^{-i\omega t}$. Henceforth, for the amplitudes of the vibrations, we shall use the notations that are used for the quantities themselves. Here, we shall dispense with the word "amplitude" and speak simply of the potential, velocity, etc., instead of the amplitude of the potential, the amplitude of the velocity, etc. This is more convenient in that we need not introduce new symbols, and thus we simplify the terminology; when used cautiously, it cannot lead to misunderstanding. Transforming from the general equations for the oscillations to the equations for steady-state oscillations will then be quite simple. When we introduce the time dependence by a factor $e^{-i\omega t}$, the transformation is reduced to the substitution

$$\partial/\partial t \to -i\omega \, . \tag{2}$$

The retention of this notation is also justified by the fact that in a steady state, the amplitude field gives a complete solution to the entire problem by determining, in particular, the fields of the vibrating quantities themselves.

According to the definition of velocity potential, the projection of the velocity of motion of the medium in the direction l is

$$v_l = \partial u/\partial l \, . \tag{3}$$

The velocity of the medium is related to the pressure p in the medium as shown in eqs. (12) of Chapter VI, which on the basis of (2) can be written in the form

$$v_l = -\frac{i}{\rho\omega}\frac{\partial p}{\partial l}, \tag{4}$$

where ρ is the density of the medium. It now follows from eqs. (3) and (4) that

$$\frac{\partial p}{\partial l} = i\rho\omega\frac{\partial u}{\partial l} = i\rho\omega v_l . \tag{5}$$

Since acoustic vibrations are small vibrations, we will not introduce any significant error in the determination of p in (5) by assuming the quantity ρ to be equal to the density ρ_0 of the undisturbed medium and, under this assumption, we can integrate it with respect to l. This gives us

$$p = i\rho_0\omega u + \text{constant} .$$

It then follows from this and eq. (1) that

$$\Delta p + k^2 p = 0 ; \tag{6}$$

that is, the pressure (the amplitude of the pressure oscillations) satisfies the Helmholtz equation.

In acoustic theory, we have the concept of the intensity I of sound. It is defined by the equation

$$I = |p|^2/2\rho a . \tag{7}$$

The intensity of a sound wave is equal to the rate at which energy transmitted by a wave flows through a unit area perpendicular to the direction of propagation of the wave.

2. *The acoustic field of a vibrating cylinder*

Let us suppose that a cylinder of radius r_0 undergoes small harmonic oscillations of amplitude b in a direction perpendicular to its axis. Let us find the steady-state acoustic field that arises in this situation.

As was shown in section 1, the field of the pressure of a sound wave satisfies the Helmholtz equation (6) so that the mathematical statement of the problems that we are considering consist now in establishing the boundary conditions. We introduce cylindrical coordinates (r, φ, z) with the z-axis directed along the axis of the cylinder at that instant at which the cylinder is in its equilibrium position. We put the plane from which the angles φ are measured into coincidence with the plane of vibration of the cylinder. If the cylinder vibrates with a velocity that does not exceed the velocity of sound in the medium, as we shall assume, no vacuum will be formed between the medium and the cylinder. Therefore, the velocity of the motion of the medium in the direction perpendicular to the surface of the cylinder coincides with the radial velocity of the motion of this surface:

$$v_r = \omega b \cos\varphi .$$

From formula (5), we then conclude that the boundary condition of the problem in question is

$$\left.\frac{\partial p}{\partial r}\right|_{r=r_0} = i\rho_0\omega^2 b \cos\varphi \ . \tag{8}$$

Also, at infinity, the radiation condition (62) of Chapter XXV must be satisfied:

$$\lim_{r\to\infty} p = 0 \ , \qquad \lim_{r\to\infty} r\left(\frac{\partial p}{\partial r} - ikp\right) = 0 \ . \tag{9}$$

To find the solution p, we use the results of section 4 of Chapter XXV, where we saw that particular solutions of the Helmholtz equation can be represented by formula (50):

$$u = AZ_n(\mu r)\cos(n\varphi + \psi_n)\cos(\nu z + \psi_\nu) \qquad (\nu \equiv \sqrt{k^2 - \mu^2}) \ .$$

One of these will be the solution to our problem.

In the case in question, there is no dependence on z. Therefore, $\nu = 0$ and $\mu = k$. Furthermore, on the basis of the boundary condition (8) the dependence on φ must be expressed only in terms of the factor $\cos\varphi$, which is possible only if $n = 1$ and $\psi_n = 0$. Finally, in order to have the radiation condition satisfied as r becomes infinitely great, we choose the Hankel function of the first kind $H_1^{(1)}(kr)$ as our solution $Z_1(kr)$ of Bessel's equation. Thus, the choice of a solution from among a number of solutions of the form (50) of Chapter XXV is unique and we obtain

$$p = AH_1^{(1)}(kz)\cos\varphi \ , \tag{10}$$

where A is a constant to be determined from the boundary condition (8). This gives us

$$A\left(\frac{dH_1^{(1)}}{dr}\right)_{r=r_0}\cos\varphi = i\rho_0\omega^2 b\cos\varphi \ ,$$

from which,

$$A = \frac{i\rho_0\omega^2 b}{(dH_1^{(1)}/dr)_{r=r_0}} \ . \tag{11}$$

If

$$kr_0 = 2\pi r_0/\lambda \ll 1 \ , \tag{12}$$

where λ is the wavelength of the oscillation of angular frequency ω, that is, if the length of the sound wave is much greater than the perimeter of the cross section of the cylinder (wire), we may compute the value of A by using the relations (18) and (61) of Chapter XII. Then, for $n = 1$ and small values of $x = kr = \omega r/a$, we obtain

$$H_1^{(1)}(kr) \approx -\frac{2i}{\pi kr} = -\frac{2ia}{\pi\omega r} \ ,$$

where a is the velocity of sound in the medium. Then,

$$\left(\frac{dH_1^{(1)}}{dr}\right)_{r=r_0} \approx \frac{2ia}{\pi\omega r_0^2}$$

and

$$A \approx \frac{\pi\omega^3 r_0{}^2 b\rho_0}{2a}. \tag{13}$$

Let us investigate the behaviour of the solution that we have obtained at large distances from the cylinder, that is, for

$$kr = 2\pi r/\lambda \gg 1 .$$

If we use the asymptotic representation (64) of Chapter XII, we obtain

$$H_1{}^{(1)}(kr) \approx \sqrt{2/\pi kr}\, \exp\left[\mathrm{i}(kr - \tfrac{3}{4}\pi)\right] .$$

Then from eq. (10),

$$p = A\sqrt{2/\pi}\, \frac{\exp\left[\mathrm{i}(kr-\frac{3}{4}\pi)\right]}{\sqrt{kr}} \cos\varphi . \tag{14}$$

If we apply formula (4) and note that $k = \omega/a$, we can find the components of the vector velocity of the medium at great distances from the cylinder. The radial component is

$$v_r = \frac{A}{\rho_0 a}\sqrt{2/\pi}\, \frac{\exp\left[\mathrm{i}(kr-\frac{3}{4}\pi)\right]}{\sqrt{kr}} \left(1 - \frac{1}{2\mathrm{i}}\,\frac{1}{kr}\right) \cos\varphi$$

and the tangential component is

$$v_\varphi = -\mathrm{i}\,\frac{A}{\rho_0\omega}\sqrt{2/\pi}\, \frac{\exp\left[\mathrm{i}(kr-\frac{3}{4}\pi)\right]}{\sqrt{kr^3}} \sin\varphi .$$

Discarding terms of higher order of smallness than $r^{-\frac{1}{2}}$, we obtain

$$v_r \approx \sqrt{2/\pi}\, \frac{A}{\rho_0\omega}\, \frac{\exp\left[\mathrm{i}(kr-\frac{3}{4}\pi)\right]}{\sqrt{kr}} \cos\varphi , \qquad v_\varphi \approx 0 , \tag{15}$$

that is, at a great distance from the cylinder, the motion of the medium is basically radial and the tangential components of the velocity decrease faster (as $\sqrt{r^{-3}}$) than do the radial components, which go as $\sqrt{r^{-1}}$.

Finally, let us determine the intensity I of the acoustic field – the quantity that is of the greatest practical interest. From formulae (7), (14), and (15), we obtain

$$I = \frac{A^2}{\pi\omega\rho_0 r} \cos^2\varphi \tag{16}$$

or, for a string,

$$I = \frac{\pi\rho_0 b^2 r_0{}^4 \omega^5}{4a^2 r} \cos^2\varphi . \tag{17}$$

Thus, the energy flux of the acoustic field of a vibrating cylinder decreases in proportion to the *first power* of the distance. It falls very rapidly with a decrease in the cross section of the cylinder or of the frequency. This last fact explains one of the reasons why the base strings of musical instruments must be thick. In a plane perpendicular to the plane of the vibrations

of the cylinder, there is no acoustic field (or, more precisely, it is determined only by a rapidly decreasing tangential component).

Problems

1. Compute the acoustic field of a pulsating infinite cylinder (that is, of a cylinder whose radius changes periodically).
2. Determine the energy emitted from a unit of area of a vibrating and of a pulsating cylinder.
   Method: Integrate the expression for I and apply the results to a unit of area.

3. *The acoustic field of a pulsating sphere. Point sources*

Consider a harmonically pulsating sphere, that is, a sphere whose radius oscillates harmonically. Suppose that r_0 is the average radius of the sphere, that b is the amplitude of the vibrations, and that ω is their angular frequency.

As we know (section 1), the pressure field outside the sphere must satisfy the Helmholtz equation (1). If the vibrations of the sphere take place with subsonic velocity, the boundary condition consists in the equality of the radial velocities of the surface of the sphere and the medium adjacent to it (for greater detail, see section 2). If we introduce spherical coordinates (r, θ, φ) with origin at the center of the sphere, on the basis of formula (5) this boundary condition can be written in the form

$$\left.\frac{\partial p}{\partial r}\right|_{r=r_0} = i\rho\omega^2 b \ . \tag{18}$$

In order to exclude waves that converge on the sphere from an infinite distance, we require that the radiation condition (62) of Chapter XXV be satisfied:

$$\lim_{r\to\infty} r\left(\frac{\partial p}{\partial r} - ikp\right) = 0 \ . \tag{19}$$

The solution of this problem is contained directly in one of the particular solutions of the Helmholtz equations in spherical coordinates (56) of Chapter XXV:

$$\frac{1}{\sqrt{r}} Z_{n+\frac{1}{2}}(kr)\ P_{nm}(\cos\theta)\ \cos(m\varphi + \psi_m) \ .$$

In order that the solutions do not depend on either θ or φ when $r = r_0$, it is obviously necessary that $m = 0$ and that $n = 0$. Also, for the radiation condition to be satisfied, we need to choose a Hankel function of the first kind as a cylindrical function. This singles out the solution

$$p = \frac{A}{\sqrt{r}} H_{\frac{1}{2}}^{(1)}(kr) \ , \tag{20}$$

where A is a constant, from among all the solutions that we are considering. The uniqueness theorem for the Helmholtz equation (Chapter XXV, section 8) assures the uniqueness of this solution. Remembering (from Chapter XII, section 6) that

$$H_{\frac{1}{2}}^{(1)}(x) = -\mathrm{i}\sqrt{2\pi/x}\ \mathrm{e}^{\mathrm{i}x} \ ,$$

we rewrite the solution obtained in the form

$$p = -\mathrm{i}A\sqrt{2\pi k}\ \frac{\mathrm{e}^{\mathrm{i}kr}}{kr} \, . \tag{21}$$

Differentiating this expression, setting $r = r_0$, and comparing the results with the expression (18), we find that

$$A = \frac{1}{\sqrt{2\pi}}\ \frac{\omega^2 r_0{}^2 b\rho\sqrt{k}}{1 - \mathrm{i}kr_0}\ \mathrm{e}^{-\mathrm{i}kr_0} \ ,$$

and consequently,

$$p = -\mathrm{i}\,\frac{\omega^2 r_0{}^2 b\rho}{(1 - \mathrm{i}kr_0)r}\ \mathrm{e}^{\mathrm{i}k(r-r_0)} \, . \tag{22}$$

The quantity $4\pi r_0{}^2 b$ represents the amplitude of the pulsation of the volume of the sphere. Therefore, the quantity

$$Q_0 = 4\pi r_0{}^2 b\omega \tag{23}$$

represents the amplitude of the volumetric velocity of pulsation. It is called the *generator* or *equivalent strength* of the spherical source of sound. By combining eqs. (22) and (23), we obtain

$$p = \frac{1}{4\pi\mathrm{i}}\ \frac{\omega\rho Q_0}{1 - \mathrm{i}kr_0}\ \frac{\mathrm{e}^{\mathrm{i}k(r-r_0)}}{r} \, . \tag{24}$$

Now, let us suppose that the radius of the sphere is negligibly small in comparison with the length λ of the emitted wave, that is, that

$$kr_0 = 2\pi\,\frac{r_0}{\lambda} \ll 1 \, .$$

In this highly important case, the pulsating sphere approximates in its properties an idealized emitter, namely, a point source the pressure field of which, as is clear from formula (24), is determined by the relationship

$$p = \frac{\omega\rho Q_0}{4\pi\mathrm{i}}\ \frac{\mathrm{e}^{\mathrm{i}kr}}{r} = \frac{Q_0}{4\pi}\ \frac{\omega^2\rho}{a}\ \frac{\mathrm{e}^{\mathrm{i}kr}}{\mathrm{i}kr} \, . \tag{25}$$

On the basis of formula (4), the velocity of the motion of the medium at great distances from a point source is

$$v = v_r = -\frac{\mathrm{i}}{\rho\omega}\,\frac{\partial p}{\partial r} = \frac{1}{4\pi}\ \frac{\omega^2 Q_0}{a^2}\ \frac{\mathrm{e}^{\mathrm{i}kr}}{\mathrm{i}kr} \, .$$

Finally, the intensity of the acoustic field of a point source is

$$I = \frac{\omega^2 \rho}{32\pi^2 a} \frac{Q_0{}^2}{r^2}. \tag{26}$$

Thus, the flow of energy in the field of a point source (and in the field of a small pulsating sphere) decreases at a great distance from the source in proportion to the square of the distance.

The importance of the concept of a point source consists in the fact that an extended source of sound can be replaced by an equivalent system of distributed point sources with generator $q(x)\mathrm{d}V$. Then, the field of the pressure can be found by means of the following formula, which is a direct consequence of formula (25):

$$p(x) = \frac{\omega\rho}{4\pi \mathrm{i}} \iiint_V q \frac{\mathrm{e}^{\mathrm{i}kr}}{r} \mathrm{d}V, \tag{27}$$

where V is the volume occupied by the source and r is the distance from the point x at which the field is determined to a point ξ belonging to the element of volume $\mathrm{d}V$. This formula gives a representation of the pressure field in terms of a volume vibrational potential. The relative phase of the vibrations of different point sources can easily be computed by introducing complex values for q.

Problem

Compute the energy emitted by a point source.

4. *Emission from an opening in a plane wall*

As a simple example, let us show how an approximative calculation can be made for an acoustic field by replacing an emitter with a system of distributed point sources.

Let us assume that in an infinite plane wall there is a circular opening through which a plane sound wave falling on one side of the wall penetrates into the space on the other side. Leaving aside the question of the intensity of the acoustic field emitted from the opening, let us see how it is distributed in the space in front of the wall at a sufficient distance away from it.

We shall consider each element $\mathrm{d}S$ of the area of the opening as a point source of sound with generator $4\pi q\mathrm{d}S$, where q is a constant (that is, the distribution of the intensity of emission is uniform over the opening). We introduce spherical coordinates (r, θ, φ) with origin at the center of the opening and with polar axis perpendicular to its plane. From formula (25), the pressure created by the source $\mathrm{d}S$ at a point $x(r, \theta, \varphi)$ is equal to

$$\mathrm{d}p = \frac{q\omega^2\rho}{a} \frac{\mathrm{e}^{\mathrm{i}kR}}{\mathrm{i}kR} \mathrm{d}S, \tag{28}$$

where R is the distance between a point $\xi(r', \frac{1}{2}\pi, \varphi')$ belonging to $\mathrm{d}S$ and the

point of observation $x(r,\theta,\varphi)$. By using a familiar formula of analytic geometry, we find that

$$R^2 = (r\sin\theta\cos\varphi - r'\cos\varphi')^2 + (r\sin\theta\sin\varphi - r'\sin\varphi')^2 + r^2\cos^2\theta\ .$$

Assuming that we are considering the field at a great distance from the wall ($r \gg r'$), we neglect the terms containing r'^2. After some simple transformations, this gives

$$R^2 \approx r^2 - 2rr'\sin\theta\cos\psi \qquad (\psi = \varphi - \varphi')\ ,$$

so that

$$R \approx r - r'\sin\theta\cos\psi\ . \tag{29}$$

We can, without causing any significant error, take $R \approx r$ in the denominator of the right side of eq. (28). In the exponent, however, we must, as will be clear from what follows, keep the more exact expression given by (29). This gives us

$$\mathrm{d}p \approx \frac{q\omega^2\rho}{a}\,\mathrm{e}^{ikr}\,\frac{\mathrm{e}^{ikr'\sin\theta\cos\psi}}{ikr}\,\mathrm{d}S\ .$$

Noting that $\mathrm{d}S = r'\,\mathrm{d}r'\,\mathrm{d}\psi$, we obtain the expression

$$p = \frac{q\omega^2\rho}{a}\,\frac{\mathrm{e}^{ikr}}{ikr}\int_0^{r_0'} r'\,\mathrm{d}r'\int_0^{2\pi}\mathrm{e}^{-ikr'\sin\theta\cos\psi}\,\mathrm{d}\psi\ ,$$

where r_0' is the radius of the opening, for determining the pressure from the entire set of point sources on S. From (58) of Chapter XII, the inner integral yields:

$$\int_0^{2\pi}\mathrm{e}^{-ikr'\sin\theta\cos\psi}\,\mathrm{d}\psi = 2\pi J_0(kr'\sin\theta)\ .$$

Therefore,

$$p = \frac{2\pi q\omega^2\rho}{a}\,\frac{\mathrm{e}^{ikr}}{ikr}\int_0^{r_0'} J_0(kr'\sin\theta)\,r'\,\mathrm{d}r' = \frac{2\pi q\omega^2\rho r_0'^2}{a}\,\frac{J_1(kr_0'\sin\theta)}{kr_0'\sin\theta}\,\frac{\mathrm{e}^{ikr}}{ikr}\,.$$

Substituting this expression into eq. (7), we find that the intensity of the acoustic field at a great distance from the opening is

$$I = \frac{4\pi^4q^2a\rho r_0'^2}{r^2\lambda^2}\left\{\frac{2J_1(2\pi(r_0'/\lambda)\sin\theta)}{2\pi(r_0'/\lambda)\sin\theta}\right\}^2\ ,$$

written in terms of the wavelength λ of the emitted sound instead of the angular frequency ω.

The function $2J_1(x)/x$ is close to unity when $x \leq 1.5$; thereafter, its variation acquires an oscillatory character with rapidly falling amplitude, Therefore, if $2\pi r_0'/\lambda \leq 1.5$, that is, if $r_0' < \frac{1}{4}\lambda$, the emission from the opening into the space beyond the wall will be propagated very nearly uniformly (the case of long waves). As the wavelength decreases, the emission will

acquire an ever more directional nature and as θ varies from 0 to $\frac{1}{2}\pi$, the appearance of some maxima and minima in intensity is possible (side petals).

5. *The acoustic field due to arbitrary oscillation of the surface of a sphere*

In this section, we turn to the more complicated problem of determining the field of a sphere whose surface is undergoing harmonic oscillations of a definite frequency with amplitude and phase varying arbitrarily from point-to-point. This problem is more general than it might seem at first glance. For if we construct around an arbitrary source of sound a spherical surface that is large enough to contain the source within it, the field on this surface will determine uniquely the field in the space outside it because of the uniqueness of the solution to the boundary-value problem for the Helmholtz equation. Therefore, if we consider a sphere with *arbitrarily* vibrating surface, we obtain a number of results that apply to the field at a great distance from the source that are valid for an arbitrary source of sound.

We first recall how the change in the phase of the vibrations from point to point on the sphere can be calculated. When a phase shift is involved, the vibrations are described not by the expression $a(\theta,\varphi)e^{-i\omega t}$, where $a(\theta,\varphi)$ is the amplitude of the vibrations, but by the expression

$$a(\theta,\varphi)e^{-i[\omega t+q(\theta,\varphi)]},$$

where $q(\theta,\varphi)$ is the phase shift. Let us set

$$a(\theta,\varphi)\ e^{-iq(\theta,\varphi)} \equiv b(\theta,\varphi)\ . \tag{30}$$

Then, as in Chapter XXIV, we formally return to the expression $b(\theta,\varphi)e^{-i\omega t}$. This is of the same form as when there is no phase shift, but now the number $b(\theta,\varphi)$ is *complex*. By introducing the complex amplitude $b(\theta,\varphi)$ given by the identity (30), we allow for a phase shift.

As in sections 2 and 3, we shall examine the field of the excess pressure p, which, as we know (section 1), satisfies the Helmholtz equation

$$\Delta p + k^2 p = 0 \qquad (k^2 = \omega^2/a^2)\ , \tag{31}$$

where a is the velocity of sound in the medium and ω is the angular velocity of the oscillations. The requirement that the radial velocities of the surface of the sphere and the points of the medium adjacent to it be equal gives the boundary condition. In spherical coordinates r, θ, φ, with origin at the center of the sphere, it can be written in the following way:

$$\left.\frac{\partial p}{\partial r}\right|_{r=r_0} = i\rho\omega^2 b(\theta,\varphi)\ , \tag{32}$$

where r_0 is the radius of the sphere (at equilibrium). Noting that the derivative with respect to r coincides up to sign with the normal derivative at the surface of the sphere, we see that we are dealing with the external Neumann problem for the Helmholtz equation. To ensure that its solution be unique, we require that the radiation condition be satisfied:

$$\lim_{r\to\infty} p = 0\ , \qquad \lim_{r\to\infty} r\left(\frac{\partial p}{\partial r} - ikp\right) = 0\ , \tag{33}$$

which then excludes waves coming from infinity.

Let us seek a solution in the form of the series (94) of Chapter XXV:

$$p = \sum_{n=0}^{\infty} h_n(kr) \sum_{m=0}^{n} P_{nm}(\cos\theta)\,[A_{nm}\cos m\varphi + B_{nm}\sin m\varphi]\ , \tag{34}$$

where

$$h_n(x) \equiv \sqrt{\tfrac{1}{2}\pi}\,\frac{1}{\sqrt{x}} H^{(1)}_{n+\frac{1}{2}}(x)\ . \tag{35}$$

On the basis of formulae (28), (62), and (64) of Chapter XII, we immediately observe the following two limiting relationships:

(1) As x approaches zero,

$$h_n(x) \to -i\,\frac{(2n-1)!!}{x^{n+1}} \qquad [(2n-1)!! \equiv 1\times1\times3\times5\times\ldots(2n-1)]\ , \tag{36}$$

(2) As x approaches ∞,

$$h_n(x) \to (-i)^n\,\frac{e^{ix}}{ix}\ , \tag{37}$$

which we shall be using below. The error made by replacing the values of $h_n(x)$ with their limiting values for small x is of the order of $1/x^n$, and for large x, it is of the order of $(x^2-n^2)^{-\frac{1}{2}}$. It then follows in particular that as x becomes infinitely great $h_n(x)$ does not approach a limit in the same manner for all n. Therefore, at large values of x ($x \gg 1$), the replacement of $h_n(x)$ by the asymptotic form is possible only if $x \gg n$.

Let us differentiate the series (34) termwise with respect to r. We use the formula (section 6, chapter XII)

$$\frac{d}{dx}\left[\frac{1}{\sqrt{x}} H^{(1)}_{n+\frac{1}{2}}(x)\right] = \frac{1}{2n+1}\left[\frac{n}{\sqrt{x}} H^{(1)}_{n+\frac{1}{2}}(x) - \frac{n+1}{\sqrt{x}} H^{(1)}_{n+\frac{3}{2}}(x)\right]\ ,$$

which gives us

$$\frac{d}{dr} h_n(kr) = k\left[\frac{n}{2n+1} h_{n-1}(kr) - \frac{n+1}{2n+1} h_{n+1}(kr)\right]\ , \tag{38}$$

so that under the assumption that after differentiation the series converges uniformly for $r = r_0$, we obtain

$$\left.\frac{\partial p}{\partial r}\right|_{r=r_0} = -kA_0 h_1(kr_0) + k\sum_{n=1}^{\infty} h'_n(kr_0) \times\left[\sum_{m=0}^{n} P_{nm}(\cos\theta)\,(A_{nm}\cos m\varphi + B_{nm}\sin m\varphi)\right]\ , \tag{39}$$

where, for brevity, we define

$$h'(kr) = \frac{n}{2n+1} h_{n-1}(kr) - \frac{n+1}{2n+1} h_{n+1}(kr)\ . \tag{40}$$

If the real and imaginary parts of the function $b(\theta,\varphi)$ have continuous first-order partial derivatives, we can expand this function in a series of spherical functions:

$$b(\theta,\varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_{nm}(\cos\theta)\,(a_{nm}\cos m\varphi + b_{nm}\sin m\varphi)\ , \qquad (41)$$

where the coefficients a_{nm} and b_{nm} can be determined from formulae (30) of Chapter XXI. By comparing the series (39) and (41) and taking into consideration the relationship (31) and the boundary condition (32), we conclude that this last will be observed if we take

$$A_{nm} = \mathrm{i}\rho\omega a\,\frac{a_{nm}}{h_n'(kr_0)}\ , \qquad B_{nm} = \mathrm{i}\rho\omega a\,\frac{b_{nm}}{h_n'(kr_0)}\ . \qquad (42)$$

We introduce the notation

$$Y_n(\theta,\varphi) \equiv a_{n0}P_n(\cos\theta) + \sum_{m=1}^{n} P_{nm}(\cos\theta)\,(a_{nm}\cos m\varphi + b_{nm}\sin m\varphi)\ . \qquad (43)$$

The functions $Y_n(\theta,\varphi)$ represent the same spherical functions that appear in the expansion

$$b(\theta,\varphi) = \sum_{n=0}^{\infty} Y_n(\theta,\varphi)\ . \qquad (44)$$

With this notation, the series (39) can finally be written in the form

$$p = \mathrm{i}\rho\omega a \sum_{n=0}^{\infty} \frac{h_n(kr)}{h_n'(kr_0)}\, Y_n(\theta,\varphi)\ . \qquad (45)$$

Thus, the formal solution of the problem posed is found in the form of an infinite series. Let us consider the convergence of this series. We shall show that it converges absolutely for any arbitrary finite value of r greater than r_0.

Let us discard a finite number n_0 of terms of the series (45) by choosing n_0 so that

$$n_0 > \sqrt{6.25(n_0+\tfrac{1}{2})^{\frac{2}{3}} + (kr)^2}\ , \qquad n_0 \gg kr\ . \qquad (46)$$

We note that the first of these inequalities implies that n_0 is greater than 16.

Let us now use the following asymptotic representation for Hankel functions of the first kind *:

$$H_\nu^{(1)}(x) = -\,\mathrm{i}\sqrt{2/\pi s}\,\exp\,[-s + \nu\tanh^{-1}(s/p)]\,[1 + \mathrm{O}(1/s)] \qquad (47)$$

$$(s \equiv \sqrt{\nu^2 - x^2})\ ,$$

which is valid if

* See V. I. Smirnov [1], Vol. 3, Part 2, p. 152.

$$\nu/x \gg 1\,, \qquad \nu \gg 1\,. \tag{48}$$

On the basis of the inequalities (46), the inequality (48) will hold if $x = kr$ and ν is greater than n_0. By using the fact that $kr \ll n_0$, we have

$$s/\nu \approx 1 - \tfrac{1}{2}(kr/\nu)^2\,.$$

Since s/ν is close to unity, we have

$$\xi \equiv \tanh^{-1}(s/\nu) \gg 1$$

and

$$\frac{s}{\nu} \equiv \tanh^{-1}\xi = \frac{e^{\xi} - e^{-\xi}}{e^{\xi} + e^{-\xi}} = \frac{1 - e^{-2\xi}}{1 + e^{-2\xi}} \approx (1 - e^{-2\xi})^2 \approx 1 - 2e^{-2\xi}\,,$$

and hence,

$$\xi = \tanh^{-1}\frac{s}{\nu} \approx -\frac{1}{2}\ln\frac{1}{2}\left(1 - \frac{s}{\nu}\right) = -\frac{1}{2}\ln\frac{1}{4}\left(\frac{kr}{\nu}\right)^2 = \ln\frac{2\nu}{kr}\,,$$

$$-s + \nu\tanh^{-1}\frac{s}{\nu} \approx -\nu + \frac{\nu}{2}\left(\frac{kr}{\nu}\right)^2 + \nu\ln\frac{2\nu}{kr} \approx \nu\left(\ln\frac{2\nu}{kr} - 1\right) \approx \ln\left(\frac{2\nu}{kr}\right)^{\nu}\,.$$

If we substitute the approximative expressions that we have found into formula (47), we obtain

$$H_{\nu}^{(1)}(kr) \approx -\,i\sqrt{2/\pi\nu}\,(2\nu/kr)^{\nu}\,, \tag{49}$$

from which, because of the definition (35) of the functions $h_n(x)$, it follows that

$$h_n(kr_0) \equiv \sqrt{\tfrac{1}{2}\pi}\,\frac{1}{\sqrt{kr_0}}\,H_{n+\frac{1}{2}}^{(1)}(kr_0) \approx -\frac{i\sqrt{2}}{kr_0}\left(\frac{2n}{kr_0}\right)^{n}\,. \tag{50}$$

Let us now set $r = r_0$. Then, on the basis of formula (40), if we discard the small terms, we obtain

$$h_n'(kr_0) \approx \frac{i}{\sqrt{2}}\,\frac{1}{kr_0}\left(\frac{2n}{kr_0}\right)^{n} \qquad (n > n_0)\,. \tag{51}$$

If we substitute these expressions into the series (45) without the first n_0 terms, we see that the problem in question is reduced to investigating the convergence of the series

$$R_n = \sum_{n=n_0+1}^{\infty}\left(\frac{r_0}{r}\right)^{n} Y_n(\theta,\varphi)\,.$$

But this series is known to converge absolutely since r_0 is less than r and the series

$$\sum_{n=0}^{\infty} Y_n(\theta,\varphi)$$

converges by hypothesis.

6. *Investigation of the field of a sphere with arbitrary vibration of its surface. Acoustic or vibrational multipoles*

Let us investigate the solution obtained in the preceding section.

It follows from eq. (45) that points for which $h_n'(kr) = 0$ when $r = r_0$ can be singular points. This, however, is impossible because the functions $h_n'(x)$, like Hankel functions, do not have real zeros. This is easy to show for small values of kr_0 that satisfy the inequality

$$kr_0 = \frac{2\pi}{\lambda} r_0 \ll 1 ,$$

where λ is the length of the waves emitted by the system. For small values of kr_0, it follows from formulae (36) and (40) that

$$h_n'(kr_0) \approx \mathrm{i}(n+1)\frac{(2n-1)!!}{(kr_0)^{n+2}} \quad (n = 0, 1, 2, \ldots) . \tag{52}$$

Substituting these relationships into formula (42), we obtain

$$A_{nm} = \rho\omega a(kr_0)^2 \frac{(kr_0)^n}{(n+1)(2n-1)!!} a_{nm} ,$$

$$B_{nm} = \rho\omega a(kr_0)^2 \frac{(kr_0)^n}{(n+1)(2n-1)!!} b_{nm} ,$$

from which the validity of the assertion made follows since all the numbers A_{nm} and B_{nm} are known to be bounded.

The expressions (52) also make it possible to obtain an approximative representation of the series (45) when the dimensions of the source are small. If we substitute the expression (52) into the series (45), we obtain the series

$$p = \frac{\omega^3 \rho r_0{}^2}{a} \sum_{n=0}^{\infty} \frac{(kr_0)^n}{(n+1)(2n-1)!!} h_n(kr)\, Y_n(\theta, \varphi) , \tag{53}$$

which usually converges rapidly.

Let us now clarify the physical meaning of the individual terms of the series (45).

Consider the first term, which, in view of the fact (see Chapter XII, section 6) that

$$h_0(x) \equiv \sqrt{\tfrac{1}{2}\pi}\,\frac{1}{\sqrt{x}}\, H_{\frac{1}{2}}^{(1)}(x) = \frac{\mathrm{e}^{\mathrm{i}x}}{\mathrm{i}x} ,$$

can be written in the form

$$\frac{\omega^3 \rho r_0{}^2}{a} \frac{\mathrm{i}c_0}{h_0'(kr_0)(kr_0)^2} \frac{\mathrm{e}^{\mathrm{i}kr}}{\mathrm{i}kr} .$$

Let us set

$$\frac{ic_0}{h_0'(kr_0)(kr_0)^2} \equiv \alpha_0 + i\beta_0 = \alpha_0 + \beta_0 \, e^{\frac{1}{2}i\pi} ,$$

where α_0 and β_0 are real numbers. Then, the first term can be broken into two terms

$$\frac{\omega^3 \rho r_0^2}{a} \alpha_0 \frac{e^{ikr}}{ikr} \quad \text{and} \quad \frac{\omega^3 \rho r_0^2}{a} \beta_0 \frac{e^{ikr+\frac{1}{2}i\pi}}{ikr} .$$

Comparing these with the expression (25), we see that they correspond to the fields of point sources with generators

$$Q_{0\alpha} = 4\pi r_0^2 \omega \alpha_0 \quad \text{and} \quad Q_{0\beta} = 4\pi r_0^2 \omega \beta_0$$

and a mutual phase shift of $\frac{1}{2}\pi$.

To show the physical meaning of the second term, we find the corresponding distribution of the radial velocities v_r on the surface of the sphere. From formulae (4) and (45),

$$v_r = -\frac{i}{\rho\omega}\frac{\partial p}{\partial r} = -\frac{i}{\rho\omega}\frac{\partial}{\partial r} i\rho\omega a \frac{h_n(kr)}{h_n'(kr_0)} Y_1(\theta,\varphi) = \frac{h_n'(kr)}{h_n'(kr_0)} \omega Y_1(\theta,\varphi) ,$$

from which,

$$v_r|_{r=r_0} = \omega Y_1(\theta,\varphi) .$$

But, from (43),

$$Y_1(\theta,\varphi) = a_{10} P_1(\cos\theta) + P_{11}(\cos\theta)(a_{11}\cos\varphi + b_{11}\sin\varphi)$$
$$= a_{10}\cos\theta + \sin\theta\,(a_{11}\cos\varphi + b_{11}\sin\varphi) .$$

If we represent each of the complex quantities a_{10}, a_{11}, and b_{11} in the form

$$a_{10} = \alpha_1 + \gamma_1 e^{\frac{1}{2}i\pi} , \qquad a_{11} = \alpha_{11} + \gamma_{11} e^{\frac{1}{2}i\pi} , \qquad b_{11} = \beta_{11} + \delta_{11} e^{\frac{1}{2}i\pi} ,$$

we break the expression $\omega Y_1(\theta,\varphi)$ into two terms. The first of these is equal to

$$\omega Y_{1b}(\theta,\varphi) = \omega[\alpha_1\cos\theta + \sin\theta\,(\alpha_{11}\cos\varphi + \beta_{11}\sin\varphi)] . \tag{54}$$

In particular, for $\alpha_{11} = \beta_{11} = 0$, we obtain

$$v_r|_{r=r_0} = \omega\alpha_1\cos\theta .$$

But it is easy to show that just this distribution of radial velocity exists in the case of harmonic vibrations of a rigid sphere that have an angular frequency ω and an amplitude α_1 along the polar axis of a spherical system of coordinates. With this in mind, let us examine the case of vibrations of the sphere with amplitude s_1 along the axis whose direction is determined by the angles $\theta = \theta'$ and $\varphi = \varphi'$. In this case, the radial velocity v_r of a point on the surface of the sphere in question with coordinates θ, φ (equal to the projection of the velocity of the vibrations of the center of the sphere in the direction given by the angles θ and φ) is determined by the equation

$$v_r|_{r=r_0} = (\omega s_1 \cos\theta')\cos\theta + \sin\theta\,[(\omega s_1 \sin\theta'\cos\varphi')\cos\varphi + (\omega s_1 \sin\theta'\sin\varphi')\sin\varphi]\,,$$

which coincides with eq. (54) if

$$s_1\cos\theta' = \alpha_1\,,\qquad s_1\sin\theta'\cos\varphi' = \alpha_{11}\,,\qquad s_1\sin\theta'\sin\varphi' = \beta_{11}\,,$$

that is, if

$$s_1^2 = \alpha_1^2 + \alpha_{11}^2 + \beta_{11}^2\,,\qquad \tan\theta' = \frac{\sqrt{\alpha_{11}^2+\beta_{11}^2}}{\alpha_1}\,,\qquad \tan\varphi' = \frac{\beta_{11}}{\alpha_{11}} \tag{55}$$

In just the same way, we find that the second term in $\omega Y_1(\theta,\varphi)$ which contains a factor $e^{\frac{1}{2}i\pi}$, also gives a field corresponding to a harmonic vibration of a solid sphere but, generally speaking, one with a different amplitude and along a different axis. We can determine the amplitude and direction of the vibration by making the substitutions in formula (55):

$$\alpha_1 \to \gamma_1\,,\qquad \alpha_{11}\to\gamma_{11}\qquad \beta_{11}\to\delta_{11}\,.$$

Also, the phase of the second vibration is displaced by an amount $\frac{1}{2}\pi$ relative to the phase of the first.

In analogy to our study of the first term of the series (45), we can introduce a point object, which is called an acoustic or vibrational dipole. By this we mean a sphere which is harmonically vibrating in some direction and whose cross section is negligibly small in comparison with the length of the emitted wave. The reader will find no difficulty in showing that if r exceeds r_0, the second term of the expansion (45) corresponds to the field of *two* properly oriented acoustic dipoles with phases that are mutually displaced by an amount $\frac{1}{2}\pi$.

By considering in succession the terms of the series (45), we could continue the construction of acoustic multipoles. However, we prefer some other approach to the matter, all the more so since the model (a small vibrating sphere) of only an acoustic dipole is of any great practical value.

Each of the spherical functions $Y_n(\theta,\varphi)$ and of the terms appearing in its composition gives a distribution of the vibrations of the surface of the sphere that is characterized, in the first place, by a definite degree of symmetry – that is, by a definite number and by the relative positions of the axes and the planes of symmetry – and, in the second place, by a definite orientation of these figures of symmetry in space. A particularity of each of these figures of symmetry is that the corresponding patterns of the field on all the spherical surfaces concentric with the source are similar, which, generally speaking, is not the case with an arbitrary pattern of the field.

We encountered an analogous situation when we examined the electrostatic fields of an arbitrary system of charges. As was shown in Chapter XIX, sections 3 and 4, the field of an arbitrary system of charges can be represented in the form of an expansion in multipoles of different orders:

$$\sum_{k=0}^{\infty}\frac{Y_k(\theta,\varphi)}{r^{k+1}} \tag{56}$$

With increasing r, the role of the different terms varies. Therefore, if, for some value of r, the term with $n = n_1$ for example plays a special role, this role will, with increasing r, be transferred sooner or later to the term with $n = n_2 < n_1$ (provided all the terms with n less than n_1 are not identically equal to zero). The pattern of the field approximates more and more closely a symmetric one corresponding to the multipole of lowest order for which the multipole moment of the system is not equal to zero. Thus, at a sufficient distance from an arbitrary system of charges with net charge not equal to zero, the field of this system approximates the spherically symmetric field of a point charge. If the net charge of the system is equal to zero but the dipole moment is different from zero, then at a sufficiently great distance the field will approximate the field of a dipole, etc.

On the other hand, with increasing r, the fields of each individual multipole $Y_k(\theta, r)/r^{k+1}$ can only decrease; its patterns are similar on all spherical surfaces with center at $r = 0$. Because the system of spherical functions is closed, the multipoles (or the systems that, through the field they create, are equivalent to a single multipole) exhaust all systems having this property. To show this, suppose that some system not reducible to any multipole has this property. Then, according to sections 3 and 4 of Chapter XIX, it can be represented as the sum (finite or infinite) of multipoles of different orders situated at the point $r = r_0$. But fields of multipoles of different orders change according to different laws as r increases and so the system cannot have the required property. This contradiction proves the assertion.

In fact, a multipole may be characterized by the ability to create fields which are similar on each member an infinite system of concentric spherical surfaces of arbitrary radius. In particular, by an *acoustic* or *vibrational* multipole, we mean a point source that creates in a homogeneous medium a field with the following properties: (a) It satisfies the Helmholtz equation; (b) the phase of vibrations of the field depends only on the distance from the source (so that, in particular, the phase of the vibrations is the same at every point of an arbitrary spherical surface with center at the point at which the source is located; (c) the fields are similar at all spherical surfaces with center at the point at which the source is located; and (d) the radiation condition is satisfied. An emitting system whose field, beginning at some distance, is similar to a multipole field can be called reducible to a multipole.

Let us now continue our comparison of electrostatic and acoustic fields.

If we substitute the expression (36) for $h_n(kr)$ for small values of the argument into the series (53), we obtain the expansion

$$p = -\, i\omega^2 \rho r_0 \sum_{n=0}^{\infty} \frac{1}{n+1} Y_n(\theta, \varphi)\, (r_0/r)^{n+1} \qquad (kr \ll 1)\,. \tag{57}$$

If we include here the factors $(-i\omega^2 \rho r_0^{n+2}/(n+1))$ in the composition of the functions $Y_n(\theta, \varphi)$, we obtain a series that coincides in form exactly with the series (56). Therefore, *within the limits of applicability of the series*

(57), what was said about the field of a system of electric charges applies to the field p.

There are, however, significant differences in the natures of the two fields. In the first place, when $kr \gg n$, the field of the pressure of the acoustic multipoles of all orders falls off only as $1/r$. This follows from the relationship (37). Therefore, in contrast with the electrostatic field, in which the role of the multipole moments of higher orders becomes arbitrarily small with increasing distance from the system of charges, in an acoustic field, as r increases, the difference in the rate of decrease in the fields of multipoles of different orders becomes less and less perceptible. As a result, the dependence of the pressure in an acoustic field on the angular coordinate, generally speaking, approaches with increasing r not the dependence corresponding to some one acoustic multipole but a dependence that is distinct in each case. This is clearly shown by the following: Suppose that $n_0 \ll kr$. Then, the sum of the terms of the series (45) with n less than n_0 is, on the basis of (37), approximately equal to

$$p_{n_0} = \frac{i\rho\omega a}{h_n'(kr_0)} \frac{e^{ikr}}{ikr} \sum_{n=0}^{n_0} (-i)^n \frac{h_0'(kr_0)}{h_n'(kr_0)} Y_n(\theta, \varphi) \ . \tag{58}$$

The series

$$\psi(\theta, \varphi, n_0) \equiv \sum_{n=0}^{n_0} (-i)^n \frac{h_0'(kr_0)}{h_n'(kr_0)} Y_n(\theta, \varphi) \tag{59}$$

appearing in this expression does not depend on r and it defines, as kr and n_0 become infinitely great, the limiting angular dependence.

Thus, if the electrostatic field approaches arbitrarily closely the field of a multipole of some order with increasing distance from the source (that is, if an arbitrary system of motionless electric charges is reduced to a multipole), this phenomenon will not generally take place in an acoustic field. However, the role of multipoles of higher order in an acoustic field is ordinarily not great, as we might expect in view of the rapid decrease of the terms in the series (53) caused by the presence of the factor $1/(2n-1)!!$.

In the second place, the difference between electrostatic and acoustic fields appears in the fact that acoustic fields (and in general, vibrational fields) can differ one from the other not only in amplitude but also in phase, which is not the case with an electric field. Therefore, every term of the expansions (45) and (53) corresponds to the field of not one but two multipoles, differing from each other in phase of vibrations.

From formulae (4) and (58), the radial velocity of the medium, determined by multipoles with $n \leqslant n_0$ and $kr \gg n_0$, is equal to

$$v_{rn_0} \approx \frac{\omega^3 r_0^2}{a^2} \frac{e^{ikr}}{ikr} \psi(\theta, \varphi, n_0) + O(1/r^2) \ . \tag{60}$$

The tangential components of the velocity v_θ and v_φ are of the order of $1/r^2$, as is clear from the expressions for the differential operators in the directions of the tangents to the curves φ = constant and θ = constant:

$$\frac{1}{r}\frac{\partial}{\partial\theta}, \qquad \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}.$$

Therefore, at a great distance from an emitting system, when the role of multipoles of higher orders is small, the motion of the medium is basically radial.

By using the relationship (7) and assuming that the pressure of the acoustic field is sufficiently well represented in the form

$$p \approx \frac{i\rho\omega a}{h_0'(kr_0)}\frac{e^{ikr}}{ikr}\psi(\theta,\varphi,\infty),$$

we find that the intensity of the field at a great distance from an emitter is equal to

$$I = \frac{|p^2|}{2\rho a} = \frac{\omega^4\rho r_0{}^4 c_0{}^2}{2ar^2}F(\theta,\varphi) = \frac{\omega^2\rho}{32\pi^2 a}\frac{Q_0{}^2}{r^2}F(\theta,\varphi),$$

where

$$F(\theta,\varphi) = \frac{1}{c_0{}^2}|\psi(\theta,\varphi,\infty)|^2. \tag{61}$$

This is the so-called angular distribution function of the intensity of emission and

$$Q_0 = 4\pi r_0{}^2\omega c_0 \tag{62}$$

is the "averaged equivalent strength" of the source. For a uniformly pulsating sphere, $F(\theta,\varphi) = 1$.

Problems

1. Investigate the field of emission of a point source of low frequency that is located on the surface of a sphere.
   Method: It is convenient to locate the source at the pole of the sphere ($\theta = 0$). Then,

   $$a_{n0} = \frac{1}{2\pi}(n+\tfrac{1}{2})\frac{Q^2}{r_0{}^2}, \qquad a_{nm} = b_{nm} = 0,$$

   where Q is the equivalent strength of the source. Then consider the point source as the limiting case of a source with the form of a circular area of radius r' and equivalent strength $(b\omega\pi r'^2)$.

2. Investigate the field of an acoustic quadrupole.

7. *The scattering of sound*

If a sound wave meets an obstacle, it is partially reflected from it and partially transmitted through it. As a result, the initial direction of propagation of the wave is changed. This process is called scattering or diffraction of the sound waves.

Let us consider a steady-state sound wave that is set up in a homogeneous medium characterized by a density ρ and sound velocity a in which there is a homogeneous body of density ρ_i and sound velocity a_i. As above, we shall characterize the sound wave by the pressure p and the angular frequency ω of the acoustic vibrations. We shall assume that the medium occupies all space R_E with the exception of the volume V occupied by the body.

We shall call the field $p_0(x)$ that would exist if the body were not there the *incident* wave; we shall call the field $p_i(x)$ within the body the *refracted* wave; and we shall call the field $p_e(x)$, which on combining with the incident wave $p_0(x)$ gives the actual acoustic field $p(x)$ in the medium, the *reflected* or *scattered* wave.

Because of the homogeneity of the body and of the medium, the pressure at their internal points satisfies the Helmholtz homogeneous equations

$$\Delta p_i + k_i^2 p_i = 0 , \quad k_i^2 \equiv \omega^2/a_i^2 , \quad x \in V - \mathcal{F}V ,$$

$$\Delta p + k^2 p = 0 , \quad k^2 \equiv \omega^2/a^2 , \quad x \in R_E - V .$$

Since $p_e(x) = p(x) - p_0(x)$ and the incident wave $p_0(x)$ also satisfies the Helmholtz equation, we have

$$\Delta p_e + k^2 p_e = 0 \quad \text{when} \quad x \in R_E - V .$$

At infinity, the scattered wave $p_e(x)$ must obviously satisfy the radiation condition:

$$\lim_{r\to\infty} r\left(\frac{\partial p_e}{\partial r} - ikp_e\right) = 0 , \quad \lim_{r\to\infty} p_e = 0 .$$

Finally, on the boundary $\mathcal{F}V$ of the body, the pressure and velocity of vibrations in the body and in the medium must coincide. On the basis of formula (4), this leads to the following conjugacy conditions:

$$p_i = p_e + p_0 , \quad \frac{1}{\rho_i}\frac{dp_i}{dn} = \frac{1}{\rho}\frac{dp_e}{dn} + \frac{1}{\rho}\frac{dp_0}{dn} \quad \text{when} \quad x \in \mathcal{F}V ,$$

where d/dn indicates differentiation in the direction of the outer normal to the boundary of the region V.

By combining these results and assuming that the incident wave is given, we arrive at the following problem:

$$\begin{gathered}
\Delta p_i + k_i^2 p_i = 0 \quad \text{when} \quad x \in V - \mathcal{F}V ; \\
\Delta p_e + k^2 p_e = 0 \quad \text{when} \quad x \in R_E - V ; \\
\lim_{r\to\infty} r\left(\frac{\partial p_e}{\partial r} - ikp_e\right) = 0 , \quad \lim_{r\to\infty} p_e = 0 , \\
p_i = p_e + p_0 , \quad \frac{1}{\rho_i}\frac{dp_i}{dn} = \frac{1}{\rho}\frac{dp_e}{dn} + \frac{1}{\rho}\frac{dp_0}{dn} \quad \text{when} \quad x \in \mathcal{F}V ,
\end{gathered} \tag{63}$$

which represents one of the simplest problems in the mathematical theory of diffraction.

Let us consider this problem with the assumption that the region V is a sphere and that the incident wave is plane.

We introduce spherical coordinates r, θ, and φ with origin at the center of the sphere V and with polar axis directed toward the incident wave. The incident wave can then be represented by the expression

$$p_0(r,\theta) = A\, e^{ikr\cos\theta} , \tag{64}$$

which does not depend on the coordinate φ. Because of the symmetry of the pattern of the scattering relative to the polar axis, the solution of the diffraction problem (63) will also be independent of φ.

Consequently, the expansion of the scattered wave in a series of the form (94) of Chapter XXV in particular solutions of the Helmholtz equation will take the form

$$p_e = \sum_{\alpha=0}^{\infty} a_\alpha h_\alpha(kr)\, P_\alpha(\cos\theta) , \qquad h_\alpha(kr) \equiv \sqrt{\pi/2kr}\, H^{(1)}_{\alpha+\frac{1}{2}}(kr) , \tag{65}$$

where $H^{(1)}_{\alpha+\frac{1}{2}}(kr)$ is a Hankel function of the first kind. We seek a solution for the refracted wave in the form of an analogous series:

$$p_i = \sum_{\alpha=0}^{\infty} b_\alpha j_\alpha(k_i r)\, P_\alpha(\cos\theta) , \qquad j_\alpha(kr) \equiv \sqrt{\pi/2k_i r}\, J_{\alpha+\frac{1}{2}}(k_i r) , \tag{66}$$

in which instead of Hankel functions we have Bessel functions. These ensure that the terms of the series are bounded for $r = 0$. It follows from the expression (56) of Chapter XXV for particular solutions of the Helmholtz equation that the terms of the latter series satisfy the Helmholtz equation.

To determine the unknown coefficients a_α and b_α (for $\alpha = 0, 1, 2, 3, \ldots$), we use the expansion that we are familiar with from the theory of Bessel functions for a plane wave in spherical functions:

$$e^{ikr\cos\theta} = \sum_{\alpha=0}^{\infty} (2\alpha+1)\, i^\alpha j_\alpha(kr)\, P_\alpha(\cos\theta) .$$

If we substitute these series into the conjugacy conditions and equate the coefficients of the $P_\alpha(\cos\theta)$, we obtain the following equations for determining the coefficients a_α and b_α:

$$\begin{aligned} b_\alpha j_\alpha(k_i r_0) - a_\alpha h_\alpha(kr_0) &= A(2\alpha+1)\, i^\alpha j_\alpha(kr_0) , \\ \frac{k_i}{\rho_i} b_\alpha j'(k_i r_0) - \frac{k}{\rho} a_\alpha h'_\alpha(kr_0) &= A\frac{k}{\rho}(2\alpha+1)\, i^\alpha j'_\alpha(kr_0) , \end{aligned} \tag{67}$$

where r_0 is the radius of the boundary surface $\mathcal{F}V$. If we find the coefficients a_α and b_α from this system of equations and substitute them into the series (65) and (66), we shall have the solution of the diffraction problem that we are examining.

Problems

1. Formulate and solve the problem of the scattering of a plane sound wave (64) from an absolutely rigid immovable sphere V of radius r_0.
Answer: The formulation of the problem,

$$\Delta p_e + k^2 p_e = 0 \quad \text{when} \quad r > r_0 ;$$

$$\lim_{r\to\infty} r\left(\frac{\partial p_e}{\partial r} - \mathrm{i}kp_e\right) = 0 , \qquad \lim_{r\to\infty} p_e = 0 ,$$

$$\frac{\mathrm{d}p_e}{\mathrm{d}r} = A \, \mathrm{i}k \cos\theta \, \mathrm{e}^{\mathrm{i}kr\cos\theta} \quad \text{when} \quad r = r_0 .$$

The solution,

$$p_e = - A \sum_{\alpha=0}^{\infty} (2\alpha+1) \, \mathrm{i}^{\alpha} \frac{j'_\alpha(kr_0)}{h'_\alpha(kr_0)} h_\alpha(kr) \, P_\alpha(\cos\theta) .$$

This solution is obtained by setting $\rho_i = \infty$ in the second of eqs. (67) and substituting the values of the coefficient a_α thus found into the series (65).

2. Show that when the wavelength is great in comparison with the dimensions of the sphere V $(kr_0 \ll 1)$, the solution of the preceding problem at a great distance from the sphere $(kr \gg 1)$ can be represented by the approximating formula

$$p_e = - \frac{Ak^2 r_0{}^2}{3r} (1 - \tfrac{3}{2}\cos\theta) \, \mathrm{e}^{\mathrm{i}kr} .$$

Method: Use the approximative expressions for Bessel functions for small values of the argument.

Chapter XXVII*

COMMENTS ON EQUATIONS OF THE ELLIPTIC TYPE IN THE GENERAL FORM

1. *The general form of equations of the elliptic type*

In accordance with the definition given in section 4 of the Introduction, we shall say that the equation

$$\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta} \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \sum_{\alpha=1}^{n} b_\alpha \frac{\partial u}{\partial x_\alpha} + cu = f , \tag{1}$$

where a_{ij}, b_i, c, and f are functions defined in a region V, belongs to the elliptic type in this region if the quadratic form

$$\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}\, q_\alpha q_\beta$$

retains the same sign in this region and does not vanish.

The number n is the dimensionality of the region V. In this chapter, we shall consider only three-dimensional regions ($n = 3$); however, all the results will be equally applicable to planes ($n = 2$) and to many-dimensional regions ($n > 3$).

We shall assume that the functions a_{ij}, b_i, c, and f are continuous and that both the functions a_{ij} and the functions

$$e_i \equiv b_i - \sum_{\beta=1}^{n} \frac{\partial a_{i\beta}}{\partial x_\beta}$$

have continuous first derivatives. Under these conditions, eq. (1) can be transformed into the form

$$\sum_{\alpha,\beta=1}^{n} \frac{\partial}{\partial x_\alpha} a_{\alpha\beta} \frac{\partial u}{\partial x_\beta} + \sum_{\alpha=1}^{n} e_\alpha \frac{\partial u}{\partial x_\alpha} + cu = f . \tag{2}$$

We shall denote the differential expressions on the left side of eqs. (1) and (2) by $\mathcal{M}u$. With this notation, these equations can be written in the form

$$\mathcal{M}u = f . \tag{3}$$

2. *The basic boundary-value problem*

In the first part of this book, we examined the physical quantities that satisfy equations of the hyperbolic type. All these quantities characterized time-dependent processes. In connection with this, for the equations that were examined in the first part, there was a characteristic problem in which the unknown quantity had to satisfy, in addition to the equation, certain boundary and initial conditions. We also saw that when a process that is described in the general case by a hyperbolic-type equation is of a steady-state character, it is described by an equation of the elliptic type because the terms containing time derivatives vanish.

This allows us to assume that equations of the elliptic type are naturally connected with physical problems involving steady states. It is natural to assume in turn that steady states of physical objects are dependent only on the conditions at their boundaries and not on the succession of preliminary states. The solutions of equations describing such states must be completely determed when just their *boundary* conditions are given.

To be more precise, in problems involving equations of the hyperbolic type, we had to state *one* relationship (on the boundary of the region in question) involving the unknown function, its first derivative, and certain given functions and *two* relationships involving both the unknown function and its first derivative with respect to time at the initial instant. Consequently, we must expect that the solution of a second-order partial differential equation of the elliptic type will be completely determined when *one* relationship pertaining to the boundary of the region in question is given, that is when a *boundary condition* involving given functions, the unknown function, and its first derivative is given. This supposition is justified under the natural supplementary requirements regarding the smoothness of the unknown function and its behaviour at infinity (if the solution is being sought for an infinite region). For internal problems, these requirements can be reduced to the regularity of the solution.

We shall not deal with the question of the conditions at infinity but shall confine ourselves to a consideration of the internal problem only. In its most general form, the boundary-value problem for an equation of the elliptic type can be formulated as the problem of finding functions which
(a) are regular solutions of eq. (3) in the region V in question, and
(b) on the boundary $\mathcal{F}V$ of the region V satisfy the boundary condition

$$\alpha \frac{\mathrm{d}u}{\mathrm{d}l} + \beta u = \varphi \quad \text{when} \quad x \in \mathcal{F}V \,, \tag{4}$$

where α, β, and φ are given functions defined on $\mathcal{F}V$ such that $|\alpha| + |\beta| > 0$ and $\mathrm{d}/\mathrm{d}l$ denotes differentiation in the direction l, this direction being given at every point on $\mathcal{F}V$ at which $\alpha \neq 0$.

Up to now, we have not fully studied a boundary-value problem in such a general setting. The particular boundary-value problems that we have already encountered are the most important ones for mathematical physics. It is convenient to classify them in a manner suitable for an equation of the elliptic type in the general form, that is, in a different way from that used

in Chapter XVIII. We shall define them in a way depending on the form of the boundary condition as follows:

1. Dirichlet's problem or the first boundary-value problem:

$$\mathcal{M}u = f \quad \text{when} \quad x \in V - \mathcal{F}V \ ; \qquad u = \psi \quad \text{when} \quad x \in \mathcal{F}V \ . \tag{5}$$

2. Neumann's problem or the second boundary-value problem:

$$\mathcal{M}u = f \quad \text{when} \quad x \in V - \mathcal{F}V \ ; \qquad \alpha \frac{du}{d\nu} + \beta u = \psi \quad \text{when} \quad x \in \mathcal{F}V \ , \tag{6}$$

where the coefficient α does not vanish on the surface $\mathcal{F}V$ and $d/d\nu$ denotes differentiation in the direction of the conormal to $\mathcal{F}V$ (Chapter XVII, section 3).

3. The mixed or third boundary-value problem:

$$\mathcal{M}u = f \quad \text{when} \quad x \in V - \mathcal{F}V \ ; \qquad \alpha \frac{du}{d\nu} + \beta u = \psi \quad \text{when} \quad x \in \mathcal{F}V \ , \tag{7}$$

where the coefficient α is equal to zero at some but not all points of the surface $\mathcal{F}V$.

We note that, with our new definition of boundary-value problems, Neumann's problem embraces both Neumann's problem and in part the mixed boundary-value problem in the original meaning of the word.

These boundary-value problems are called internal or external depending on whether they apply to regions within or without the finite closed surface $\mathcal{F}V$. In this chapter, we shall deal with only the internal Dirichlet and Neumann problems. This classification of boundary-value problems can be extended to the case of two variables.

3. *Conjugate boundary-value problems*

Let us consider the Neumann boundary-value problem:

$$\alpha \frac{du}{d\nu} + \beta u = \varphi \quad \text{when} \quad x \in \mathcal{F}V \quad (\alpha \neq 0) \ . \tag{8}$$

If we multiply this equation by the ratio a/α, where a is the normalizing divisor defined by formula (11) of Chapter XVII, we obtain

$$a \frac{du}{d\nu} + gu = \psi \quad \text{when} \quad x \in \mathcal{F}V \ , \tag{9}$$

where g and ψ are known functions. Comparing this relationship with the first identity of eqs. (14) of Chapter XVII, we see that it can be written in the form

$$\mathcal{P}u = \psi \quad \text{when} \quad x \in \mathcal{F}V \ . \tag{10}$$

The boundary condition

$$Qv \equiv a \frac{dv}{d\nu} + (g - b)v = \psi^* \quad \text{when} \quad x \in \mathcal{F}V \ , \tag{11}$$

where the differential expression Qv is defined by the second of the identities (14) of Chapter XVII and ψ^* is some function defined on $\mathcal{F}V$, is called the *conjugate* to the boundary condition (10).

We shall also call the following boundary-value problems *conjugate*

$$\mathcal{M}u = f \quad \text{when} \quad x \in V - \mathcal{F}V; \qquad \mathcal{P}u = \psi \quad \text{when} \quad x \in \mathcal{F}V, \tag{12}$$

$$\mathcal{N}v = f^* \quad \text{when} \quad x \in V - \mathcal{F}V; \qquad Qv = \psi^* \quad \text{when} \quad x \in \mathcal{F}V,$$

where $\mathcal{M}u$ and $\mathcal{N}v$ are conjugate differential expressions, $\mathcal{P}u$ and Qv are the expressions corresponding to them defined by the identities (14) of Chapter XVII, and f and f^* on the one hand and ψ and ψ^* on the other are functions defined respectively in the region V in question and on its boundary $\mathcal{F}V$.

In the case of Dirichlet's boundary-value problem,

$$\mathcal{M}u = f \quad \text{when} \quad x \in V - \mathcal{F}V; \qquad u = \psi \quad \text{when} \quad x \in \mathcal{F}V, \tag{14}$$

the conjugate problem is

$$\mathcal{N}v = f^* \quad \text{when} \quad x \in V - \mathcal{F}V; \qquad v = \psi^* \quad \text{when} \quad x \in \mathcal{F}V.$$

It is easy to see that the property of conjugacy is mutual and that mutually conjugate problems always have analogous boundary conditions; that is, either they are both Dirichlet's problems or they are both Neumann's problems.

If both functions f and ψ are identically zero, the corresponding problem is said to be *homogeneous*. Similarly, the conjugate problem is said to be homogeneous if both functions f^* and ψ^* are identically equal to zero. To every boundary-value problem these is one conjugate homogeneous problem.

4. *Fundamental solutions. Green's function*

Along with the regular solutions of equations of the elliptic type, an important role is played by the so-called *fundamental solutions*.

A fundamental solution of the equation $\mathcal{M}L = 0$ is a Lévy function $L(\xi, x)$ (see Chapter XVII, section 4) which, for $\xi \neq x$, satisfies this equation for the coordinates of one of the points x or ξ and depends on the coordinates of the other point as parameters. We shall write $\mathcal{M}_\xi L$ and $\mathcal{M}_x L$ depending on whether we are considering the coordinates of the point ξ or those of the point x as the variables with respect to which we are differentiating. We shall use the expressions $\mathcal{M}_x u$ and $\mathcal{M}_\xi u$ to indicate $\mathcal{M}_x u(x)$ and $\mathcal{M}_\xi u(\xi)$.

Consider Dirichlet's problem:

$$\mathcal{M}_x u = f \quad \text{when} \quad x \in V - \mathcal{F}V; \qquad u(x) = \psi \quad \text{when} \quad x \in \mathcal{F}V, \tag{15}$$

where f and ψ are continuous functions.

Let us suppose that both the solutions $u(x)$ of the problem (15) and the Lévy function $L(\xi, x)$ of the differential expression $\mathcal{M}_x u$ are continuous in a closed region V along with their first derivatives. If we apply the Green-Stokes formula (39) of Chapter XVII to the function $u(x)$, we obtain

$$u(x) = \iint\limits_{\mathcal{F}V} (L\mathcal{P}_\xi u - \psi Q_\xi L)\, dS_\xi - \iiint\limits_V (Lf - u\mathcal{N}_\xi L)\, dV_\xi \,. \tag{16}$$

If there is a fundamental solution to the homogeneous problem

$$\begin{aligned} \mathcal{N}_\xi G(\xi, x) &= 0 \quad \text{when} \quad \xi \in V - \mathcal{F}V - x \quad , \\ G(\xi, x) &= 0 \quad \text{when} \quad \xi \in \mathcal{F}V \,, \quad x \in V - \mathcal{F}V \,, \end{aligned} \tag{17}$$

which is conjugate to Dirichlet's problem (15), and if this solution and its first derivatives are continuous in V, we may set $L(\xi, x) = G(\xi, x)$. Formula (16) then takes the form

$$u(x) = - \iint\limits_{\mathcal{F}V} \psi(\xi)\, Q_\xi G(\xi, x)\, dS_\xi - \iiint\limits_V f(\xi)\, G(\xi, x)\, dV_\xi \,. \tag{18}$$

Thus, if there exist both a solution to Dirichlet's problem (15) and a fundamental solution to the homogeneous conjugate problem and if these solutions and their partial derivatives with respect to the coordinates of the point ξ are continuous in the region V, then we can seek, instead of the solution to the problem (15), the fundamental solution of the homogeneous conjugate problem, after which the solution of the problem (15) will be defined by formula (18). This principle is at the basis of Green's method of solving Dirichlet's problem.

The fundamental solution of the homogeneous problem (17) is called Green's function of Dirichlet's problem (15).

In an analogous manner, we can get Green's function for Neumann's problem. Consider the problem

$$\mathcal{M}_x u = f \quad \text{when} \quad x \in V - \mathcal{F}V \,; \qquad \mathcal{P}u = \psi \quad \text{when} \quad x \in \mathcal{F}V \,. \tag{19}$$

Suppose that $u(x)$ is a solution to this problem continuous in a closed region V along with its first derivatives. If we apply the Green-Stokes' formula, we get

$$u(x) = \iint\limits_{\mathcal{F}V} (L\psi - uQ_\xi L)\, dS_\xi - \iiint\limits_V (Lf - v\mathcal{N}_\xi L)\, dV_\xi \,.$$

Suppose that $G(\xi, x)$ is a fundamental solution of the homogeneous problem

$$\begin{aligned} \mathcal{N}_\xi G(\xi, x) &= 0 \quad \text{when} \quad \xi \in V - \mathcal{F}V - x \quad , \\ Q_\xi G(\xi, x) &= 0 \quad \text{when} \quad \xi \in \mathcal{F}V \,, \quad x \in V - \mathcal{F}V \,, \end{aligned} \tag{20}$$

which is conjugate to the problem (19). If this solution and its first derivatives are continuous in the region V, then, by setting $L(\xi, x)$ equal to $G(\xi, x)$ we obtain

$$u(x) = \iint\limits_{\mathcal{F}V} \psi(\xi)\, G(\xi, x)\, dS_\xi - \iiint\limits_V f(\xi)\, G(\xi, x)\, dV_\xi \,. \tag{21}$$

Thus, if the function $G(\xi, x)$ is found by some method or other and if it satisfies the smoothness requirements, the solution of the problem (19) which,

together with its first derivatives, is continuous in the closed region V can be found by means of formula (21).

The fundamental solution of the problem (20) is called Green's function of the problem (19). The names "second Green's function" and "characteristic Neumann function" are also used.

Let us consider two mutually conjugate boundary-value problems and let us assume that their Green's functions $G(\xi, x)$ and $G^*(\xi, x)$ exist. By definition,

$$\mathcal{M}_\xi G^*(\xi, x) = 0 , \quad \mathcal{N}_\xi G(\xi, x) = 0 \quad \text{when} \quad \xi \in V - \mathcal{F}V - x , \tag{22}$$

$$G^*(\xi, x) = 0 , \quad G(\xi, x) = 0 \quad \text{when} \quad \xi \in \mathcal{F}V , \quad x \in V + \mathcal{F}V , \tag{23}$$

or

$$\mathcal{P}_\xi G^*(\xi, x) = 0 , \quad Q_\xi G(\xi, x) = 0 \quad \text{when} \quad \xi \in \mathcal{F}V , \quad x \in V - \mathcal{F}V . \tag{24}$$

The first boundary condition is satisfied for Dirichlet's problem and the second for Neumann's problem.

Let us suppose also that the functions $G(\xi, x)$ and $G^*(\xi, x)$ have continuous first derivatives with respect to the coordinates of the point ξ in the region $V - x$. Then, if we fix two points $x = x'$ and $x = x''$ (where $x' \neq x''$), we may apply Green's theorem (15) of Chapter XVII to the functions $G(\xi, x')$ and $G(\xi, x'')$ in the region $V - V_1(x', \rho) - V_1(x'', \rho)$, where $V_1(x', \rho)$ and $V_1(x'', \rho)$ are ellipsoidal neighbourhoods of the points x' and x'' defined by inequalities of the form (27) of Chapter XVII. Recalling the relationship (22) - (24), we obtain

$$\iint\limits_{\mathcal{F}V_1(x', \rho) + \mathcal{F}V_1(x'', \rho)} [G(\xi, x') \, \mathcal{P}_\xi G^*(\xi, x'') - G^*(\xi, x'') \, Q_\xi G(\xi, x')] \, dS_\xi = 0 .$$

Let us take the limit as ρ approaches zero. On the basis of the considerations made in the derivation of the Green-Stokes' formula (Chapter XVII, section 5), we note that the following limiting relationships are valid:

$$\lim_{\rho \to 0} \iint\limits_{\mathcal{F}V_1(x', \rho)} [G(\xi, x') \, \mathcal{P}_\xi G^*(\xi, x'') - G^*(\xi, x'') \, Q_\xi G(\xi, x')] \, dS_\xi = -G^*(x', x'') ,$$

$$\lim_{\rho \to 0} \iint\limits_{\mathcal{F}V_1(x'', \rho)} [G(\xi, x') \, \mathcal{P}_\xi G^*(\xi, x'') - G^*(\xi, x'') \, Q_\xi G(\xi, x')] \, dS_\xi = G(x'', x') .$$

We then obtain the formula

$$G^*(x', x'') = G(x'', x') , \tag{25}$$

which connects Green's functions of the conjugate boundary-value problem. In particular, if the differential expression $\mathcal{M}u$ is self-conjugate then, $G^*(\xi, x) = G(\xi, x)$, and it then follows from formula (25) that

$$G(x', x'') = G(x'', x') . \tag{26}$$

Thus, if, for a self-conjugate boundary-value problem stated for the region V, there exists a Green's function $G(\xi, x)$ that is continuous along with

its first derivative in the region $V-x$, then this function is symmetric with respect to the point ξ and x.

5. *Uniqueness theorem*

If we set $v = 1$ and $u = w^2$ in Green's theorem (15) of Chapter XVII, we obtain, after some manipulations,

$$\iiint_V \Big(w\mathcal{M}w + \sum_{\alpha,\beta=1}^{3} a_{\alpha\beta}\frac{\partial w}{\partial x_\alpha}\frac{\partial w}{\partial x_\beta}\Big)\,dV$$
$$= \iiint_V \Big(c - \tfrac{1}{2}\sum_{\alpha=1}^{3}\frac{\partial e_\alpha}{\partial x_\alpha}\Big) w^2\,dV + \iint_{\mathcal{F}V} w\mathcal{P}w\,dS + \iint_{\mathcal{F}V}(\tfrac{1}{2}b - g)\,w^2\,dS\,. \tag{27}$$

Suppose that u_1 and u_2 are two solutions to Dirichlet's problem:

$$\mathcal{M}u = f \quad \text{when} \quad x \in V - \mathcal{F}V\,; \qquad u = \psi \quad \text{when} \quad x \in \mathcal{F}V\,, \tag{28}$$

satisfying the conditions under which Green's formula is valid. The difference $w = u_1 - u_2$ of these solutions is a solution to the homogeneous Dirichlet problem:

$$\mathcal{M}w = 0 \quad \text{when} \quad x \in V - \mathcal{F}V\,, \qquad w = 0 \quad \text{when} \quad x \in \mathcal{F}V\,,$$

which satisfies the same requirements. If we substitute the difference w in formula (27), we obtain

$$\iiint_V \sum_{\alpha,\beta=1}^{3} a_{\alpha\beta}\frac{\partial w}{\partial x_\alpha}\frac{\partial w}{\partial x_\beta}\,dV = \iiint_V \Big(c - \tfrac{1}{2}\sum_{\alpha=1}^{3}\frac{\partial e_\alpha}{\partial x_\alpha}\Big) w^2\,dV\,. \tag{29}$$

The left side of this equation is non-negative because of the inequality

$$\sum_{\alpha,\beta=1}^{3} a_{\alpha\beta}\,\lambda_\alpha\lambda_\beta > 0 \quad \text{for} \quad \sum_{\alpha=1}^{3}\lambda_\alpha^2 > 0\,.$$

If

$$c - \tfrac{1}{2}\sum_{\alpha=1}^{3}\frac{\partial e_\alpha}{\partial x_\alpha} \leqslant 0\,, \tag{30}$$

the right side will be non-positive, and so eq. (29) is possible only if both sides are equal to zero. Because of the continuity of the function w and the zero boundary condition, it follows that the function $w = 0$, that is, that $u_1 = u_2$ in the region V.

Thus, when the inequality (30) holds, Dirichlet's problem has no more than one solution that is continuous and has continuous first derivatives in the region.

We might prove this uniqueness theorem in another manner by showing that the requirement that the derivatives of the solution in the closed region V be continuous is superfluous and that it is sufficient to require continuity of the solution itself.

Consider Neumann's problem:

$$\mathcal{M}u = f \quad \text{when} \quad x \in V - \mathcal{F}V \; ; \qquad \mathcal{P}u = \psi \quad \text{when} \quad x \in \mathcal{F}V \; . \tag{31}$$

Suppose that u_1 and u_2 are two solutions of the problem that satisfy the conditions under which Green's theorem is applicable. The difference $w = u_1 - u_2$ is a solution of the homogeneous problem

$$\mathcal{M}w = 0 \quad \text{when} \quad x \in V - \mathcal{F}V \; ; \qquad \mathcal{P}w = a\,\frac{dw}{d\nu} + gw = 0 \quad \text{when} \quad x \in \mathcal{F}V \; . \tag{32}$$

Applying formula (27) to the difference w, we obtain

$$\iiint\limits_V \sum_{\alpha,\beta=1}^{3} a_{\alpha\beta}\,\frac{\partial w}{\partial x_\alpha}\,\frac{\partial w}{\partial x_\beta}\,dV = \iiint\limits_V \left(c - \tfrac{1}{2}\sum_{\alpha=1}^{3}\frac{\partial e_\alpha}{\partial x_\alpha}\right) w^2\,dV + \iint\limits_{\mathcal{F}V} \left(\tfrac{1}{2}b - g\right) w^2\,dS \; . \tag{33}$$

If

$$c - \tfrac{1}{2}\sum_{\alpha=1}^{3}\frac{\partial e_\alpha}{\partial x_\alpha} \leqslant 0 \; , \qquad \tfrac{1}{2}b - g \leqslant 0 \; , \tag{34}$$

then, as can easily be seen from this integral relationship,

$$\frac{\partial w}{\partial x_i} = 0 \qquad (i = 1, 2, 3) \qquad \text{when} \qquad x \in V \; , \tag{35}$$

so that the problem (32) takes the form

$$\mathcal{M}w = cw = 0 \quad \text{when} \quad x \in V - \mathcal{F}V \; ; \qquad \mathcal{P}w = gw = 0 \quad \text{when} \quad x \in \mathcal{F}V \; .$$

It then follows that, if the inequality (34) holds, then if at least one of the functions g and c is not identically equal to zero, w must be equal to zero. It also follows from eq. (33) that, since the function w is continuous, $w = 0$ if at least one of the inequalities (34) is a strict inequality. However, if neither of these conditions holds, it follows from (35) that w = constant.

Thus, when $g \neq 0$ and the conditions (34) are satisfied, Neumann's problem has no more than one continuous solution with a continuous first derivative in the region V. For $g = 0$, the solutions to Neumann's problem can differ by only a constant term. If at least one of the inequalities (34) is a strict inequality or if the function c is not identically equal to zero, this constant must be zero.

Problems

1. Show that if Dirichlet's problem has only one solution admitting the application of Green's theorem, its conjugate problem also has no more than one such solution.

2. Show that the self-conjugate Dirichlet problem has no more than one solution admitting application of Green's theorem if c is non-positive and

that the self-conjugate Neumann problem has no more than one such solution if in addition g is positive.

6. *Conditions of solubility of boundary-value problems*

Up to now, we have been considering boundary-value problems under the assumption that their solutions exist and satisfy certain additional requirements. The proof of the existence of solutions to boundary-value problems is an extremely complicated question which requires the development of a special mathematical apparatus that goes far beyond the framework of that usually used in the study of specific physical situations. As a consequence, except for certain solubility conditions that follow immediately from Green's theorem, we shall confine ourselves to an exposition of the basic results regarding the existence of solutions to boundary-value problems and shall omit the proof.

Suppose that u is a solution to Dirichlet's problem

$$\mathcal{M}u = f \quad \text{when} \quad x \in V - \mathcal{F}V \; ; \qquad u = \psi \quad \text{when} \quad x \in \mathcal{F}V \; , \tag{36}$$

that is continuous and has continuous first derivatives in the region V and that v is a solution of the homogeneous conjugate problem

$$\mathcal{N}v = 0 \quad \text{when} \quad x \in V - \mathcal{F}V \; ; \qquad v = 0 \quad \text{when} \quad x \in \mathcal{F}V \; , \tag{37}$$

satisfying the same conditions. We shall assume that the functions f and ψ are continuous. Under these assumptions, we may apply Green's theorem (15) of Chapter XVII. Making the corresponding substitutions from eqs. (36) and (37) in it, we obtain

$$\iiint\limits_V fv \, \mathrm{d}V + \iint\limits_{\mathcal{F}V} a\psi \, \frac{\mathrm{d}v}{\mathrm{d}\nu} = 0 \; . \tag{38}$$

In the same way, for Neumann's problem,

$$\mathcal{M}u = f \quad \text{when} \quad x \in V - \mathcal{F}V \; ; \qquad \mathcal{P}u = \psi \quad \text{when} \quad x \in \mathcal{F}V \; ,$$

we obtain

$$\iiint\limits_V fv \, \mathrm{d}V - \iint\limits_{\mathcal{F}V} v\psi \, \mathrm{d}S = 0 \; , \tag{39}$$

where v is a solution of the homogeneous conjugate problem

$$\mathcal{N}v = 0 \quad \text{when} \quad x \in V - \mathcal{F}V \; ; \qquad \mathcal{P}v = 0 \quad \text{when} \quad x \in \mathcal{F}V \; , \tag{40}$$

which, along with its first derivatives, is continuous in the region V.

Thus, in connection with the solutions of the boundary-value problems that admit application of Green's theorem, we encounter the following alternatives: Either the solutions of the homogeneous conjugate problems that are continuous and have continuous first derivatives in the region in question are identically equal to zero or the boundary-value problems are soluble only when the appropriate condition (38) or (39) is satisfied.

This property is closely associated with the uniqueness conditions. For if an inhomogeneous boundary-value problem has no more than one solution, then, as we saw in the preceding section, the solution of the homogeneous problem corresponding to it is identically equal to zero. Therefore, the uniqueness of the solution to the problem conjugate with the boundary-value problem that we are considering implies the vanishing of the function v and the identical satisfaction of the corresponding condition (38) or (39). In particular, if the self-conjugate problem has no more than one solution, the condition (38) or (39) is identically satisfied.

The case in which the uniqueness theorem does not hold is of great interest. Before stating the results that are applicable here, let us recall the definition of a Hölder condition. A function φ is said to satisfy this condition in a region V if the ratio $|\varphi(x') - \varphi(x'')|/r^{\lambda}$, where r is the distance between the points x' and x'' and λ is some positive number, is bounded for an arbitrary choice of the points x' and x'' belonging to V.

Consider the elliptic-type equation

$$\mathcal{M}u \equiv \sum_{\alpha,\beta=1}^{3} \frac{\partial}{\partial x_\alpha} a_{\alpha\beta} \frac{\partial u}{\partial x_\beta} + \sum_{\alpha=1}^{3} e_\alpha \frac{\partial u}{\partial x_\alpha} + cu = f\,, \tag{41}$$

in which the coefficients a_{ij}, e_i, and c (where $i,j = 1,2,3$) and the free term f are defined in a closed region V and the first derivatives of the coefficients a_{ij} and e_i and the coefficient c are continuous and satisfy a Hölder condition in the region V. Suppose also that the free term f is continuous in V and that it satisfies a Hölder condition in the region $V - \mathcal{F}V$.

Under these conditions, if c is non-positive, the Dirichlet problem

$$\mathcal{M}u = f \quad \text{when} \quad x \in V - \mathcal{F}V\,; \qquad u = \psi \quad \text{when} \quad x \in \mathcal{F}V \tag{42}$$

has a unique solution if the function ψ is continuous on the boundary $\mathcal{F}V$. If the condition (30) is satisfied, the Dirichlet problem (42) and its conjugate problem

$$\mathcal{N}u = f^* \quad \text{when} \quad x \in V - \mathcal{F}V\,; \qquad u = \psi^* \quad \text{when} \quad x \in \mathcal{F}V\,,$$

where the functions f^* and ψ^* have the same properties as the functions f and ψ respectively, have unique solutions.

If the coefficient c is positive, one of the following two alternatives holds: Either the homogeneous mutually conjugate problems

$$\mathcal{M}u = 0 \quad \text{when} \quad x \in V - \mathcal{F}V\,; \qquad u = 0 \quad \text{when} \quad x \in \mathcal{F}V\,, \tag{43}$$

$$\mathcal{N}v = 0 \quad \text{when} \quad x \in V - \mathcal{F}V\,; \qquad v = 0 \quad \text{when} \quad x \in \mathcal{F}V\,, \tag{44}$$

both have no solution not identically equal to zero (in which case the Dirichlet problem (42) has a unique solution) or these problems have each the same number m of linearly independent solutions $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_m$ (in which case the Dirichlet problem (42) is soluble only if the integral relationships of the form (38) are satisfied for each of the solutions $v_1, v_2, \ldots, v_m$). When this last condition is satisfied, Dirichlet's problem has an infinite number of solutions. If u is one of them, all the remaining ones can be represented in the form

$$u + \sum_{\alpha=1}^{m} c_\alpha u_\alpha ,$$

where the c_α are constants. This last remark shows that a solution u of Dirichlet's problem that is orthogonal to all the solutions $u_1, u_2, \ldots, u_m$ of the homogeneous problem (43) is unique For suppose that the solution u satisfies the orthogonality condition

$$\iiint\limits_V u u_i \, dV = 0 , \qquad i = 1, 2, \ldots, m .$$

Any other solution $\tilde{u}$ of the problem can, according to what has been said, be represented in the form

$$\tilde{u} = u + \sum_{\alpha=1}^{m} c_\alpha u_\alpha .$$

But for at least one coefficient $c_i \neq 0$, the integral

$$\iiint\limits_V \tilde{u} u_i \, dV = c_i \iiint\limits_V u_i^2 \, dV$$

is different from zero, from which the assertions made follows.

If the region V is sufficiently small, Dirichlet's problem (42) will always have a unique solution.

Let us turn now to a formulation of the conditions under which Green's functions of Dirichlet's problem exist.

If Dirichlet's problem (42) has a unique solution $u(x)$, then Green's function $G(\xi, x)$ of this problem will exist and

$$u(x) = - \iint\limits_{\mathcal{F}V} \psi(\xi) \, Q_\xi(\xi, x) \, dS_\xi - \iiint\limits_V f(\xi) \, G(\xi, x) \, dV_\xi . \tag{18}$$

If Dirichlet's problem has a non-unique solution, we can still construct a function $G(\xi, x)$, called the generalized Green's function, so that the solution is still given by formula (18). This function is not uniquely defined. For example, one may take for the generalized Green's function the fundamental solutions of the boundary-value problem

$$\mathcal{M}_x G(\xi, x) = \sum_{\alpha=1}^{m} v_\alpha(\xi) \, v_\alpha(x) \quad \text{when} \quad x \in V - \mathcal{F}V ; \tag{45}$$

$$G(\xi, x) = 0 \quad \text{when} \quad x \in \mathcal{F}V ,$$

satisfying the additional requirement of orthogonality:

$$\iiint\limits_V u_i(\xi) \, G(\xi, x) \, dV_\xi = 0 \quad (i = 1, 2, \ldots, m) . \tag{46}$$

Then, the function $u(x)$, defined by formula (18) is a solution that is ortho-

gonal to all the solutions $u_1, u_2, \ldots, u_m$ of the homogeneous problem (43). As we have shown, such a solution is unique.

Analogous results hold for the Neumann problem

$$\mathcal{M}u = f \quad \text{when} \quad x \in V - \mathcal{F}V\,; \qquad \mathcal{P}u = a\,\frac{du}{d\nu} + gu = \psi \quad \text{when} \quad x \in \mathcal{F}V\,, \tag{47}$$

if the functions ψ and g are continuous on $\mathcal{F}V$.

If $c \leqslant 0, g > 0$ or $c < 0, g \geqslant 0$, problem (47) will have a unique solution. However, if inequalities (34) hold and at least one of the functions c and g is not identically equal to zero, the problem (47) and its conjugate problem will, with continuous boundary conditions, have the same solution.

If these conditions are not satisfied the following alternatives exist: Either the homogeneous conjugate problems

$$\mathcal{M}u = 0 \quad \text{when} \quad x \in V - \mathcal{F}V\,; \qquad \mathcal{P}u = 0 \quad \text{when} \quad x \in \mathcal{F}V\,, \tag{48}$$

$$\mathcal{N}v = 0 \quad \text{when} \quad x \in V - \mathcal{F}V\,; \qquad Qu = 0 \quad \text{when} \quad x \in \mathcal{F}V\,, \tag{49}$$

have no solutions not identically equal to zero (in which case, the problem (47) has a unique solution), or these problems have each the same number m of linearly independent solutions $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_m$ (and then the problem (47) is soluble only when the integral relationships of the form (39) are satisfied for each of the solutions $v_1, v_2, \ldots, v_m$). When this last condition is satisfied, the problem (47) has an infinite number of solutions and all of them can be represented in the form

$$u + \sum_{\alpha=1}^{m} c_\alpha u_\alpha\,,$$

where the c_α are constants and u is any solution to the problem (47). The solution u is orthogonal to all the functions u_i and is thus unique.

If $c = 0$ and $g = 0$, then $m = 1$, $u_1 =$ constant, and the solution of the problem (47) is determined up to an additive constant.

Green's function of the problem (47) always exists when the solution to the problem (47) is unique. This solution can be represented by means of formula (21). If the solution to the problem (47) is not unique, the various solutions can yet be represented in the form (21) by suitably defined generalized Green's functions.

PART III

EQUATIONS OF THE PARABOLIC TYPE

Chapter XXVIII

THE SIMPLEST PROBLEMS LEADING TO THE HEAT-FLOW EQUATION. SOME GENERAL THEOREMS

1. *The heat-flow equation in an isotropic solid body. Initial and boundary conditions*

Consider a solid body whose temperature at a point (x,y,z) at an instant of time t is determined by a function $u(x, y, z, t)$. If the different parts of the body are at different temperatures, heat will flow from the hotter portions to the cooler ones. Let us take some surface S within the body and some small element ΔS of that surface near a point $M(x, y, z)$. It is assumed in the theory of heat-flow that the amount of heat ΔQ passing through the element ΔS in the time Δt is proportional to $\Delta t \Delta S$ and to the normal derivative of the temperature $\partial u/\partial n$, that is, that

$$\Delta Q = -k(x, y, z)\ \Delta t \cdot \Delta S \frac{\partial u}{\partial n}, \tag{1}$$

where $k(x, y, z)$, the internal heat conductivity, is positive and n is the normal to the element of the surface ΔS in the direction of decreasing temperature. We also assume that the body in question is isotropic with regard to heat conductivity, that is, that the function $k(x, y, z)$ does not depend on the direction of the normal to the surface S at the point (x, y, z).

We denote by q the heat-flow, that is, the amount of heat passing through a unit of area of the surface in unit time. Then, (1) can be written in the form

$$q = -k \frac{\partial u}{\partial n}. \tag{2}$$

To derive the heat-flow equation, we take an arbitrary volume V within the body that is bounded by a smooth closed surface S and we observe the variation in the quantity of heat in that volume during the interval of time (t_1, t_2). It is easy to see that the amount of heat flowing through the surface S in the interval (t_1, t_2) is, according to (1), equal to

$$Q_1 = -\int_{t_1}^{t_2} \mathrm{d}t \iint_S k(x, y, z) \frac{\partial u}{\partial n} \mathrm{d}S,$$

where n is the inner normal to the surface S.

Consider an element of volume ΔV. An amount of heat

$$\Delta Q_2 = [u(x, y, z, t + \Delta t) - u(x, y, z, t)]\, c(x, y, z)\, \rho(x, y, z)\, \Delta V ,$$

where $\rho(x, y, z)$ is the density of the substance in question, and $c(x, y, z)$ is its specific heat, is needed to change the temperature of this volume by an amount Δu in the time interval Δt. Thus, the amount of heat needed to change the temperature of the volume V by an amount $\Delta u = u(x, y, z, t_2) - u(x, y, z, t_1)$, is equal to

$$Q_2 = \iiint_V [u(x, y, z, t_2) - u(x, y, z, t_1)]\, c\rho \, \mathrm{d}V$$

or

$$Q_2 = \int_{t_1}^{t_2} \mathrm{d}t \iiint_V c\rho \frac{\partial u}{\partial t} \mathrm{d}V ,$$

since

$$u(x, y, z, t_2) - u(x, y, z, t_1) = \int_{t_1}^{t_2} \frac{\partial u}{\partial t} \mathrm{d}t .$$

Suppose that there are sources of heat within the body. We denote by $F(x, y, z, t)$ the density of heat sources (that is, the amount of heat absorbed or liberated per unit of time in a unit of volume of the body). Then, the amount of heat liberated or absorbed in the volume V in the time interval (t_1, t_2) will be equal to

$$Q_3 = \int_{t_1}^{t_2} \mathrm{d}t \iiint_V F(x, y, z, t) \mathrm{d}V .$$

Let us now set up the equations for thermal equilibrium for the volume V that we are considering. Obviously, $Q_2 = Q_1 + Q_3$; that is,

$$\int_{t_1}^{t_2} \mathrm{d}t \iiint_V c\rho \frac{\partial u}{\partial t} \mathrm{d}V = - \int_{t_1}^{t_2} \mathrm{d}t \iint_S k \frac{\partial u}{\partial n} \mathrm{d}S + \int_{t_1}^{t_2} \mathrm{d}t \iiint_V F(x, y, z, t)\, \mathrm{d}V ,$$

or, by applying Ostrogradskii's formula to the integral over the surface S, we obtain

$$\int_{t_1}^{t_2} \mathrm{d}t \iiint_V \left[c\rho \frac{\partial u}{\partial t} - \mathrm{div}\,(k \,\mathrm{grad}\, u) - F(x, y, z, t)\right] \mathrm{d}V = 0 .$$

Since the integrand is continuous, and since the volume V and the interval (t_1, t_2) are arbitrary, it follows that for an arbitrary point (x, y, z) in the body and for an arbitrary instant t,

$$c\rho \frac{\partial u}{\partial t} = \mathrm{div}\,(k \,\mathrm{grad}\, u) + F(x, y, z, t) \tag{3}$$

or

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial u}{\partial z}\right) + F(x, y, z, t) . \tag{3a}$$

This equation is called the heat-flow equation for an inhomogeneous isotropic body.

If the body is homogeneous, the quantities c, ρ, and k will be constant and eq. (3a) can be written in the form

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + f(x, y, z, t) , \tag{4}$$

where $a^2 = k/c\rho$ is the diffusivity, and

$$f(x, y, z, t) = \frac{F(x, y, z, t)}{c\rho} .$$

If there are no sources of heat in the homogeneous body that we are considering, that is, if $F(x, y, z, t) \equiv 0$, we obtain the *homogeneous heat-flow equation*

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) . \tag{5}$$

In the particular case in which the temperature depends only on the coordinates, x, y, and t, which, for example, is the case for the conduction of heat in a very thin homogeneous sheet, eq. (4) becomes

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f(x, y, t) . \tag{6}$$

Finally, for a body that is essentially one-dimensional, for example, for a homogeneous rod, the heat-flow equation takes the form

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x, t) . \tag{7}$$

Note that in this form eqs. (6) and (7) do not take into account the heat exchange between the surface of the sheet or rod and the neighbouring space.

To find the temperature within the body at an arbitrary instant, eq. (3a) is alone insufficient. As one can see from physical considerations, it is also necessary to know the temperature distribution within the body at an initial instant (the initial condition) and the conditions on the temperature at the surface of the body (the boundary condition).

The boundary condition can be given in different ways:

(1) At every point of the surface S, the temperature may be given:

$$u|_S = \psi_1(\mathrm{P}, t) , \tag{8a}$$

where $\psi_1(\mathrm{P}, t)$ is a known function of the point P on the surface S and of the time for $t \geqslant 0$.

(2) The heat flow on the surface S

$$q = -k \frac{\partial u}{\partial n}$$

may be given. Then,

$$\left.\frac{\partial u}{\partial n}\right|_S = \psi_2(\mathrm{P}, t) , \tag{8b}$$

where $\psi_2(P, t)$ is a known function expressed in terms of the given heat flow by the formula

$$\psi_2(P, t) = -\frac{q(P, t)}{k} .$$

(3) The temperature u_0 may be given for the medium surrounding the surface of the solid body. The law of the heat exchange between the surface of the body and this medium is very complicated, but to simplify the problem it can be assumed to have the form of Newton's law.

According to Newton's law, the amount of heat transmitted in unit time from a unit of surface area to the surrounding medium is proportional to the temperature difference between the surface of the body and the surrounding medium: $q = \alpha(u - u_0)$, where α is the coefficient of heat exchange. The coefficient of heat exchange depends on the temperature difference, $u - u_0$, and on the natures of the surface and the surrounding medium. (It may vary from point to point on the surface of the body.) We shall assume the coefficient of heat exchange to be constant with respect to temperature and uniform for the entire surface of the body.

From the law of conservation of energy, this amount of heat must be equal to the amount that is transmitted through a unit of area of the surface in unit time as a result of the internal thermal conductivity. This leads us to the boundary condition

$$\alpha(u - u_0) = -k \frac{\partial u}{\partial n} \quad (\text{on } S) ,$$

where n is the outer normal to the surface S. Setting $h = \alpha/k$, we have

$$\frac{\partial u}{\partial n} + h(u - u_0)|_S = 0 . \tag{8c}$$

Thus, the problem of the propagation of heat in an isotropic solid body is seen to be that of finding the solution to eq. (3a) thst satisfies the initial condition

$$u|_{t=0} = \varphi(x, y, z) \tag{9}$$

and one of the boundary conditions (8).

2. *The diffusion equation*

The basic law of diffusion in a motionless medium is Fick's law, which states that the diffusional flow is proportional to the gradient of the concentration

$$q = -D \frac{\partial C}{\partial n} , \tag{10}$$

where C is the concentration of the diffusing substance and q is the diffusional flow, that is, the amount of the substance that passes through a unit of area of a surface in unit time. D is called the coefficient of diffusion.

Diffusion in a motionless medium can be observed in pure form only in solid bodies because in liquids and gases the motion of the liquid or gas – free or forced convection – will necessarily be superimposed on these processes.

It is obvious from formula (10) that Fick's law of diffusion is analogous to Fourier's hypothesis in the theory of heat conduction (see formula (2)). Following the same reasoning as in the derivation of the heat-flow equation, we obtain the following equation for the diffusion in a motionless medium:

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial z}\left(D\frac{\partial C}{\partial z}\right) + F(x, y, z, t) , \tag{11}$$

where F is the density of the sources of matter, that is, the amount of matter that is formed as a result of chemical reaction in unit volume in unit time.

If the coefficient of diffusion D is constant and $F = 0$, then eq. (11) will take the form

$$\frac{\partial C}{\partial t} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right) . \tag{12}$$

3. *The heat-flow equation in a torus*

Suppose that we have a torus whose cross section is sufficiently small that the temperature can be assumed to be the same at all points of a given cross section. The initial temperature of the torus is given and we are to determine the temperature at an arbitrary subsequent instant after it has started cooling as a result of the internal heat conduction and the heat exchange with the surrounding medium.

We denote by x the coordinate determining the position of a point on the circle passing through the centers of the cross sections of the torus as measured from some fixed point on that circle. Consider that part of the ring between two near-by cross sections x_1 and x_2. The amount of heat flowing into this part of the torus through the sections $x = x_1$ in unit time is equal to

$$\left(-k\sigma\frac{\partial u}{\partial x}\right)_{x=x_1} ,$$

where σ is the area of the generating cross section. The amount of heat flowing out of this portion through the cross sections $x = x_2$ in unit time is equal to

$$\left(-k\sigma\frac{\partial u}{\partial x}\right)_{x=x_2} .$$

Hence, the heat increment in this portion of the torus that is caused by the flow of heat through the two cross sections in unit time is expressed by

$$\left(k\sigma \frac{\partial u}{\partial x}\right)_{x=x_2} - \left(k\sigma \frac{\partial u}{\partial x}\right)_{x=x_1} = \int_{x_1}^{x_2} k\sigma \frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x \,.$$

The heat loss due to the exchange on the surface of this part of the torus is equal to

$$\int_{x_1}^{x_2} \alpha(u - u_0)\, p\, \mathrm{d}x \,,$$

where α is the coefficient of heat exchange, u_0 is the temperature of the surrounding medium, and p is the perimeter of the cross section of the torus. Thus, the net heat increment in this portion of the torus in unit time is obviously equal to

$$\int_{x_1}^{x_2} k\sigma \frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x - \int_{x_1}^{x_2} \alpha(u - u_0)\, p\, \mathrm{d}x \,. \tag{13}$$

On the other hand, this amount of heat is equal to

$$\int_{x_1}^{x_2} c\rho\sigma \frac{\partial u}{\partial t}\,\mathrm{d}x \,, \tag{14}$$

where c is the specific heat and ρ is the density of the torus. Equating the expressions (13) and (14), we obtain

$$\int_{x_1}^{x_2} \left[c\rho\sigma \frac{\partial u}{\partial t} - k\sigma \frac{\partial^2 u}{\partial x^2} + \alpha(u - u_0)p\right] \mathrm{d}x = 0 \,,$$

from which, because of the arbitrariness of x_1 and x_2, we obtain the differential equation for the heat flow in the torus:

$$c\rho\sigma \frac{\partial u}{\partial t} = k\sigma \frac{\partial^2 u}{\partial x^2} - \alpha p(u - u_0)$$

or

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} - b(u - u_0) \,, \tag{15}$$

where

$$a^2 = k/c\rho \,, \qquad b = p\alpha/c\rho\sigma .$$

If the external temperature u_0 is constant, eq. (15) will, by the simple substitution

$$u = u_0 + v\, \mathrm{e}^{-bt} \,, \tag{16}$$

lead to the equation

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2} \,. \tag{17}$$

It should be noted that these considerations are applicable to an arbi-

trary dense rod of small cross sections, the axis of which forms a closed curve which does not intersect itself, and if the coefficient of internal heat conductivity may not be considered constant, we arrive at the following equation:

$$c\rho\sigma \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k\sigma \frac{\partial u}{\partial x}\right) - p\alpha(u - u_0) . \tag{18}$$

4. *An extreme-value theorem. The uniqueness of the solution to the first boundary-value problem*

Consider the homogeneous heat-flow equation in a bounded region Ω of space (x, y, z):

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{19}$$

with the initial and boundary conditions

$$u|_{t=0} = \varphi(x, y, z) \quad \text{(in the region } \Omega) \ ; \qquad u|_S = \Psi(\mathrm{P}, t) \quad (t \geqslant 0) \ , \tag{20}$$

where S is the boundary of the region Ω. The functions φ and Ψ are continuous; so, the values of Ψ coincide for $t = 0$ with the values of φ on the boundary S. P is a point on the surface S.

The problem of finding the solution to eq. (19) with the conditions (20) is called the first boundary-value problem for the heat-flow equation.

In the four-dimensional space (x, y, z, t), let us denote by D the cylinder whose base is the region Ω and whose generators are parallel to the t-axis. Let D_T be that portion of the cylinder bounded below by the plane $t = 0$ and above by the plane $t = T$ (where T is positive). We denote by Γ the lower base ($t = 0$) and the lateral surface of the cylinder.

THEOREM. *The function $u(x, y, z, t)$ that satisfies the homogeneous heat-flow equation* (19) *within the cylinder D_T and that is continuous up to its boundary has its largest and its smallest value on Γ, that is, either where $t = 0$ (the lower base) or on the boundary S of the region Ω.*

Since the theorem for the minimum reduces to the theorem for the maximum if we change the sign of $u(x, y, z, t)$, we can confine ourselves to proving the maximum theorem.

Let us denote by M the greatest value of the function $u(x, y, z, t)$ within the cylinder D_T and let us denote by m the largest value of $u(x, y, z, t)$ on Γ. Let us suppose that there exists a solution $u(x, y, z, t)$ for which M exceeds m, that is, for which the maximum theorem does not hold. Suppose that this function takes the value M at the point (x_0, y_0, z_0, t_0), where (x_0, y_0, z_0) belongs to Ω and $0 < t_0 \leq T$.

Consider the function

$$v(x, y, z, t) = u(x, y, z, t) + \frac{M - m}{6d^2}\left[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right] ,$$

where d is the diameter of the region Ω.

On the lateral surface of the cylinder D_T and on its lower base,

$$v(x, y, z, t) \leqslant m + \tfrac{1}{6}(M - m) = \tfrac{1}{6}M + \tfrac{5}{6}m = \theta M ,$$

where $0 < \theta < 1$ and

$$v(x_0, y_0, z_0, t_0) = M .$$

Consequently, the function $v(x, y, z, t)$, like the function $u(x, y, z, t)$, does not have its maximum value either on the letaral surface of D_T or on it's lower base. Suppose that $v(x,y,z,t)$ has it greatest value at the point (x_1, y_1, z_1, t_1), where (x_1, y_1, z_1) belongs to Ω and $0 < t_1 \leqslant T$. Then, at that point, the second derivatives

$$\frac{\partial^2 v}{\partial x^2}, \quad \frac{\partial^2 v}{\partial y^2}, \quad \frac{\partial^2 v}{\partial z^2}$$

are non-positive and $\partial v/\partial t$ is non-negative. (If t_1 is less than T, we have $\partial v/\partial t = 0$ and if $t_1 = T$, we have $\partial v/\partial t \geqslant 0$.) It then follows that at the point (x_1, y_1, z_1, t_1),

$$\frac{\partial v}{\partial t} - a^2 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \geqslant 0 . \tag{21}$$

On the other hand,

$$\frac{\partial v}{\partial t} - a^2 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \frac{\partial u}{\partial t} - a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - a^2 \frac{M - m}{d^2}$$
$$= -a^2 \frac{M - m}{d^2} < 0 .$$

This contradicts the inequality (21) and proves the theorem.

The following two facts follow immediately from this theorem:

(1) *The solution of the first boundary-value problem* (19) - (20) *in the cylinder D_T is unique*. For if we had two solutions of the problem u_1 and u_2, their difference $\omega = u_1 - u_2$ would satisfy the homogeneous equation (19), and would vanish for $t = 0$ and on the surface S of the region Ω. But then, because of the extreme-value theorem, it would follow that ω must be identically equal to zero in the region Ω for $0 \leqslant t \leqslant T$; that is, u_1 is identically equal to u_2.

(2) *The solution of the first boundary-value problem* (19) - (20) *is a continuous function of the right hand side of the initial and boundary conditions*. For if we change the functions φ and Ψ in (20) so that the difference between the old and new functions appearing respectively in the initial and boundary conditions does not exceed in absolute value some positive number ϵ. Then the difference $\omega = u_1 - u_2$ in the corresponding solutions, being a solution of the homogeneous heat-flow equation with small initial and boundary, would also not exceed ϵ in absolute value anywhere throughout the cylinder D_T.

5. *The uniqueness of the solution to the Cauchy problem*

The Cauchy problem is to find $u(x, t)$ (for non-negative t and $-\infty < x < +\infty$) satisfying the heat-flow equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (t > 0,\ -\infty < x < \infty) \tag{22}$$

and the initial condition

$$u|_{t=0} = \varphi(x) \qquad (-\infty < x < \infty)\ . \tag{23}$$

Let us prove the uniqueness of the solution to the Cauchy problem under the assumption that the solution $u(x, t)$ is bounded throughout the entire region, that is, that there exists a number M such that $|u(x, t)| < M$ for all $-\infty < x < +\infty$ and for arbitrary non-negative t.

Suppose that $u_1(x, t)$ and $u_2(x, t)$ are two solutions to eq. (22), both satisfying the initial condition (23). Then, the difference

$$\omega(x, t) = u_1(x, t) - u_2(x, t)$$

will satisfy eq. (22) and the initial condition

$$\omega|_{t=0} = 0\ .$$

Also, the function $\omega(x, t)$ is bounded throughout the entire region

$$|\omega(x, t)| \leq |u_1(x, t)| + |u_2(x, t)| \leq 2M\ .$$

We cannot immediately use the extreme-value theorem for an unbounded region because the function $\omega(x, t)$ may not have a maximum or a minimum anywhere. To use this theorem, we consider a finite region

$$|x| \leq L\ , \qquad 0 \leq t \leq T\ . \tag{24}$$

Let us take the function

$$v(x, t) = \frac{4M}{L^2}\left(\tfrac{1}{2}x^2 + a^2 t\right)\ ,$$

which is a solution of the heat-flow equation (22). It is easy to see that

$$v(x, 0) \geq \omega(x, 0) = 0\ , \qquad v(\pm L, t) \geq 2M \geq |\omega(\pm L, t)|\ .$$

By applying the extreme-value theorem to the difference between the functions $v(x, t)$ and $\pm\omega(x, t)$ in the region (24), we obtain

$$v(x, t) - \omega(x, t) \geq 0\ , \qquad v(x, t) + \omega(x, t) \geq 0\ ,$$

so that

$$-v(x, t) \leq \omega(x, t) \leq v(x, t)$$

or

$$|\omega(x, t)| \leq v(x, t) = \frac{4M}{L^2}\left(\tfrac{1}{2}x^2 + a^2 t\right)\ .$$

By fixing the values of (x, t) and letting L become infinite, we obtain $\omega(x, t) \equiv 0$, which proves the uniqueness of the solution to the Cauchy problem.

Chapter XXIX

HEAT-FLOW IN AN INFINITE ROD

1. *Heat-flow in an infinite rod*

The problem of heat-flow in an infinite rod whose lateral surface is thermally insulated is mathematically formulated as follows:

Find a bounded function $u(x,t)$ (where $t \geqslant 0$, $-\infty < x < \infty$) that satisfies the heat-flow equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (t > 0,\ -\infty < x < \infty) \tag{1}$$

and the initial condition

$$u|_{t=0} = \varphi(x) \qquad (-\infty < x < \infty) . \tag{2}$$

Let us first find particular solutions to eq. (1) of the form

$$u = T(t)\ X(x) . \tag{3}$$

Substituting this expression into eq. (1), we get

$$T'(t)\ X(x) = a^2\ T(t)\ X(x)$$

or

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 ,$$

where λ^2 is a constant. We thus obtain

$$T'(t) + a^2\lambda^2 T(t) = 0 , \qquad X''(x) + \lambda^2 X(x) = 0 ,$$

so that, if we discard the constant factor in the expression we find upon integrating for $T(t)$, we have

$$T(t) = e^{-a^2\lambda^2 t} , \qquad X(x) = A \cos \lambda x + B \sin \lambda x ,$$

where the constants A and B may depend on λ. Since there are no boundary conditions, the parameter λ is completely arbitrary.

From eq. (3), we obtain

$$u_\lambda(x,t) = e^{-a^2\lambda^2 t}\,[A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x] , \tag{4}$$

which is a particular solution to eq. (1) for arbitrary $A(\lambda)$ and $B(\lambda)$. If we integrate eq. (4) with respect to the parameter λ, we again obtain a solution to eq. (1):

$$u(x,t) = \int_{-\infty}^{\infty} e^{-a^2\lambda^2 t}\,[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,d\lambda \tag{5}$$

provided this integral converges and can be differentiated under the integral sign once with respect to t and twice with respect to x.

Let us choose $A(\lambda)$ and $B(\lambda)$ so that the initial condition (2) will be satisfied. If we set $t = 0$ in eq. (5), we obtain, from eq. (2)

$$\varphi(x) = \int_{-\infty}^{\infty} [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,d\lambda\ . \tag{6}$$

If we compare the integral on the right side with the Fourier integral for the function $\varphi(x)$:

$$\begin{aligned}\varphi(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} \varphi(\xi)\cos\lambda(\xi - x)\,d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\cos\lambda x \int_{-\infty}^{\infty} \varphi(\xi)\cos\lambda\xi\,d\xi + \sin\lambda x\int_{-\infty}^{\infty}\varphi(\xi)\sin\lambda\xi\,d\xi\right] d\lambda\ ,\end{aligned}$$

we see that we may satisfy eq. (6) by setting

$$A(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\xi)\cos\lambda\xi\,d\xi\ , \qquad B(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\xi)\sin\lambda\xi\,d\xi\ . \tag{7}$$

Substituting eq. (7) into eq. (5), we obtain

$$\begin{aligned}u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\int_{-\infty}^{\infty}\varphi(\xi)\,e^{-a^2\lambda^2 t}\cos\lambda(\xi - x)\,d\xi \\ &= \frac{1}{\pi}\int_{0}^{\infty} d\lambda\int_{-\infty}^{\infty}\varphi(\xi)\,e^{-a^2\lambda^2 t}\cos\lambda(\xi - x)\,d\xi\end{aligned}$$

or, by reversing the order of integration,

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(\xi)\,d\xi\int_{0}^{\infty} e^{-a^2\lambda^2 t}\cos\lambda(\xi - x)\,d\lambda\ . \tag{8}$$

The inner integral can be evaluated. Let us set

$$a\lambda\sqrt{t} = z\ , \qquad \lambda(\xi - x) = \mu z\ ,$$

so that

$$d\lambda = \frac{dz}{a\sqrt{t}}\,, \qquad \mu = \frac{\xi - x}{a\sqrt{t}}\ .$$

Therefore,

$$\int_0^\infty e^{-a^2\lambda^2 t}\cos\lambda(\xi - x)\,d\lambda = \frac{1}{a\sqrt{t}}\int_0^\infty e^{-z^2}\cos\mu z\,dz = \frac{1}{a\sqrt{t}}J(\mu)\,. \tag{9}$$

Differentiating the integral $J(\mu)$ with respect to the paremeter μ, we see that

$$J'(\mu) = -\int_0^\infty e^{-z^2} z\sin\mu z\,dz\,.$$

This differentiation is permissible because of the uniform convergence of the integral obtained. If we now integrate by parts, we obtain

$$J'(\mu) = -\tfrac{1}{2}\mu\int_0^\infty e^{-z^2}\cos\mu z\,dz = -\tfrac{1}{2}\mu\,J(\mu)\,,$$

so that

$$J(\mu) = C\,e^{-\frac{1}{4}\mu^2}$$

To find the constant C, let us set $\mu = 0$. This gives us

$$C = J(0) = \int_0^\infty e^{-z^2}\,dz = \tfrac{1}{2}\sqrt{\pi}\,.$$

Therefore,

$$J(\mu) = \tfrac{1}{2}\sqrt{\pi}\,e^{-\frac{1}{4}\mu^2}\,,$$

and, from eq. (9),

$$\int_0^\infty e^{-a^2\lambda^2 t}\cos\lambda(\xi - x)\,d\lambda = \frac{\sqrt{\pi}}{2a\sqrt{t}}\,e^{-(\xi-x)^2/4a^2 t}\,.$$

When we substitute this into eq. (8), we finally obtain

$$u(x,t) = \int_{-\infty}^{\infty}\varphi(\xi)\,\frac{1}{2a\sqrt{\pi t}}\,e^{-(\xi-x)^2/4a^2 t}\,d\xi\,. \tag{10}$$

We note that the function

$$v(x,t) = \frac{1}{2a\sqrt{\pi t}}\,e^{-(\xi-x)^2/4a^2 t}\,, \tag{11}$$

regarded as a function of x and t is a solution to eq. (1) for $x \neq \xi$ and positive t. The function (11) is called the fundamental solution of the heat-flow equation (1).

Let us show that for an arbitrary continuous and bounded function $\varphi(x)$ the function (10) satisfies the heat-flow equation (1) and the initial condition (2).

It is easy to verify that the integral (10) and the integrals that are obtained by differentiating it any number of times under the integral sign with respect to x and t converge uniformly in a neighbourhood of an arbitrary

point (x, t) if $t \geqslant t_0 > 0$. It then follows that for $t \geqslant t_0 > 0$, the function $u(x, t)$ defined by formula (10) and its partial derivatives with respect to x and t of all orders exist. Since the integrand satisfies eq. (1) for $t \geqslant t_0 > 0$, it follows that the function $u(x, t)$ itself satisfies this equation for $t \geqslant t_0 > 0$.

To verify that the initial condition (2) is satisfied, we introduce, in place of ξ, a new variable α, defined by

$$\alpha = \frac{\xi - x}{2a\sqrt{t}},$$

where we are assuming, of course, that t is positive. Formula (10) then takes the form

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(x + 2a\alpha\sqrt{t})\; e^{-\alpha^2}\, d\alpha\,. \tag{12}$$

It is easy to show then that the solution $u(x, t)$ for positive t is bounded if $|\varphi(x)| \leqslant M$ for all x. For

$$|u(x, t)| \leqslant \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} |\varphi(x + 2a\alpha\sqrt{t})|\; e^{-\alpha^2}\, d\alpha \leqslant \frac{M}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2}\, d\alpha = M\,,$$

because, as we know,

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2}\, d\alpha = 1\,. \tag{13}$$

Multiplying eq. (13) by $\varphi(x)$ and subtracting the result from eq. (12), we obtain

$$u(x, t) - \varphi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} [\varphi(x + 2a\alpha\sqrt{t}) - \varphi(x)]\; e^{-\alpha^2}\, d\alpha\,,$$

so that

$$|u(x, t) - \varphi(x)| \leqslant \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} |\varphi(x + 2a\alpha\sqrt{t}) - \varphi(x)|\; e^{-\alpha^2}\, d\alpha\,. \tag{14}$$

Because the function $\varphi(x)$ is bounded for arbitrary values of x, t, and α, we have

$$|\varphi(x + 2a\alpha\sqrt{t}) - \varphi(x)| \leqslant 2M\,.$$

Let ϵ be any small positive number. We can choose some positive number N sufficiently large that, because of the convergence of the integral (13),

$$\frac{2M}{\sqrt{\pi}} \int_{-\infty}^{-N} e^{-\alpha^2}\, d\alpha \leqslant \tfrac{1}{3}\epsilon\,, \qquad \frac{2M}{\sqrt{\pi}} \int_{N}^{\infty} e^{-\alpha^2}\, d\alpha \leqslant \tfrac{1}{3}\epsilon\,.$$

It then follows from (14) that

$$|u(x,t) - \varphi(x)| \leqslant \tfrac{2}{3}\epsilon + \frac{1}{\sqrt{\pi}} \int_{-N}^{N} |\varphi(x+2a\alpha\sqrt{t}) - \varphi(x)|\ e^{-\alpha^2}\, d\alpha .$$

Since the function $\varphi(x)$ is continuous, we may assert that for all t sufficiently close to zero and for $|\alpha| \leqslant N$, we have

$$|\varphi(x+2a\alpha\sqrt{t}) - \varphi(x)| < \tfrac{1}{3}\epsilon ,$$

and this inequality then gives

$$|u(x,t) - \varphi(x)| \leqslant \tfrac{2}{3}\epsilon + \tfrac{1}{3}\epsilon \frac{1}{\sqrt{\pi}} \int_{-N}^{N} e^{-\alpha^2}\, d\alpha$$

and hence

$$|u(x,t) - \varphi(x)| \leqslant \tfrac{2}{3}\epsilon + \tfrac{1}{3}\epsilon \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2}\, d\alpha ,$$

that is, on the basis of eq. (13), we have

$$|u(x,t) - \varphi(x)| < \epsilon$$

for all t sufficiently close to zero. Since ϵ is arbitrary, it follows that

$$\lim_{t\to 0} u(x,t) = \varphi(x) .$$

We have thus shown that the function

$$u(x,t) = \int_{-\infty}^{\infty} \varphi(\xi) \frac{1}{2a\sqrt{\pi t}} e^{-(\xi-x)^2/4a^2t}\, d\xi \tag{10}$$

is bounded and that it satisfies the heat-flow equation (1) and the initial condition (2).

The uniqueness of this solution for a continuous bounded function $\varphi(x)$ follows from the theorem proven in Chapter XXVIII, section 5.

It follows from formula (10) that heat flows along a rod not with any finite velocity but instantaneously. For suppose that the initial temperature $\varphi(x)$ is positive for $\alpha \leqslant x \leqslant \beta$ and equal to zero outside that interval. Then, for this temperature distribution, we have

$$u(x,t) = \int_{\alpha}^{\beta} \varphi(\xi) \frac{1}{2a\sqrt{\pi t}} e^{-(\xi-x)^2/4a^2t}\, d\xi ,$$

from which it is clear that for arbitrarily small positive values of t and for arbitrarily large values of x, the function $u(x,t)$ is positive. This is explained by the inexactness of the physical premises that lie at the basis of the theory of heat flow.

Let us also note one very important fact. The solution to the problem (1)-(2) (the Cauchy problem) is a function that is infinitely many times continuously differentiable with respect to x and t independently of whether the function $\varphi(x)$ has derivatives or not. This smoothness of the solutions to the

homogeneous heat-flow equation is in sharp contrast, for example, with the equation for the vibration of a string.

Let us now clarify the *physical* meaning of the fundamental solution (11) of the homogeneous heat-flow equation (1).

Let us take a small element of the rod $(x_0 - h, x_0 + h)$ about a point x_0 and let us suppose that the function $\varphi(x)$ giving the initial temperature distribution is equal to zero outside the interval $(x_0 - h, x_0 + h)$ and has a constant value u_0 within it. This would be the physical set-up if at the initial instant of time we gave this element an amount of heat $Q = 2hc\rho u_0$, which caused a temperature rise of u_0 in this section of the rod. At subsequent instants of time, the temperature distribution in the rod would be given by formula (10) which in this case would take the form

$$u(x, t) = \int_{x_0-h}^{x_0+h} u_0 \frac{1}{2a\sqrt{\pi t}} e^{-(\xi - x)^2/4a^2 t} \, d\xi$$

$$= \frac{Q}{c\rho 2a\sqrt{\pi t}} \frac{1}{2h} \int_{x_0-h}^{x_0+h} e^{-(\xi - x)^2/4a^2 t} \, d\xi .$$

If we now decrease the value of h to zero, that is, if we assume that this amount of heat Q is being distributed over an ever smaller portion and, in the limit, is being given to the rod at the point $x = x_0$, we arrive at the concept of an instantaneous point source of heat of strength Q located at the point $x = x_0$ at the instant $t = 0$. From the action of such an instantaneous point source of heat in the rod, we obtain the temperature distribution from the formula

$$\lim_{h \to 0} \frac{Q}{2c\rho a\sqrt{\pi t}} \frac{1}{2h} \int_{x_0-h}^{x_0+h} e^{-(\xi - x)^2/4a^2 t} \, d\xi . \tag{15}$$

By using the theorem of the mean, we obtain

$$\frac{1}{2h} \int_{x_0-h}^{x_0+h} e^{-(\xi - x)^2/4a^2 t} \, d\xi = e^{-(\xi_0 - x)^2/4a^2 t} ,$$

where

$$x_0 - h < \xi_0 < x_0 + h ,$$

and since ξ_0 approaches x_0 as h approaches zero, the expression (15) takes the following form:

$$\frac{Q}{c\rho} \frac{1}{2a\sqrt{\pi t}} e^{-(x_0 - x)^2/4a^2 t} .$$

Thus, the fundamental solution (11) gives the temperature distribution caused by an instantaneous point heat source of strength $Q = c\rho$ placed at the point $x = \xi$ of a rod at the initial instant of time $t = 0$.

The graphs of the fundamental solution

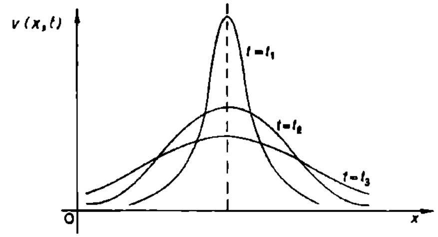

Fig. 64.

$$v(x,t) = \frac{1}{2a\sqrt{\pi t}}\, e^{-(\xi - x)^2/4a^2 t} \tag{11}$$

for fixed ξ as a function of x at various instants of time $0 < t_1 < t_2 < t_3 < \ldots$ are shown in fig. 64. The area under each of these curves is equal to

$$\int_{-\infty}^{\infty} \frac{1}{2a\sqrt{\pi t}}\, e^{-(\xi - a)^2/4a^2 t}\, d\xi = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2}\, d\alpha = 1 .$$

This means that the amount of heat $Q = c\rho$ in the rod remains the same with the passage of time. It is clear from the figure that almost all of the area bounded by the curve (11) and the x-axis falls within the interval $(\xi-\epsilon, \xi+\epsilon)$, where ϵ is an arbitrarily small number, if t is a sufficiently small positive number. This area multiplied by $c\rho$ is equal to the amount of heat added at the initial instant of time. Thus, for small positive values of t, almost all the heat is concentrated in a small neighbourhood of the point $x = \xi$. It follows from this that at the instant $t = 0$, all the heat is located at the point $x = \xi$; that is, we have an instantaneous point source of heat.

Now, it is easy to give a physical interpretation to the solution (10). Specifically, in order to give the cross section $x = \xi$ of the rod a temperature $\varphi(\xi)$ at the initial instant, we must distribute over a small element $d\xi$ around this point an amount of heat $dQ = c\rho\,\varphi(\xi)\,d\xi$ or, what amounts to the same thing, place at the point ξ, instantaneously, a point source of heat of strength dQ. The temperature distribution caused by this instantaneous point source will, according to formula (11), be

$$\varphi(\xi)\, d\xi\, \frac{1}{2a\sqrt{\pi t}}\, e^{-(\xi - x)^2/4a^2 t} .$$

The overall effect of the initial temperature $\varphi(\xi)$ at all points of the rod is the sum of the effects of these individual elements. This gives us the solution (10) that we obtained above:

$$u(x,t) = \int_{-\infty}^{\infty} \varphi(\xi)\, \frac{1}{2a\sqrt{\pi t}}\, e^{-(\xi - x)^2/4a^2 t}\, d\xi .$$

We have been examining the flow of heat in an infinite rod. In a com-

pletely analogous way, we have the equation for the heat flow in the case of infinite space

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

and, correspondingly, the initial condition

$$u|_{t=0} = \varphi(x, y, z) .$$

Instead of formula (10), the solution will be

$$u(x,y,z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(\xi,\eta,\zeta) \frac{1}{(2a\sqrt{\pi t})^3} \exp\left[-\frac{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2}{4a^2 t}\right] d\xi\, d\eta\, d\zeta .$$

2. *Heat-flow in a semi-infinite rod*

Let us consider the problem of heat-flow in a semi-infinite rod the lateral surface of which is thermally insulated. Suppose that the end $x = 0$ is maintained at a given temperature, which we can vary with the passage of time. Then, the problem is reduced to solving the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \infty,\ t > 0) , \tag{16}$$

with the boundary condition

$$u|_{x=0} = \psi(t) \qquad (t \geqslant 0) \tag{17}$$

and the initial condition

$$u|_{t=0} = \varphi(x) \qquad (x \geqslant 0) . \tag{18}$$

We seek a solution to the problem (16) - (18) in the form of a sum

$$u = v + w , \tag{19}$$

where v and w are solutions to the following problem:

$$\begin{aligned} &\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2} , & & \qquad \frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2} , & \\ &v|_{x=0} = 0 , & \text{(I)} & \qquad w|_{x=0} = \psi(t) , & \text{(II)} \\ &v|_{t=0} = \varphi(x) , & & \qquad w|_{t=0} = 0 & \end{aligned}$$

Let us first solve the problem (I). The solution to this problem can be obtained from the solution that we have found for an infinite rod. We rewrite formula (10) in the form

$$v(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_0^\infty \left[\varphi(\xi)\, e^{-(\xi-x)^2/4a^2 t} + \varphi(-\xi)\, e^{-(\xi+x)^2/4a^2 t}\right] d\xi . \tag{20}$$

When we satisfy the boundary condition, we have

$$v|_{x=0} = \frac{1}{2a\sqrt{\pi t}} \int_0^\infty e^{-\xi^2/4a^2t} [\varphi(\xi) + \varphi(-\xi)]\, d\xi \ . \tag{21}$$

This condition will indeed be satisfied if we set *

$$\varphi(-\xi) = -\varphi(\xi) \qquad (0 \leq \xi < \infty) \ ; \tag{22}$$

that is, the function $\varphi(x)$ must be extended to the interval $(-\infty, +\infty)$ by treating it as an odd function.

Substituting eq. (22) into eq. (21), we obtain a solution to the problem (I) in the form

$$v(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_0^\infty \varphi(\xi) \left[e^{-(\xi-x)^2/4a^2t} - e^{-(\xi+x)^2/4a^2t}\right] d\xi \ . \tag{23}$$

If for example, the initial temperature is constant:

$$v|_{t=0} = \varphi(x) = v_0 \ ,$$

we have, from eq. (23),

$$v(x,t) = \frac{v_0}{2a\sqrt{\pi t}} \int_0^\infty \left[e^{-(\xi-x)^2/4a^2t} - e^{-(\xi+x)^2/4a^2t}\right] d\xi \ .$$

If we split this integral into its separate terms and introduce the new variables of integration,

$$\alpha = \frac{\xi - x}{2a\sqrt{t}}, \qquad \beta = \frac{\xi + x}{2a\sqrt{t}} \ ,$$

we obtain

$$v(x,t) = \frac{v_0}{\sqrt{\pi}} \left\{ \int_{-x/2a\sqrt{t}}^\infty e^{-\alpha^2}\, d\alpha - \int_{x/2a\sqrt{t}}^\infty e^{-\beta^2}\, d\beta \right\}$$

$$= \frac{v_0}{\sqrt{\pi}} \int_{-x/2a\sqrt{t}}^{x/2a\sqrt{t}} e^{-\alpha^2}\, d\alpha = \frac{2v_0}{\sqrt{\pi}} \int_0^{x/2a\sqrt{t}} e^{-\alpha^2}\, d\alpha$$

or

$$v(x,t) = v_0 \theta(x/2a\sqrt{t}) \ , \tag{24}$$

where

$$\theta(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\alpha^2}\, d\alpha \tag{25}$$

is the error function.

Let us now solve problem (II). We begin with the particular case of $\psi(t) = 1$, that is,

* The solution to the problem (I) in the class of bounded functions is unique.

$$w|_{x=0} = 1\,. \tag{26}$$

It is easy to see that the function

$$w(x,t) = 1 - \theta(x/2a\sqrt{t}) \tag{27}$$

will be a solution to problem (II) for this particular case. Suppose now that at the end $x=0$, the temperature is held equal to zero until the instant τ and then it is raised to unity. In this case, we denote the solution by $w_T(x,t)$. Obviously, up until the instant $t=\tau$, we shall have $w_T = 0$ and subsequently, w_T will coincide with the solution (27) if we replace t with $t-\tau$ in that solution. This gives us

$$w_T(x,t) = \begin{cases} 0 & \text{for}\quad 0 \leqslant t \leqslant \tau, \\ 1 - \theta(x/2a\sqrt{t-\tau}) & \text{for}\quad t \geqslant \tau\,. \end{cases}$$

But then it is evident that if, at the end $x=0$, a temperature equal to unity is maintained only during the interval of time $(\tau, \tau+\mathrm{d}\tau)$ and is equal to zero at all remaining instants of time, the corresponding temperature distribution along the rod will be

$$w_T(x,t) - w_{T+\mathrm{d}T}(x,t) = -\frac{\partial w_T}{\partial \tau}\,\mathrm{d}\tau\,.$$

However, if at the end $x=0$ a temperature equal to $\varphi(\tau)$ instead of unity is maintained during the interval $(\tau, \tau+\mathrm{d}\tau)$, we shall obtain

$$-\psi(\tau)\,\frac{\partial w_T}{\partial \tau}\,\mathrm{d}\tau\,,$$

from which it is clear that if a temperature $\psi(\tau)$ is maintained at the end $x=0$ for all τ, as τ varies from zero to t we shall obtain the net effect by adding all the elementary effects. This gives us the desired solution to problem (II) in the form

$$w(x,t) = -\int_0^t \psi(\tau)\,\frac{\partial w_T}{\partial \tau}\,\mathrm{d}\tau\,,$$

or since, for $t \geqslant \tau$,

$$-\frac{\partial w_T}{\partial \tau} = \frac{\partial}{\partial \tau}\,\theta\!\left(\frac{x}{2a\sqrt{t-\tau}}\right) = \frac{\partial}{\partial \tau}\,\frac{2}{\sqrt{\pi}}\int_0^{x/2a\sqrt{t-\tau}} \mathrm{e}^{-\alpha^2}\,\mathrm{d}\alpha$$

$$= \frac{x}{2a\sqrt{\pi}(t-\tau)^{\frac{3}{2}}}\,\mathrm{e}^{-x^2/4a^2(t-\tau)}\,,$$

we finally obtain

$$w(x,t) = \frac{x}{2a\sqrt{\pi}}\int_0^t \frac{\psi(\tau)}{(t-\tau)^{\frac{3}{2}}}\,\mathrm{e}^{-x^2/4a^2(t-\tau)}\,\mathrm{d}\tau\,. \tag{28}$$

We replace τ by a new variable of integration ξ defined by

$$\xi = \frac{x}{2a\sqrt{t-\tau}}\,.$$

Then, formula (28) can be written in the form

$$w(x,t) = \frac{2}{\sqrt{\pi}} \int_{x/2a\sqrt{t}}^{\infty} \psi\left(t + \frac{x^2}{4a^2\xi^2}\right) e^{-\xi^2}\, d\xi \ .$$

For $x = 0$, we obtain

$$w(0,t) = \psi(t) \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-\xi^2}\, d\xi = \psi(t) \ ;$$

that is, the solution (28) satisfies the boundary condition (17).

Problems

1. Show that the inhomogeneous equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t)$$

with the initial condition

$$u|_{t=0} = 0$$

has a solution of the form

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} f(\xi,\tau) \frac{1}{2a\sqrt{\pi(t-\tau)}} e^{-(\xi-x)^2/4a^2(t-\tau)}\, d\xi\, d\tau \ .$$

Method: Use the method explained in Chapter VI, section 8 for the inhomogeneous wave equation.

2. By using the method explained in section 1 of this chapter, show that the temperature of a thin plate of infinite area is given by the formula

$$u(x,y,t) = \frac{1}{4\pi a^2 t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\xi,\eta) \exp\left[-\frac{(\xi-x)^2 + (\eta-y)^2}{4a^2 t}\right] d\xi\, d\eta \ .$$

3. Suppose that the initial temperature of a semi infinite rod with thermally insulated lateral surface is known. At the end $x = 0$, a free exchange of heat takes place with the surrounding medium. Find the temperature distribution in the rod at an arbitrary subsequent instant of time.
Answer:

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_0^{\infty} \left\{\varphi(\xi)\left[e^{-(\xi-x)^2/4a^2t} - e^{-(\xi+x)^2/4a^2t}\right]\right.$$

$$\left. - 2h\, e^{-h\xi} \int_0^{\xi} e^{h\mu} \varphi(\mu)\, d\mu\right\} d\xi \ .$$

4. Suppose that the initial temperature of a semi-infinite rod is equal to zero and that that of a second rod is equal to a constant u_0. Suppose that the ends of the two rods are joined at the initial instant. Determine the temperature distribution along each rod at an arbitrary subsequent instant.

Answer:

$$u_1(x, t) = \frac{2u_0\sigma}{\sqrt{\pi}(1+\sigma)} \int_{-x/2a_1\sqrt{t}}^{\infty} e^{-\mu^2}\, d\mu \qquad (x < 0)$$

$$u_2(x, t) = \frac{u_0\sigma}{1+\sigma}\left[1 + \frac{1}{\sigma}\theta\left(\frac{x}{2a_2\sqrt{t}}\right)\right] \qquad (x > 0) ,$$

where $\sigma = k_2a_1/k_1a_2$.

Method: The problem reduces to integrating the equations

$$\frac{\partial u_1}{\partial t} = a_1^2 \frac{\partial^2 u_1}{\partial x^2} \quad (x < 0) , \qquad \frac{\partial u_2}{\partial t} = a_2^2 \frac{\partial^2 u_2}{\partial x^2} \quad (x > 0) , \qquad a_i^2 = \frac{k_i}{c_i\rho_i} \quad (i = 1, 2)$$

under the conditions

$$u_1(0, t) = u_2(0, t) , \qquad k_1 \frac{\partial u_1(0, t)}{\partial x} = k_2 \frac{\partial u_2(0, t)}{\partial x} ,$$

$$u_1(x, 0) = 0 , \qquad u_2(x, 0) = u_0 .$$

Chapter **XXX**

THE APPLICATION OF THE FOURIER METHOD TO THE SOLUTION OF BOUNDARY-VALUE PROBLEMS

1. *Heat-flow in a finite rod*

1. Heat-flow in a rod whose ends are kept at zero temperature. The problem consists in finding solutions to the heat-flow equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

with the boundary conditions

$$u|_{x=0} = 0 , \qquad u|_{x=l} = 0 \tag{2}$$

and the initial condition

$$u|_{t=0} = \varphi(x) , \tag{3}$$

where $\varphi(x)$ is continuous, has a piecewise-continuous derivative, and vanishes at $x = 0$ and $x = l$.

Following the Fourier method, we seek particular solutions to eq. (1) in the form

$$u(x, t) = X(x)\ T(t) . \tag{4}$$

Substituting eq. (4) into eq. (1), we have

$$X(x)\ T'(t) = a^2\ T(t)\ X''(x)$$

or

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda ,$$

from which we obtain the two equations

$$T'(t) + a^2\lambda T(t) = 0 , \tag{5}$$

$$X''(x) + \lambda X(x) = 0 . \tag{6}$$

To obtain a non-trivial solution to eq. (1) in the form (4) satisfying the boundary conditions (2), we need to find a non-trivial solution to eq. (6) satisfying the boundary conditions

$$X(0) = 0 , \qquad X(l) = 0 . \tag{7}$$

Thus, to determine the function $X(x)$, we are led to the eigenvalue problem

$$X''(x) + \lambda X(x) = 0 , \qquad X(0) = 0 , \qquad X(l) = 0 , \tag{8}$$

which we examined when studying the vibration of a string of finite length (see Chapter VIII, section 1). It was shown there that only for values of the parameter λ that are equal to

$$\lambda_n = (n\pi/l)^2 \qquad (n = 1, 2, 3, \ldots) , \tag{9}$$

are there non-trivial solutions to the problem (8):

$$X_n(x) = \sin \frac{n\pi x}{l} . \tag{10}$$

To the values $\lambda = \lambda_n$ correspond the solutions to eq. (5):

$$T_n(t) = a_n \, e^{-(n\pi a/l)^2 t} , \tag{11}$$

where the a_n are arbitrary constants. Thus, all the functions

$$u_n(x, t) = X_n(x) \; T_n(t) = a_n \, e^{-(n\pi a/l)^2 t} \sin \frac{n\pi x}{l} \tag{12}$$

satisfy eq. (1) and the boundary conditions (2) for arbitrary values of the constants a_n.

Let us construct the series

$$u(x, t) = \sum_{n=1}^{\infty} a_n \, e^{-(n\pi a/l)^2 t} \sin \frac{n\pi x}{l} . \tag{13}$$

When we satisfy the initial condition (3), we obtain

$$u(x, 0) = \varphi(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} . \tag{14}$$

This series is an expansion of the given function $\varphi(x)$ in a Fourier sine series for the interval $(0, l)$. The coefficients a_n are determined from the familiar formula

$$a_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l} \, dx . \tag{15}$$

Since we have assumed that the function $\varphi(x)$ is continuous, has a piecewise-continuous derivative, and vanishes at $x = 0$ and $x = l$, the series (14) with the coefficients a_n determined from formula (15) converges uniformly and absolutely to $\varphi(x)$, as is known from the theory of trigonometric series [34].

Since, for non-negative t,

$$0 < e^{-(n\pi a/l)^2 t} \leq 1 ,$$

the series (13) also converges absolutely and uniformly for non-negative t. Therefore, the function $u(x, t)$, defined by the series (13) is continuous in the region $0 \leq x \leq l$, $t \geq 0$ and satisfies the initial and boundary conditions. It remains to show that the function $u(x, t)$ satisfies eq. (1) in the region $0 < x < l$, $t > 0$. For this, it is sufficient to show that the series obtained by differentiating eq. (13) termwise once with respect to t and the series ob-

tained by differentiating eq. (13) termwise twice with respect to x converge absolutely and uniformly in the region $0 < x < l$, $t > 0$. But this follows from the fact that for arbitrary positive t,

$$0 < (n\pi a/l)^2 \, e^{-(n\pi a/l)^2 t} < 1 , \qquad 0 < (n\pi/l)^2 \, e^{-(n\pi a/l)^2 t} < 1$$

if n is sufficiently great.

In exactly the same way, we can show that the function $u(x, t)$ has continuous derivatives of arbitrary order with respect to x and t in the region $0 < x < l$, $t > 0$.

2. Heat-flow in a rod the temperatures at the ends of which are given functions of time. This problem reduces to solving the heat-flow equation (1) with the boundary conditions

$$u|_{x=0} = \psi_1(t) , \qquad u|_{x=l} = \psi_2(t) \tag{16}$$

and the initial condition

$$u|_{t=0} = \varphi(x) , \tag{17}$$

where $\psi_1(t)$, $\psi_2(t)$ and $\varphi(x)$ are given functions. We seek a solution in the form of a series

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{l} , \tag{18}$$

where

$$T_n(t) = \frac{2}{l} \int_0^l u(x, t) \sin \frac{n\pi x}{l} \, dx . \tag{19}$$

If we twice integrate by parts, we obtain

$$T_n(t) = \frac{2}{n\pi} [u(0, t) - (-1)^n u(l, t)] - \frac{2l}{n^2\pi^2} \int_0^l \frac{\partial^2 u}{\partial x^2} \sin \frac{n\pi x}{l} \, dx .$$

Since $u(x, t)$ satisfies eq. (1) and the boundary conditions (16),

$$T_n(t) = \frac{2}{n\pi} [\psi_1(t) - (-1)^n \psi_2(t)] - \frac{2l}{n^2\pi^2 a^2} \int_0^l \frac{\partial u}{\partial t} \sin \frac{n\pi x}{l} \, dx . \tag{20}$$

If we now differentiate eq. (19) with respect to t, we obtain

$$\frac{dT_n(t)}{dt} = \frac{2}{l} \int_0^l \frac{\partial u}{\partial t} \sin \frac{n\pi x}{l} \, dx . \tag{21}$$

If we eliminate the integral from eqs. (20) and (21), we obtain the following equations for determining the coefficients $T_n(t)$:

$$\frac{dT_n(t)}{dt} + \left(\frac{n\pi a}{l}\right)^2 T_n(t) = \frac{2n\pi a^2}{l^2} [\psi_1(t) - (-1)^n \psi_2(t)] . \tag{22}$$

The general solution of this equation is of the form

$$T_n(t) = e^{-(n\pi a/l)^2 t}\left[C_n + \frac{2n\pi a^2}{l^2}\int_0^t e^{(n\pi a/l)^2\tau}\,(\psi_1(\tau) - (-1)^n\psi_2(\tau))\,d\tau\right], \tag{23}$$

where, obviously, $C_n = T_n(0)$. To satisfy the initial condition (17), we require that

$$u(x,0) = \sum_{n=1}^{\infty} T_n(0)\sin\frac{n\pi x}{l} = \varphi(x)$$

and, consequently,

$$T_n(0) = C_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}\,dx\,. \tag{24}$$

Thus, the series (18), where the $T_n(t)$ are determined by eqs. (23) and (24), is a solution to the problem (1), (16) - (17).

Let us examine the particular case in which the ends of the rod are kept at constant temperature; that is,

$$\psi_1(t) = u_1 = \text{constant}\,, \qquad \psi_2(t) = u_2 = \text{constant}\,.$$

Then, eq. (23) takes the form

$$T_n(t) = \frac{2}{n\pi}[u_1 - (-1)^n u_2]\,[1 - e^{-(n\pi a/l)^2 t}] + \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}\,dx\; e^{-(n\pi a/l)^2 t}$$

Substituting $T_n(t)$ into the series (18), we obtain

$$\begin{aligned} u(x,t) = \frac{2u_1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi x/l)}{n} &+ \frac{2u_2}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(n\pi x/l)}{n} \\ &+ \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n u_2 - u_1}{n}\,e^{-(n\pi a/l)^2 t}\sin\frac{n\pi x}{l} \\ &+ \frac{2}{l}\sum_{n=1}^{\infty} e^{-(n\pi a/l)^2 t}\sin\frac{n\pi x}{l}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}\,dx\,. \end{aligned}$$

On the basis of known relationships,

$$\sum_{n=1}^{\infty}\frac{\sin n\xi}{n} = \begin{cases}\frac{1}{2}(\pi - \xi) & \text{for}\quad 0 < \xi < 2\pi\,,\\ 0 & \text{for}\quad \xi = 0, 2\pi,\end{cases}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin n\xi}{n} = \begin{cases}\frac{1}{2}\xi & \text{for}\quad -\pi < \xi < \pi\,,\\ 0 & \text{for}\quad \xi = -\pi, \pi\,,\end{cases}$$

we finally obtain

$$u(x,t) = u_1 + (u_2 - u_1)\frac{x}{l} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n u_2 - u_1}{n} e^{-(n\pi a/l)^2 t} \sin\frac{n\pi x}{l}$$

$$+ \frac{2}{l}\sum_{n=1}^{\infty} e^{-(n\pi a/l)^2 t} \sin\frac{n\pi x}{l}\int_0^l \varphi(x) \sin\frac{n\pi x}{l}\, dx\,. \qquad (25)$$

3. Heat-flow in a rod at the ends of which a free exchange of heat takes place with the adjacent medium. The problem consists in finding the solution to the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (26)$$

with boundary conditions

$$\frac{\partial u}{\partial x} - hu\Big|_{x=0} = 0\,, \qquad \frac{\partial u}{\partial x} + hu\Big|_{x=l} = 0 \qquad (27)$$

and initial condition

$$u\big|_{t=0} = \varphi(x)\,. \qquad (28)$$

Following the Fourier method, we seek particular solutions to eq. (26) in the form

$$u(x,t) = X(x)\ T(t)\,. \qquad (29)$$

Then, we obtain the equations

$$T'(t) + a^2\lambda^2 T(t) = 0\,, \qquad (30)$$

$$X''(x) + \lambda^2 X(x) = 0\,. \qquad (31)$$

In order that the particular solution (29) that is not identically equal to zero satisfy the boundary conditions (27), it is evident that

$$X'(0) - hX(0) = 0\,, \qquad X'(l) + hX(l) = 0\,. \qquad (32)$$

Thus, we arrive at the problem of the eigenvalues for eq. (31) with boundary conditions (32) *. Integrating eq. (31), we obtain

$$X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x\,. \qquad (33)$$

From the boundary conditions (32), we find that

$$hC_1 - \lambda C_2 = 0\,, \quad (h\cos\lambda l - \lambda\sin\lambda l)\, C_1 + (h\sin\lambda l + \lambda\cos\lambda l)\, C_2 = 0\,. \qquad (34)$$

This system of two homogeneous equations has the obvious solution $C_1 = C_2 = 0$ and we obtain the solution $X(x) \equiv 0$. Discarding this case, we must assume that at least one of the constants C_1, C_2 is non-zero. Then, the determinant of the system (34) must be equal to zero:

$$\begin{vmatrix} h & -\lambda \\ h\cos\lambda l - \lambda\sin\lambda l & h\sin\lambda l + \lambda\cos\lambda l \end{vmatrix} = 0\,,$$

* On the basis of Chapter VIII, section 5, all the eigenvalues of the problem (31) – (32) are positive. Therefore, instead of λ, we may write λ^2.

which, after the substitution

$$\mu = \lambda l \ , \qquad p = hl > 0 \ , \tag{35}$$

is reduced to the form

$$2 \cot \mu = \frac{\mu}{p} - \frac{p}{\mu} \ . \tag{36}$$

This equation has an infinite number of real roots, which can easily be shown by drawing the graph of the curves (fig. 65)

$$y = 2 \cot \mu \ , \qquad y = \frac{\mu}{p} - \frac{p}{\mu} \ .$$

It is clear from the figure that in each of the intervals $(0, \pi), (\pi, 2\pi), \ldots$ there is one positive root of eq. (36) and that the negative roots are equal in absolute value to the positive roots.

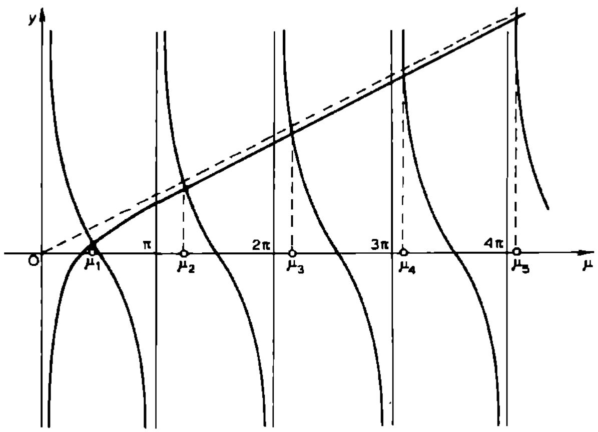

Fig. 65.

We denote by $\mu_1, \mu_2, \mu_3, \ldots$ the positive roots of eq. (36). Then, from (35), the eigenvalues will be

$$\lambda_n{}^2 = (\mu_n/l)^2 \qquad (n = 1, 2, 3, \ldots) \ . \tag{37}$$

To each eigenvalue corresponds an eigenfunction

$$X_n(x) = \cos \frac{\mu_n x}{l} + \frac{p}{\mu_n} \sin \frac{\mu_n x}{l} \ . \tag{38}$$

For $\lambda = \lambda_n$, the general solution to eq. (30) is of the form

$$T_n(t) = a_n \, \mathrm{e}^{-(\mu_n a/l)^2 t} \ , \tag{39}$$

where the a_n are arbitrary constants.

Thus, we have found particular solutions to eq. (26)

$$u_n(x, t) = X_n(x)\, T_n(t) = a_n\, e^{-(\mu_n a/l)^2 t} \left(\cos \frac{\mu_n x}{l} + \frac{p}{\mu_n} \sin \frac{\mu_n x}{l}\right),$$

satisfying the boundary conditions (27) for arbitrary values of a_n.

Let us construct the series

$$u(x, t) = \sum_{n=1}^{\infty} a_n\, e^{-(\mu_n a/l)^2 t} \left(\cos \frac{\mu_n x}{l} + \frac{p}{\mu_n} \sin \frac{\mu_n x}{l}\right). \tag{40}$$

When we satisfy the initial condition (28), we obtain

$$\varphi(x) = \sum_{n=1}^{\infty} a_n \left(\cos \frac{\mu_n x}{l} + \frac{p}{\mu_n} \sin \frac{\mu_n x}{l}\right) = \sum_{n=1}^{\infty} a_n X_n(x). \tag{41}$$

On the basis of the theory of eigenvalue problems (see Chapter VII, section 5), the eigenfunctions $X_n(x)$ are orthogonal; that is,

$$\int_0^l X_n(x)\, X_m(x)\, dx = 0 \qquad (n \neq m). \tag{42}$$

Taking the square of the norm of the eigenfunctions (38), we obtain

$$\int_0^l X_n^2(x)\, dx = \int_0^l \left(\cos \frac{\mu_n x}{l} + \frac{p}{\mu_n} \sin \frac{\mu_n x}{l}\right)^2 dx = \frac{l}{2}\, \frac{p(p+2) + \mu_n^2}{\mu_n^2}. \tag{43}$$

Assuming that the series (41) converges uniformly and taking into account eqs. (42) and (43), we find the coefficients a_n from the formula

$$a_n = \frac{2}{l}\, \frac{\mu_n^2}{p(p+2) + \mu_n^2} \int_0^l \varphi(x) \left(\cos \frac{\mu_n x}{l} + \frac{p}{\mu_n} \sin \frac{\mu_n x}{l}\right) dx.$$

Substituting this expression for the coefficients a_n in the series (40), we obtain the solution to the problem (26) - (28):

$$u(x, t) = \frac{2}{l} \sum_{n=1}^{\infty} e^{-(\mu_n a/l)^2 t}\, \frac{\mu_n \cos(\mu_n x/l) + p \sin(\mu_n x/l)}{p(p+2) + \mu_n^2} \times \int_0^l \varphi(x) \left(\mu_n \cos \frac{\mu_n x}{l} + p \sin \frac{\mu_n x}{l}\right) dx. \tag{44}$$

2. *The inhomogeneous heat-flow equation*

1. Consider the inhomogeneous equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x, t) \tag{45}$$

with initial condition

$$u|_{t=0} = 0 \tag{46}$$

and boundary conditions

$$u(0,t) = 0\,, \qquad u(l,t) = 0\,. \tag{47}$$

Let us seek a solution to this problem in the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\frac{n\pi x}{l}\,, \tag{48}$$

so that the boundary conditions (47) will be automatically satisfied. Let us suppose that the function $f(x,t)$, regarded as a function of x, can be expanded in a Fourier series:

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\frac{n\pi x}{l}\,, \tag{49}$$

where

$$f_n(t) = \frac{2}{l}\int_0^l f(x,t) \sin\frac{n\pi x}{l}\,\mathrm{d}x\,. \tag{50}$$

Substituting the series (48) into eq. (45) and taking eq. (49) into consideration, we obtain

$$\sum_{n=1}^{\infty}\left[T_n'(t) + \left(\frac{n\pi a}{l}\right)^2 T_n(t) - f_n(t)\right] \sin\frac{n\pi x}{l} = 0\,,$$

from which, by setting $n\pi a/l = \omega_n$, we obtain

$$T_n'(t) + \omega_n^2 T_n(t) = f_n(t)\,. \tag{51}$$

By using the initial condition for $u(x,t)$

$$u(x,0) = \sum_{n=1}^{\infty} T_n(0) \sin\frac{n\pi x}{l} = 0\,,$$

we obtain the initial condition for $T_n(t)$:

$$T_n(0) = 0\,. \tag{52}$$

Solving the ordinary differential equation (51) with this initial condition, we find

$$T_n(t) = \int_0^t \mathrm{e}^{-\omega_n^2(t-\tau)} f_n(\tau)\,\mathrm{d}\tau\,. \tag{53}$$

Substituting this into the series (48), we obtain the solution to the problem (45) - (47) in the form

$$u(x,t) = \sum_{n=1}^{\infty}\left\{\int_0^t \mathrm{e}^{-\omega_n^2(t-\tau)} f_n(\tau)\,\mathrm{d}\tau\right\} \sin\frac{n\pi x}{l}\,. \tag{54}$$

If the initial condition is inhomogeneous, we must add to the solution (54) the solution of the homogeneous heat-flow equation with given initial condition $u(x,0) = \varphi(x)$ and boundary conditions (47), which was obtained in section 1, paragraph 1 of this chapter.

2. Let us now consider the case in which the initial and boundary conditions are inhomogeneous; that is, we need to find the solution of the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t) \tag{55}$$

with initial condition

$$u|_{t=0} = \varphi(x) \tag{56}$$

and boundary conditions

$$u(0,t) = \psi_1(t)\ , \qquad u(l,t) = \psi_2(t)\ . \tag{57}$$

This problem is easily reduced to the problem considered in section 1, paragraph 2 and in section 2, paragraph 1. For if we set

$$u = v + w\ , \tag{58}$$

where the function v satisfies the homogeneous equation

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}\ , \tag{59}$$

the boundary conditions

$$v(0,t) = \psi_1(t)\ , \qquad v(l,t) = \psi_2(t)\ , \tag{60}$$

and the initial condition

$$v|_{t=0} = \varphi(x)\ , \tag{61}$$

and the function $w(x,t)$ satisfies the inhomogeneous equation

$$\frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2} + f(x,t)\ , \tag{62}$$

the boundary conditions

$$w|_{x=0} = 0\ , \qquad w|_{x=l} = 0\ , \tag{63}$$

and the initial condition

$$w|_{t=0} = 0\ . \tag{64}$$

Obviously, the sum (58) is the solution of the problem (55) - (57).

We note that the problem (55) - (57) is also easily reduced to the problem examined in section 2, paragraph 1 if we introduce a new unknown function $v(x,t)$ by setting

$$u(x,t) = v(x,t) + U(x,t)\ ,$$

where

$$U(x,t) = \psi_1(t) + [\psi_2(t) - \psi_1(t)]\frac{x}{l}\ .$$

3. *Heat-flow in an infinite cylinder*

In this section we shall deal with heat-flow in an infinite cylinder; we shall consider several cases.

1. Let us first consider the radial flow of heat in an infinitely long circular cylinder of radius R, the lateral surface of which is kept at zero temperature.

This problem becomes that of integrating the equation

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \tag{65}$$

with the boundary condition

$$u|_{r=R} = 0 \tag{66}$$

and the initial condition

$$u|_{t=0} = \varphi(r), \tag{67}$$

where φ is a given function of r.

Following the Fourier method, we seek particular solutions of eq. (65) in the form

$$u(r, t) = T(t)\, w(r), \tag{68}$$

and we thus obtain the two equations

$$T'(t) + a^2\lambda^2 T(t) = 0, \tag{69}$$

$$w''(r) + \frac{1}{r} w'(r) + \lambda^2 w(r) = 0, \tag{70}$$

where λ is an arbitrary parameter.

The general solution of eq. (70) is of the form (see Chapter XII, section 1)

$$w(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r).$$

Since $Y_0(\lambda r)$ becomes infinite as r approaches zero, it follows from the finiteness of the temperature along the axis of the cylinder that $C_2 = 0$. The constant λ is found from the boundary condition (66). Obviously, this constant can have any of an infinite set of values determined by the formula

$$\lambda_n = \mu_n/R, \tag{71}$$

where the μ_n are the positive roots of the equations

$$J_0(\mu) = 0. \tag{72}$$

To each eigenvalue $\lambda_n{}^2$, there corresponds an eigenfunction

$$w(r) = J_0(\mu_n r/R). \tag{73}$$

If we now take eqs. (69) and (68) into consideration, we see that the functions

$$u_n(r, t) = a_n \, e^{-(a\mu_n/R)^2 t} \, J_0(\mu_n r/R) \tag{74}$$

satisfy eq. (65) and the boundary condition (66) for arbitrary a_n.

Let us construct the series

$$u(r, t) = \sum_{n=1}^{\infty} a_n J_0(\mu_n r/R) \, e^{-(a\mu_n/R)^2 t} \tag{75}$$

and in order to satisfy the initial condition (67), we require that

$$\sum_{n=1}^{\infty} a_n J_0(\mu_n r/R) = \varphi(r) \, . \tag{76}$$

This series is an expansion of the given function $\varphi(r)$ in Bessel functions in the interval $(0, R)$. The coefficients in the expansion (76) (see Chapter XII, section 4) are determined from the formula

$$a_n = \frac{2}{R^2 J_1^2(\mu_n)} \int_0^R \rho\varphi(\rho) \, J_0(\mu_n \rho/R) \, d\rho \, . \tag{77}$$

Substituting the expression (77) for the a_n in the series (75), we obtain the solution to the problem (65) - (67) in the form

$$u(r, t) = \frac{2}{R^2} \sum_{n=1}^{\infty} \frac{J_0(\mu_n r/R)}{J_1^2(\mu_n)} \, e^{-(a\mu_n/R)^2 t} \int_0^R \rho\varphi(\rho) \, J_0(\mu_n \rho/R) \, d\rho \, . \tag{78}$$

2. Let us now consider the case in which there is an exchange of heat between the surface of the cylinder and the surrounding medium, the temperature of which is assumed equal to zero. Obviously, this problem reduces to integrating eq. (65) with the initial condition (67) and with the boundary condition

$$\frac{\partial u}{\partial r} + hu \Big|_{r=R} = 0 \, . \tag{79}$$

Repeating the reasoning of paragraph 1, we again obtain eqs. (69) and (70) and find that

$$w(r) = C_1 J_0(\lambda r) \, .$$

In satisfying the boundary condition (79), we get

$$\alpha J_0(\mu) + \mu J_0'(\mu) = 0 \, , \tag{80}$$

where we set

$$\mu = \lambda R \, , \qquad \alpha = hR \, . \tag{81}$$

In Chapter XII, section 3, it was shown that eq. (80) has only real roots.

We seek a solution of the problem in the form of a series

$$u(r, t) = \sum_{n=1}^{\infty} a_n J_0(\mu_n r/R) \, e^{-(\mu_n r/R)^2 t} \, , \tag{82}$$

where $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of eq. (80) and each term of the series satisfies the boundary condition (79).

In satisfying the initial condition (67), we obtain

$$\varphi(r) = \sum_{n=1}^{\infty} a_n J_0(\mu_n r/R) . \tag{83}$$

But this is just the particular case of the expansion (44) of Chapter XII when $\nu = 0$ and $\beta = 1$. Consequently, the coefficients a_n are determined from formula (45) of that chapter; that is,

$$a_n = \frac{2\mu_n^2}{R^2(\mu_n^2 + \alpha^2)\, J_0^2(\mu_n)} \int_0^R r\varphi(r)\, J_0(\mu_n r/R)\, dr . \tag{84}$$

Substituting this value of the coefficient a_n in the series (82), we obtain the solution to the problem in the form

$$u(r,t) = \frac{2}{R^2} \sum_{n=1}^{\infty} \frac{\mu_n^2 \int_0^R \rho\varphi(\rho)\, J_0(\mu_n\rho/R)\, d\rho}{(\mu_n^2 + h^2R^2)\, J_0^2(\mu_n)} J_0(\mu_n r/R)\, e^{-(\mu_n r/R)^2 t} . \tag{85}$$

3. Let us now consider the more general case in which the initial temperature is a function of all three coordinates: r, θ, and z.

If we limit our investigation to the case in which the temperature on the lateral surface of the cylinder is kept equal to zero, the problem is reduced to solving the equation

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{86}$$

with boundary condition

$$u|_{r=R} = 0 \tag{87}$$

and initial condition

$$u|_{t=0} = \varphi(r,\theta,z) . \tag{88}$$

Let us seek particular solutions of eq. (86) in the form of a product

$$u(r,\theta,z,t) = T(t)\, w(r)\, \Phi(\theta)\, Z(z) . \tag{89}$$

If we substitute eq. (89) into eq. (86), we obtain

$$\frac{T'(t)}{a^2 T(t)} = \frac{w''(r) + \frac{1}{r} w'(r)}{w(r)} + \frac{1}{r^2}\frac{\Phi''(\theta)}{\Phi(\theta)} + \frac{Z''(z)}{Z(z)} ,$$

from which

$$Z''(z) + \lambda^2 Z(z) = 0 , \qquad \Phi''(\theta) + m^2\Phi(\theta) = 0 ,$$

$$w''(r) + \frac{1}{r} w'(r) + \left(k^2 - \frac{m^2}{r^2}\right) w(r) = 0 ,$$

$$T'(t) + a^2(k^2 + \lambda^2)\, T(t) = 0 \ ,$$

where λ, m, and k are arbitrary constants.

The general solutions of these equations are of the following forms:

$$\begin{aligned} Z(z) &= C_1 \cos \lambda z + C_2 \sin \lambda z \ , \\ \Phi(\theta) &= C_3 \cos m\theta + C_4 \sin m\theta \ , \\ w(r) &= C_5 J_m(kr) + C_6 Y_m(kr) \ , \\ T(t) &= C_7 \exp\left[-a^2(k^2+\lambda^2)t\right] \ . \end{aligned} \tag{90}$$

Since the temperature in the cylinder is obviously a periodic function of the angle θ with period 2π, the constant m must be an integer. Furthermore, the constant C_6 must be equal to zero because otherwise the temperature would become infinitely great on the axis of the cylinder, which, of course, is impossible. The constant k is determined from the boundary condition (87), which leads to the equation

$$J_m(kR) = 0 \ .$$

It follows from this that k has an infinite set of values, defined by the formula

$$k_{mi} = \mu_{mi}/R \qquad (i = 1, 2, 3, \ldots) \ , \tag{91}$$

where $\mu_{m1}, \mu_{m2}, \mu_{m3}, \ldots$ are the positive roots of the equation

$$J_m(\mu) = 0 \ . \tag{92}$$

Since there are no restrictions on the constant λ, we may consider it completely arbitrary, taking on any real value.

Let us take the sum of all the particular solutions of the form (89). This sum will be of the form

$$u = \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} \int_{-\infty}^{\infty} \exp\left[-a^2(\lambda^2 + k_{mi}^2)t\right] \left\{ \left[A_{mi}(\lambda) \cos \lambda z + B_{mi}(\lambda) \sin \lambda z\right] \cos m\theta \right.$$
$$\left. + \left[C_{mi}(\lambda) \cos \lambda z + D_{mi}(\lambda) \sin \lambda z\right] \sin m\theta \right\} J_m(k_{mi} r)\, d\lambda \ . \tag{93}$$

To determine the functions $A_{mi}(\lambda)$, etc., let us set $t = 0$. Then, on the basis of the initial condition (88), we have

$$\varphi(r, \theta, z) = \sum_{i=1}^{\infty} \int_{-\infty}^{\infty} \left[A_{0i}(\lambda) \cos \lambda z + B_{0i}(\lambda) \sin \lambda z\right] J_0(k_{0i} r)\, d\lambda$$
$$+ \sum_{m=1}^{\infty} \left[\sum_{i=1}^{\infty} J_m(k_{mi} r) \int_{-\infty}^{\infty} \left[A_{mi}(\lambda) \cos \lambda z + B_{mi}(\lambda) \sin \lambda z\right] d\lambda \right] \cos m\theta$$
$$+ \sum_{m=1}^{\infty} \left[\sum_{i=1}^{\infty} J_m(k_{mi} r) \int_{-\infty}^{\infty} \left[C_{mi}(\lambda) \cos \lambda z + D_{mi}(\lambda) \sin \lambda z\right] d\lambda \right] \sin m\theta \ .$$

We now compare this equality with the expansion of the function $\varphi(r, \theta, z)$,

in a sine-and-cosine series over the interval $(0, 2\pi)$, in which expansion we treat φ as a function of θ. We thus obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(r,\theta,z)\,d\theta = \sum_{i=0}^{\infty} J_0(k_{0i}r)\int_{-\infty}^{\infty}[A_{0i}(\lambda)\cos\lambda z + B_{0i}(\lambda)\sin\lambda z]\,d\lambda\ ,$$

$$\frac{1}{\pi}\int_0^{2\pi} \varphi(r,\theta,z)\cos m\theta\,d\theta = \sum_{i=1}^{\infty} J_m(k_{mi}r)\int_{-\infty}^{\infty}[A_{mi}(\lambda)\cos\lambda z + B_{mi}(\lambda)\sin\lambda z]\,d\lambda,$$

$$\frac{1}{\pi}\int_0^{2\pi} \varphi(r,\theta,z)\sin m\theta\,d\theta = \sum_{i=1}^{\infty} J_m(k_{mi}r)\int_{-\infty}^{\infty}[C_{mi}(\lambda)\cos\lambda z + D_{mi}(\lambda)\sin\lambda z]\,d\lambda.$$

Each of these equations is itself the expansion of an arbitrary function $\psi(r)$ in a series of the form

$$\psi(r) = \sum_{i=1}^{\infty} a_i J_m(\mu_{mi}r/R)\ ,$$

where m is a non-negative integer. The coefficients a_i are determined from the formula

$$a_i = \frac{2}{R^2 J_{m+1}^2(\mu_{mi})}\int_0^R r\psi(r)\,J_m(\mu_{mi}r/R)\,dr\ , \tag{94}$$

where the μ_{mi} are the positive roots of eq. (92) (see Chapter XII, section 4).

It follows from formula (94) that

$$\int_{-\infty}^{\infty}[A_{0i}(\lambda)\cos\lambda z + B_{0i}(\lambda)\sin\lambda z]\,d\lambda$$
$$= \frac{1}{\pi R^2 J_1{}^2(\mu_{0i})}\int_0^R\int_0^{2\pi} r\varphi(r,\theta,z)\,J_0(\mu_{0i}r/R)\,dr\,d\theta\ ,$$

$$\int_{-\infty}^{\infty}[A_{mi}(\lambda)\cos\lambda z + B_{mi}(\lambda)\sin\lambda z]\,d\lambda$$
$$= \frac{2}{\pi R^2 J_{m+1}^2(\mu_{mi})}\int_0^R\int_0^{2\pi} r\varphi(r,\theta,z)\,J_m(\mu_{mi}r/R)\cos m\theta\,dr\,d\theta\ ,$$

$$\int_{-\infty}^{\infty}[C_{mi}(\lambda)\cos\lambda z + D_{mi}(\lambda)\sin\lambda z]\,d\lambda$$
$$= \frac{2}{\pi R^2 J_{m+1}^2(\mu_{mi})}\int_0^R\int_0^{2\pi} r\varphi(r,\theta,z)\,J_m(\mu_{mi}r/R)\sin m\theta\,dr\,d\theta\ .$$

Thus, we arrive at expressions of the form

$$\int_{-\infty}^{\infty} [a(\lambda) \cos \lambda z + b(\lambda) \sin \lambda z]\, d\lambda = \omega(z) ,$$

where $\omega(z)$ is a given function.

But we have encountered expressions of this sort in the problem of heat-flow in a rod of infinite length, and it was shown then that the functions $a(\lambda)$ and $b(\lambda)$ are of the form

$$a(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega(\xi) \cos \lambda\xi \, d\xi , \qquad b(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega(\xi) \sin \lambda\xi \, d\xi .$$

From these formulae, we have expressions for the desired functions $A_{0i}(\lambda)$, $B_{0i}(\lambda)$, $A_{mi}(\lambda)$, $B_{mi}(\lambda)$, $C_{mi}(\lambda)$, $D_{mi}(\lambda)$, namely,

$$A_{mi}(\lambda) = \frac{1}{\delta_m \pi^2 R^2 J^2_{m+1}(\mu_{mi})} \int_0^R \int_0^{2\pi} \int_{-\infty}^{\infty} r\varphi(r, \theta_1, \xi)$$

$$\times J_m(\mu_{mi} r/R) \cos m\theta_1 \cos \lambda\xi \, dr \, d\theta_1 \, d\xi ,$$

$$B_{mi}(\lambda) = \frac{1}{\delta_m \pi^2 R^2 J^2_{m+1}(\mu_{mi})} \int_0^R \int_0^{2\pi} \int_{-\infty}^{\infty} r\varphi(r, \theta_1, \xi)$$

$$\times J_m(\mu_{mi} r/R) \cos m\theta_1 \sin \lambda\xi \, dr \, d\theta_1 \, d\xi ,$$

$$C_{mi}(\lambda) = \frac{1}{\pi^2 R^2 J^2_{m+1}(\mu_{mi})} \int_0^R \int_0^{2\pi} \int_{-\infty}^{\infty} r\varphi(r, \theta_1, \xi)$$

$$\times J_m(\mu_{mi} r/R) \sin m\theta_1 \cos \lambda\xi \, dr \, d\theta_1 \, d\xi ,$$

$$D_{mi}(\lambda) = \frac{1}{\pi^2 R^2 J^2_{m+1}(\mu_{mi})} \int_0^R \int_0^{2\pi} \int_{-\infty}^{\infty} r\varphi(r, \theta_1, \xi)$$

$$\times J_m(\mu_{mi} r/R) \sin m\theta_1 \sin \lambda\xi \, dr \, d\theta_1 \, d\xi ,$$

where $\delta_0 = 2$ and $\delta_m = 1$ for positive m.

If we substitute these expressions into the series (93) and carry out the integration with respect to λ, using the integral

$$\int_{-\infty}^{\infty} e^{-\alpha^2\lambda^2} \cos \beta\lambda \, d\lambda = \frac{\sqrt{\pi}}{\alpha} e^{-\beta^2/4\alpha^2} ,$$

we obtain the solution to the problem (86) - (88) in the form

$$u = \frac{1}{a\pi R^2\sqrt{\pi t}} \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} e^{-(a\mu_{mi}/R)^2 t} \frac{J_m(\mu_{mi} r/R)}{\delta_m J_{m+1}^2(\mu_{mi})} \tag{95}$$

$$\times \int_0^R \int_0^{2\pi} \int_{-\infty}^{\infty} \rho\varphi(\rho,\theta_1,\xi)\, e^{-(z-\xi)^2/4a^2 t} J_m(\mu_{mi}\rho/R) \cos m(\theta - \theta_1)\, d\rho\, d\theta_1\, d\xi$$

4. *Heat-flow in a cylinder of finite dimensions*

Let us consider the problem of heat-flow in a circular cylinder of radius R and height $2h$. Suppose that the initial temperature of the cylinder is equal to $\varphi(r,\theta,z)$ and that the temperature of its side and basis is held at zero. The problem then reduces to solving the equation

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{96}$$

with the boundary conditions

$$u|_{z=-h} = u|_{z=h} = 0\ , \qquad u|_{r=R} = 0 \tag{97}$$

and initial condition

$$u|_{t=0} = \varphi(r,\theta,z)\ . \tag{98}$$

If we apply the Fourier method and determine the constants thus introduced in terms of the boundary conditions, we obtain the following particular solutions of eq. (96):

$$\exp\left[-a^2\left(\lambda^2 + \frac{m^2\pi^2}{4h^2}\right) t\right] J_n(\lambda r) \sin\frac{m\pi}{2h}(z+h)\,(A\cos n\theta + B\sin n\theta)\ . \tag{99}$$

Here, m denotes positive integers, as is required from the second of the boundary conditions (97). The constant λ is connected with the roots of the equation

$$J_n(\mu) = 0 \tag{100}$$

by the equation

$$\lambda = \mu/R\ . \tag{101}$$

This requirement is the third of the boundary conditions (97). The number n must be an integer since the temperature of the cylinder is a periodic function of the angle θ with period 2π.

If we take the sum of all solutions of the form (99) for all non-negative integral values of n and all positive integral values of m and for all positive roots μ_{n1}, μ_{n2}, μ_{n3} of eq. (100), we obtain a solution of the problem in the form of the series

$$u = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \exp\left[-a^2\left(\frac{\mu_{nk}^2}{R^2} + \frac{m^2\pi^2}{4h^2}\right) t\right] J_n(\mu_{nk}r/R)$$

$$\times \sin \frac{m\pi}{2h} (z+h)(A_{kmn} \cos n\theta + B_{kmn} \sin n\theta) , \qquad (102)$$

in which the coefficients A_{kmn} and B_{kmn} are still to be determined. Let us set $t = 0$ in the expansion (102). Then, taking account of the initial condition (98), we obtain

$$\varphi(r,\theta,z) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} J_n(\mu_{nk}r/R) \sin \frac{m\pi}{2h} (z+h)$$

$$\times (A_{kmn} \cos n\theta + B_{kmn} \sin n\theta) . \qquad (103)$$

Since the right side of eq. (103) is an expansion of the functions $\varphi(r,\theta,z)$ in a Fourier series in cos $n\theta$ and sin $n\theta$, the coefficients of these trigonometric functions can be determined from the familiar formulae. Thus, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi(r,\theta,z)\, d\theta = \sum_{k=1}^{\infty} \left[\sum_{m=1}^{\infty} A_{km0} \sin \frac{m\pi}{2h} (z+h)\right] J_0(\mu_{0k}r/R) ,$$

$$\frac{1}{\pi} \int_0^{2\pi} \varphi(r,\theta,z) \cos n\theta\, d\theta = \sum_{k=1}^{\infty} \left[\sum_{m=1}^{\infty} A_{kmn} \sin \frac{m\pi}{2h} (z+h)\right] J_n(\mu_{nk}r/R) ,$$

$$\frac{1}{\pi} \int_0^{2\pi} \varphi(r,\theta,z) \sin n\theta\, d\theta = \sum_{k=1}^{\infty} \left[\sum_{m=1}^{\infty} B_{kmn} \sin \frac{m\pi}{2h} (z+h)\right] J_n(\mu_{nk}r/R) .$$

Each of these equations itself is an expansion of the function treated as a function of r in a series of Bessel functions. The coefficients in these expansions may be determined from formula (43) (Chapter XII, section 5), so that

$$\sum_{m=1}^{\infty} A_{km0} \sin \frac{m\pi}{2h} (z+h) = \frac{1}{\pi R^2 J_1{}^2(\mu_{0k})} \int_0^R \int_0^{2\pi} r\varphi(r,\theta,z)\, J_0(\mu_{0k}r/R)\, dr\, d\theta, \qquad (104)$$

$$\sum_{m=1}^{\infty} A_{kmn} \sin \frac{m\pi}{2h} (z+h)$$

$$= \frac{2}{\pi R^2 J_{n+1}^2(\mu_{nk})} \int_0^R \int_0^{2\pi} r\varphi(r,\theta,z)\, J_n(\mu_{nk}r/R) \cos n\theta\, dr\, d\theta , \qquad (105)$$

$$\sum_{m=1}^{\infty} B_{kmn} \sin \frac{m\pi}{2h} (z+h)$$

$$= \frac{2}{\pi R^2 J_{n+1}^2(\mu_{nk})} \int_0^R \int_0^{2\pi} r\varphi(r,\theta,z)\, J_n(\mu_{nk}r/R) \sin n\theta\, dr\, d\theta . \qquad (106)$$

Since the functions $\sin(m\pi(z+h)/2h)$ (for $m = 1, 2, 3, \ldots$) form an orthogonal system of functions on the interval $[-h, h]$, we find, by the usual method, that the coefficients A_{kmn} and B_{kmn} in the expansions (104), (105), and (106) are given by the formulae

$$A_{km_0} = \frac{1}{\pi R^2 h J_1{}^2(\mu_{0k})} \int_0^R \int_0^{2\pi} \int_{-h}^{h} r\varphi(r,\theta,z)\, J_0(\mu_{0k} r/R) \times \sin\frac{m\pi}{2h}(z+h)\, dr\, d\theta\, dz\,,$$

$$A_{kmn} = \frac{2}{\pi h R^2 J_{n+1}^2(\mu_{nk})} \int_0^R \int_0^{2\pi} \int_{-h}^{h} r\varphi(r,\theta,z)\, J_n(\mu_{nk} r/R) \times \cos n\theta \sin\frac{m\pi}{2h}(z+h)\, dr\, d\theta\, dz\,,$$

$$B_{kmn} = \frac{2}{\pi h R^2 J_{n+1}^2(\mu_{nk})} \int_0^R \int_0^{2\pi} \int_{-h}^{h} r\varphi(r,\theta,z)\, J_n(\mu_{nk} r/R) \times \sin n\theta \sin\frac{m\pi}{2h}(z+h)\, dr\, d\theta\, dz\,.$$

When we substitute these values of the coefficients into the series (102), we obtain the final solution to the problem (96) - (98).

5. *Heat-flow in a homogeneous sphere*

Let us examine the problem of heat-flow in a homogeneous sphere of radius R with center at the origin coordinates.

1. Let us first consider the case in which the temperature at an arbitrary point of the sphere depends only on the distance r of that point from the center. We also assume that on the surface of the sphere there is an exchange of heat with the surrounding medium, the temperature of which we assume to be zero. In this case, the problem reduces to integrating the heat-flow equation

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) \tag{107}$$

with boundary condition

$$\frac{\partial u}{\partial r} + hu\Big|_{r=R} = 0 \tag{108}$$

and initial condition

$$u\big|_{t=0} = \varphi(r)\,. \tag{109}$$

Setting $v = ur$, we have

$$\frac{\partial v}{\partial t} = a^2\frac{\partial^2 v}{\partial r^2}\,, \tag{110}$$

$$v|_{r=0} = 0 \,, \qquad \frac{\partial v}{\partial r} + \left(h - \frac{1}{R}\right) v|_{r=R} = 0 \,, \tag{111}$$

$$v|_{t=0} = r\varphi(r) \,. \tag{112}$$

The problem (107) - (109) is thus reduced to the problem of heat-flow in a rod, the temperature at one end of which is held equal to zero and the other end of which exchanges heat with the surrounding medium.

Using the Fourier method, we find particular solutions of eq. (110) in the form of a product

$$v = a\, e^{-\lambda^2 a^2 t} \sin \lambda r \,. \tag{113}$$

These solutions satisfy the boundary conditions (11) for arbitrary a if λ is a root of the equation

$$\lambda R \cos \lambda R + (hR - 1) \sin \lambda R = 0 \,. \tag{114}$$

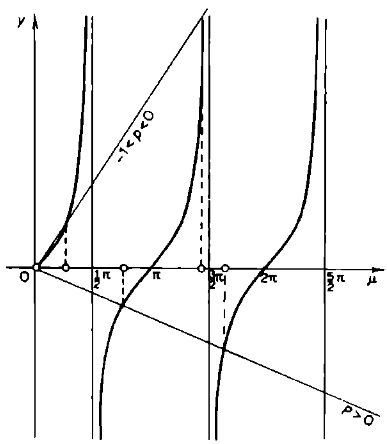

Fig. 66.

Let us set

$$\lambda R = \mu \,, \qquad p = hR - 1 > -1 \,. \tag{115}$$

Then eq. (114) takes the form

$$\mu \cos \mu + p \sin \mu = 0 \,. \tag{116}$$

This equation has an infinite number of real roots, as can easily be shown by constructing graphs of the curves (fig. 66) $y = \tan \mu$, $y = -\mu/p$.

From the figure, one can see that when $-1 < p < 0$, there will be a positive root of eq. (116) in each of the intervals $(0, \frac{1}{2}\pi), (\pi, \frac{3}{2}\pi), \ldots$ and that with increasing n they approach the values $\frac{1}{2}(2n - 1)$. When $0 < p < \infty$, the positive roots lie in the intervals $(\frac{1}{2}\pi, \pi), (\frac{1}{2}\pi, 2\pi), \ldots$ and, with increasing n, they approach the values $\frac{1}{2}(2n - 1)\pi$. We note that the negative roots of eq.

(116) are equal in absolute value to the positive roots. We denote by $\mu_1, \mu_2, \mu_3, \ldots$ the positive roots of eq. (116). Then, from (115), the eigenvalues will be

$$\lambda_n^2 = (\mu_n/R)^2 \qquad (n = 1, 2, 3, \ldots) . \tag{117}$$

To each eigenvalue corresponds an eigenfunction

$$w_n(r) = \sin(\mu_n r/R) . \tag{118}$$

Let us now construct the series

$$v(r, t) = \sum_{n=1}^{\infty} a_n \, e^{-(\mu_n a/R)^2 t} \sin \mu_n r/R . \tag{119}$$

When we satisfy the initial condition (112), we obtain

$$r\varphi(r) = \sum_{n=1}^{\infty} a_n \sin \frac{\mu_n r}{R} . \tag{120}$$

If we multiply both sides of the expansion (120) by $\sin(\mu_k r/R)$ and integrate from 0 to R, and, using the equalities

$$\int_0^R \sin \frac{\mu_n r}{R} \sin \frac{\mu_k r}{R} \, dr = \begin{cases} 0 & \text{for} \quad k \neq n , \\ \dfrac{R}{2} \dfrac{p(p+1) + \mu_n^2}{p^2 + \mu_n^2} & \text{for} \quad k = n , \end{cases}$$

we find the coefficients a_n from the formula

$$a_n = \frac{2}{R} \frac{p^2 + \mu_n^2}{p(p+1) + \mu_n^2} \int_0^R r\varphi(r) \sin \frac{\mu_n r}{R} \, dr .$$

Substituting this expression for the coefficients a_n in the series (119) and remembering that $v = ru$, we obtain the solution of the problem (107) - (109) in the form

$$u(r, t) = \frac{2}{Rr} \sum_{n=1}^{\infty} \frac{p^2 + \mu_n^2}{p(p+1) + \mu_n^2} e^{-(\mu_n a/R)^2 t} \sin \frac{\mu_n r}{R} \int_0^R \rho\varphi(\rho) \sin \frac{\mu_n \rho}{R} \, d\rho . \tag{121}$$

2. Let us now investigate the general case in which the temperature of the sphere depends on all three coordinates r, θ, and φ. Here, we assume that the temperature of the surface of the sphere is zero.

If we transform the heat-flow equation to spherical coordinates r, θ, and φ, the problem of the heat-flow in the sphere reduces to integrating the equation

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2}\right] \tag{122}$$

with the conditions

$$u|_{r=R} = 0 , \tag{123}$$

$$u|_{t=0} = f(r,\theta,\varphi) \ . \tag{124}$$

Let us seek particular solutions of eq. (122) in the form

$$u = T(t)\, v(r,\theta,\varphi) \ . \tag{125}$$

Substituting this into eq. (122), we obtain

$$\frac{\Delta v}{v} = \frac{T'(t)}{a^2 T(t)} = -k^2 \ ,$$

so that we have the two equations

$$T'(t) + a^2k^2T(t) = 0 \ , \tag{126}$$

$$\Delta v + k^2 v = 0 \ , \tag{127}$$

where

$$\Delta v = \frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial v}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\varphi^2} \ .$$

To obtain non-trivial solutions of eqs. (122) of the form (125) that satisfy the boundary condition (123), we need to find non-trivial solutions of eq. (127) that satisfy the boundary condition

$$v|_{r=R} = 0 \ . \tag{128}$$

We shall seek a solution to eq. (127) satisfying the boundary condition (128) in the form

$$v = \Phi(r)\; Y(\theta,\varphi) \ . \tag{129}$$

Substituting this into eq. (127) and separating the variables, we obtain

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} + \lambda Y = 0 \ , \tag{130}$$

$$\frac{d^2\Phi(r)}{dr^2} + \frac{2}{r}\frac{d\Phi(r)}{dr} + \left(k^2 - \frac{\lambda}{r^2}\right)\Phi(r) = 0 \ . \tag{131}$$

If we solve eq. (130) upon the condition that the solution v is bounded on the entire surface of the sphere, we obtain the eigenvalues

$$\lambda = n(n+1) \ , \tag{132}$$

to which correspond the spherical functions (see Chapter XXI, section 1)

$$P_n(\cos\theta)\ , \qquad P_{nm}(\cos\theta)\cos m\varphi\ , \qquad P_{nm}(\cos\theta)\sin m\varphi \qquad (m = 1, 2, \ldots, n) \ . \tag{133}$$

Let us now consider eq. (131). If we take eq. (132), the boundary condition (128), and the boundedness of the solution for $r = 0$ into consideration, we obtain the following boundary-value problem for the function $\Phi(r)$:

$$\frac{d^2\Phi(r)}{dr^2} + \frac{2}{r}\frac{d\Phi(r)}{dr} + \left(k^2 - \frac{n(n+1)}{r^2}\right)\Phi(r) = 0 \ , \tag{134}$$

$$\Phi(0) < \infty, \qquad \Phi|_{r=R} = 0 . \tag{135}$$

When we set

$$\Phi(r) = y(r)/\sqrt{r} , \tag{136}$$

eq. (134) becomes Bessel's equation

$$r^2 y'' + ry' + [k^2 r^2 - (n+\tfrac{1}{2})^2] y = 0 ,$$

the general solution of which is of the form (Chapter XII, section 1)

$$y = AJ_{n+\frac{1}{2}}(kr) + BY_{n+\frac{1}{2}}(kr) . \tag{137}$$

It follows from the boundedness of the solution that $B = 0$. The boundary condition (135) gives us

$$AJ_{n+\frac{1}{2}}(kR) = 0 .$$

Since we seek non-trivial solutions to eq. (134), it follows that $A \neq 0$ and, consequently,

$$J_{n+\frac{1}{2}}(kR) = 0 .$$

If we denote by $\mu_{n1}, \mu_{n2}, \mu_{n3}, \ldots$ the positive roots of the transcendental equation

$$J_{n+\frac{1}{2}}(\mu) = 0 , \tag{138}$$

we find the eigenvalues

$$k^2_{mn} = (\mu_{nm}/R)^2 \qquad (m = 1, 2, 3, \ldots ,\ n = 0, 1, 2, 3, \ldots) . \tag{139}$$

To each eigenvalue k^2_{nm} of the boundary-value problem (127) - (128) there correspond the $2n+1$ eigenfunctions

$$v_{nmj}(r, \theta, \varphi) = \frac{J_{n+\frac{1}{2}}(\mu_{nm} r/R)}{\sqrt{r}}\, Y_n^{(j)}(\theta, \varphi) \qquad (j = -n, \ldots , -1, 0, 1, \ldots , n) , \tag{140}$$

where we set

$$Y_n^{(\nu)}(\theta, \varphi) = P_{n\nu}(\cos\theta) \cos \nu\varphi , \qquad Y_n^{(-\nu)}(\theta, \varphi) = P_{n\nu}(\cos\theta) \sin \nu\varphi$$
$$(\nu = 0, 1, 2, \ldots , n) .$$

For $k = k_{mn}$, the general solution of eq. (126) is of the form

$$T_{mn}(t) = A_{mn}\, e^{-(a\mu_{nm}/R)^2 t} , \tag{141}$$

where A_{mn} is an arbitrary constant.

Thus, on the basis of (125), (140), and (141), all functions of the form

$$u_{mn}(r, \theta, \varphi, t) = \frac{J_{n+\frac{1}{2}}(\mu_{nm} r/R)\; Y_n(\theta, \varphi)}{\sqrt{r}}\, e^{-(a\mu_{nm}/R)^2 t} , \tag{142}$$

where

$$Y_n(\theta, \varphi) = a_{0m} P_n(\cos\theta) + \sum_{k=1}^{n} (a_{km} \cos k\varphi + b_{km} \sin k\varphi)\, P_{nk}(\cos\theta) \tag{143}$$

is a spherical function of order n, satisfy eq. (122) and the boundary condition (123) for arbitrary constants a_{om}, a_{km} and b_{km}.

We now construct the series

$$u(r,\theta,\varphi,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{e^{-(a\mu_{nm}/R)^2 t}}{\sqrt{r}} Y_n(\theta,\varphi)\, J_{n+\frac{1}{2}}(\mu_{nm} r/R) \; . \tag{144}$$

When we satisfy the initial condition (124), we obtain

$$f(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{Y_n(\theta,\varphi)}{\sqrt{r}}\, J_{n+\frac{1}{2}}(\mu_{nm} r/R) \; . \tag{145}$$

To find the spherical functions $Y_n(\theta,\varphi)$ in the expansion (145), we multiply both sides by $P_k(\cos\gamma)$ and integrate over the surface of the sphere of unit radius. Then, we obtain

$$\int_0^{\pi}\int_0^{2\pi} f(r,\theta',\varphi')\, P_k(\cos\gamma)\, d\sigma$$
$$= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{J_{n+\frac{1}{2}}(\mu_{nm} r/R)}{\sqrt{r}} \int_0^{\pi}\int_0^{2\pi} Y_n(\theta',\varphi')\, P_k(\cos\gamma)\, d\sigma \; . \tag{146}$$

Recalling the formula (see Chapter XXI, section 2)

$$\int_0^{\pi}\int_0^{2\pi} Y_n(\theta',\varphi')\, P_k(\cos\gamma)\, d\sigma = \begin{cases} 0 & \text{for } k \neq n \,, \\ \dfrac{4\pi}{2n+1} Y_n(\theta,\varphi) & \text{for } k = n \,, \end{cases}$$

we can rewrite eq. (146) in the form

$$\int_0^{\pi}\int_0^{2\pi} f(r,\theta',\varphi')\, P_n(\cos\gamma)\, d\sigma = \frac{4\pi}{2n+1} \sum_{m=1}^{\infty} Y_n(\theta,\varphi) \frac{J_{n+\frac{1}{2}}(\mu_{nm} r/R)}{\sqrt{r}}$$

or

$$\frac{(2n+1)\sqrt{r}}{4\pi} \int_0^{\pi}\int_0^{2\pi} f(r,\theta',\varphi')\, P_n(\cos\gamma)\, d\sigma = \sum_{m=1}^{\infty} Y_n(\theta,\varphi)\, J_{n+\frac{1}{2}}(\mu_{nm} r/R) \; . \tag{147}$$

We now compare this expansion with the expansion of the arbitrary function $F(r)$ in a series of Bessel functions (see Chapter XII, section 4):

$$F(r) = \sum_{m=1}^{\infty} a_m J_{n+\frac{1}{2}}(\mu_{nm} r/R) \; , \tag{148}$$

where

$$a_m = \frac{2}{R^2 J^2_{n+\frac{3}{2}}(\mu_{nm})} \int_0^R rF(r)\, J_{n+\frac{1}{2}}(\mu_{nm} r/R) \; ,$$

and μ_{nm} are the positive roots of eq. (138).

From a comparison of the series (147) and (148), we find the desired expression for the spherical functions $Y_n(\theta,\varphi)$, namely,

$$Y_n(\theta,\varphi) = \frac{2n+1}{2\pi R^2 J^2_{n+\frac{3}{2}}(\mu_{nm})} \int_0^R \int_0^\pi \int_0^{2\pi} r^{\frac{3}{2}} f(r,\theta',\varphi')\, P_n(\cos\gamma)$$

$$\times J_{n+\frac{1}{2}}(\mu_{nm} r/R)\, d\sigma\, dr\ . \tag{149}$$

Thus, the solution of the problem (122) - (124) is the series (144) in which the spherical functions $Y_n(\theta,\varphi)$ are determined from formula (149).

6. *Heat-flow in a rectangular plate*

Consider a thin homogeneous rectangular plate the temperature of the edge of which is kept at zero. The initial distribution of the temperature is given and we are to determine the temperature of the plate at an arbitrary subsequent instant of time under the assumption that there is no heat exchange between the large surfaces of the plate and the surrounding medium.

Obviously, this problem reduces to solving the equation

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{150}$$

with the boundary conditions

$$u|_{x=0} = u|_{x=p} = 0\ , \qquad u|_{y=0} = u|_{y=q} = 0 \tag{151}$$

and the initial condition

$$u|_{t=0} = \varphi(x,y)\ . \tag{152}$$

Using the Fourier method, let us seek particular solutions to eq. (150) in the form of a product

$$u = T(t)\ X(x)\ Y(y)\ .$$

Then, to determine the functions $X(x)$, $Y(y)$, and $T(t)$, we obtain the equations

$$X''(x) + \lambda^2 X(x) = 0\ , \qquad Y''(y) + \mu^2 Y(y) = 0\ , \qquad T'(t) + a^2(\lambda^2+\mu^2)\ T(t) = 0\ ,$$

where λ^2 and μ^2 are constants.

The general solutions of these equations are of the form

$$X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x, \qquad Y(y) = C_3 \cos\mu x + C_4 \sin\mu x\ ,$$

$$T(t) = A\ \exp\left[-a^2(\lambda^2+\mu^2)t\right]\ .$$

To satisfy the conditions (151), we need to set

$$C_1 = 0\ , \qquad C_3 = 0\ , \qquad \lambda = m\pi/p\ , \qquad \mu = n\pi/q \qquad (m,n = 1,2,3,\ldots)\ .$$

Thus, the particular solutions of eq. (150) satisfying the boundary conditions (151) are

$$u_{nm} = A_{mn} \exp\left[-a^2\pi^2\left(\frac{m^2}{p^2} + \frac{n^2}{q^2}\right) t\right] \sin\frac{m\pi}{p} x \sin\frac{n\pi}{q} y .$$

Let us construct the series

$$u(x, y, t) = \sum_{m,n=1}^{\infty} A_{mn} \exp\left[-a^2\pi^2\left(\frac{m^2}{p^2} + \frac{n^2}{q^2}\right) t\right] \sin\frac{m\pi}{p} x \sin\frac{n\pi}{q} y . \quad (153)$$

When we satisfy the initial condition (152), we obtain

$$\varphi(x, y) = \sum_{m,n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{p} \sin\frac{n\pi y}{q} .$$

This series is the expansion of the function $\varphi(x, y)$ in a double Fourier series, and the coefficients A_{mn} are determined, as can easily be seen, from the formula

$$A_{mn} = \frac{4}{pq}\int_0^p \int_0^q \varphi(x, y) \sin\frac{m\pi}{p} x \sin\frac{n\pi}{q} y \, dx \, dy . \quad (154)$$

Substituting these values of the coefficients A_{mn} in the series (153), we obtain the solution of the problem (150) - (152).

Problems

1. Consider a plate of thickness $2R$ and infinite extent at a temperature 0°. The plate is heated from both sides by the same constant heat-flow q. Find the temperature distribution from one surface of the plate to the other at an arbitrary subsequent instant of time.
   Answer:

$$u(x, t) = \frac{a^2 q}{kR}\left(t - \frac{R^2 - 3x^2}{6a^2}\right) - \frac{2qR}{k\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{k^2} e^{-(n\pi a/R)^2 t} \cos\frac{n\pi x}{R} ,$$

   where k is the thermal conductivity.
   Method: The problem is reduced to integrating the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

   with the conditions

$$k\frac{\partial u}{\partial x} + q\big|_{x=-R} = 0 , \quad -k\frac{\partial u}{\partial x} + q\big|_{x=R} = 0 , \quad \frac{\partial u(0, t)}{\partial x} = 0 , \quad u\big|_{t=0} = 0 .$$

2. Consider a thin rod of length l consisting of two parts made of different substances. The initial temperature of both parts of the rod is the same, namely 0°. At $t > 0$, one end $(x = 0)$ of the rod is raised to and held at constant temperature u_0 while the other end $(x = l)$ is kept at temperature 0°. Find the temperature of the rod at any instant of time subsequent to the initial instant.

Answer:

$$u(x,t) = \begin{cases} \displaystyle\sum_{j=1}^{\infty} \frac{k_1 a_1 u_0 A_j^2}{\mu_j} (1 - e^{-\mu_j^2 t}) \sin a_1 \mu_j x & (0 < x < \xi) , \\ \displaystyle\sum_{j=1}^{\infty} \frac{k_1 a_1 u_0 A_j^2}{\mu_j} (1 - e^{-\mu_j^2 t}) \frac{\sin a_1 \mu_j \xi}{\sin a_2 \mu_j (l-\xi)} \sin a_2 \mu_j (l - x) & \\ & (\xi < x < l) , \end{cases}$$

where the μ_j are roots of the equation

$$a_1 k_1 \cot \mu a_1 \xi + a_2 k_2 \cot a_2 (l - \xi) \mu = 0$$

and the A_j are determined from the normalization condition

$$c_1 \rho_1 \int_0^{\xi} X_{1j}^2 \, dx + c_2 \rho_2 \int_{\xi}^{l} X_{2j}^2 \, dx = 1 ,$$

$$X_{1j} = A_j \sin a_1 \mu_j x , \qquad X_{2j} = A_j \frac{\sin a_1 \mu_j \xi}{\sin a_2 \mu_j (l - \xi)} \sin a_2 \mu_j (l - x) .$$

The orthogonality condition for these functions is

$$c_1 \rho_1 \int_0^{\xi} X_{1j} X_{1k} \, dx + c_2 \rho_2 \int_{\xi}^{l} X_{2j} X_{2k} \, dx = 0 \qquad (j \neq k) .$$

Method: The problem reduces to integrating the equations

$$a_1^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (0 < x < \xi) , \qquad a_2^2 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (\xi < x < l) ,$$

$$a_i^2 = c_i \rho_i / k_i \qquad (i = 1, 2)$$

with the conditions

$$u(0, t) = u_0 , \qquad u(l, t) = 0 , \qquad u(\xi - 0, t) = u(\xi + 0, t) ,$$

$$k_1 \frac{\partial u(\xi - 0, t)}{\partial x} = k_2 \frac{\partial u(\xi + 0, t)}{\partial x} , \qquad u(x, 0) = 0 .$$

3. The temperature of the lateral surface of an infinitely long cylinder of radius R is equal to zero. The initial distribution of the temperature in the cylinder is given by the formula

$$u|_{t=0} = f(r, \theta) .$$

Show that the temperature within the cylinder at a subsequent instant of time is given by the series

$$u(r, \theta, t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (A_{mn} \cos n\theta + B_{mn} \sin n\theta) J_n(\mu_m r/R) \, e^{-(a\mu_m/R)^2 t}$$

where

$$A_{m0} = \frac{1}{\pi R^2 J_1{}^2(\mu_m)} \int_0^R \int_{-\pi}^{\pi} r f(r, \theta)\, J_0(\mu_m r/R)\, dr\, d\theta \ ,$$

$$A_{mn} = \frac{1}{\pi R^2 J_{n+1}^2(\mu_m)} \int_0^R \int_{-\pi}^{\pi} r f(r, \theta)\, J_n(\mu_m r/R) \cos n\theta\, dr\, d\theta \ ,$$

$$B_{mn} = \frac{2}{\pi R^2 J_{n+1}^2(\mu_m)} \int_0^R \int_{-\pi}^{\pi} r f(r, \theta)\, J_n(\mu_m r/R) \sin n\theta\, dr\, d\theta \ ,$$

and $\mu_1, \mu_2, \mu_3, \ldots$ are the positive roots of the equation

$$J_n(\mu) = 0 \ .$$

PART IV

SUPPLEMENTARY MATERIAL

Chapter XXXI

THE USE OF INTEGRAL OPERATORS IN SOLVING PROBLEMS IN MATHEMATICAL PHYSICS

1. *Basic definitions. Method of application of integral operators*

A transformation of the form

$$\iint\limits_{(S)} \cdots \int f(x_1, x_2, \ldots, x_n)\, K(x_1, x_2, \ldots, x_m; \gamma_1, \gamma_2, \ldots, \gamma_m)\, dx_1\, dx_2 \ldots dx_m$$

$$(m \leq n) \,, \tag{1}$$

by means of which a function $f(x_1, x_2, \ldots, x_n)$ of the n variables $x_1, x_2, x_3, \ldots, x_n$ is transformed into a function $\bar{f}(\gamma_1, \gamma_2 \ \ldots, \gamma_m; x_{m+1}, \ldots, x_n)$ of the m variables $\gamma_1, \gamma_2, \ldots, \gamma_m$ and the $n - m$ variables $x_{m+1}, x_{m+2}, \ldots, x_n$, represented by the integral (1), is called an integral operator (transformation) with kernel $K(x_1, x_2, \ldots, x_m; \gamma_1, \gamma_2, \ldots, \gamma_m)$.

Every integral operator is defined by: (1) a kernel

$$K(x_1, x_2, \ldots, x_m; \gamma_1, \gamma_2, \ldots, \gamma_m) \,,$$

(2) a region S of integration, and (3) a set Φ of functions $f(x_1, x_2, \ldots, x_n)$ on which it operates. To define a particular integral operator, one needs to specify all three of these.

The term "integral transform" is applied to the results of the transformation. Thus, the integral (1) is an integral transform with respect to the variables $x_1, x_2, \ldots, x_n$ of the function $f(x_1, x_2, \ldots, x_n)$. The integral (1) is also said to be a mapping (with respect to the variables $x_1, x_2, \ldots, x_m$) of the function $f(x_1, x_2, \ldots, x_n)$, which is called the *original* function. The variables of integration $x_1, x_2, \ldots, x_m$ in the integral (1) are called the variables of transformation. The operator (1) itself is said to be taken with respect to the variables $x_1, x_2, \ldots, x_m$.

We shall denote the transformed functions by the same symbols as were used before the transformation, but with the addition of a mark above the symbol: a macron, a double macron, a tilde, etc. The argument of the transformed function will make it clear with respect to what variable the transformation has been performed. For example, the expression $\bar{f}(\gamma_1; x_2, x_3)$ will denote the integral transform with respect to the variable x_1 of the function $f(x_1, x_2, x_3)$. We shall omit the argument x_i or γ_i in those cases in which such an omission cannot lead to misunderstanding.

The operator that retransforms $\bar{f}(\gamma_1, \gamma_2, \ldots, \gamma_m; x_{m+1}, \ldots, x_n)$ into the function $f(x_1, x_2, \ldots, x_n)$ is called the *inverse* of the integral operator (1) or simply the inverse operator. Here, the operator (1) is called the *direct*

operator (transformation). We note that the inverse operator is not always an integral operator.

We shall be using only integral operators with respect to a single variable. The general procedure for applying such integral operators to the solution of problems in mathematical physics consists in the following: By means of an integral operator, a problem is transformed in such a way as to eliminate differentiation with respect to one of the variables, thus resulting in a simpler problem involving the transformed function. Having solved the latter, we use the inverse operator to find the unknown function that is the solution to the original problem. If necessary, we use several integral transformations, successively eliminating differential operations with respect to the different variables. Having solved the transformed problem, one uses the inverse operators to obtain the desired function.

We shall consider only problems involving partial differential equations and shall not use integral operators to solve ordinary differential equations, which is the basic subject matter of the branch of mathematics known as operational calculus.

2. *Conditions allowing the use of integral operators*

Consider the problem posed, in some region, for the second-order differential equation

$$\mathfrak{M}u = f , \tag{2}$$

where

$$\mathfrak{M}u \equiv \sum_{\alpha,\beta=0}^{3} a_{\alpha\beta} \frac{\partial^2 u}{\partial x_\alpha \, \partial x_\beta} + \sum_{\alpha=0}^{3} b_\alpha \frac{\partial u}{\partial x_\alpha} + c \qquad (a_{ij} = a_{ji}) \tag{3}$$

is a second-order differential expression with variable coefficients and f is a known function of the variables x_0, x_1, x_2, and x_3 (these may be the coordinates of a point in space and time). In the general case, among the additional given conditions of a problem that ensure uniqueness of solution, there may be initial and boundary conditions, conditions at infinity, conditions of periodicity of the solutions with respect to certain coordinates, conditions of conjugacy on the boundary between media, and so forth. In making a transformation, we must, of course, transform both the given equations and the supplementary conditions.

We shall assume below that all the functions that we are subjecting to integral transformation have the properties that make such a transformation possible. In examining specific problems, we shall necessarily consider the matter of the applicability of the operator in question.

Let us find conditions sufficient to ensure transformation of a problem to a form in which it does not contain differential or integral operations with respect to the variable of transformation.

Obviously, we need to choose the limits of integration for an operator in such a way that they coincide with the limits (a, b) of variation of the variable of transformation x_i; otherwise, the values of the transformed func-

tion outside the interval of integration would not be taken into account or the integration would be extended to a region in which the transformed functions cannot be defined. Thus, if the variable of transformation varies within finite limits, the integral operator must have finite limits; in the opposite case the integral operator must have infinite limits. In view of this, we shall distinguish between finite and infinite integral operators.

If we subject eq. (2) (in the interval (a, b) of variation of the variable x_i) to an integral operator with kernel $K(x_i, \gamma)$, we obtain

$$\int_a^b \Big(\sum_{\alpha,\beta \neq i} a_{\alpha\beta} \frac{\partial^2 u}{\partial x_\alpha \, \partial x_\beta} + \sum_{\alpha \neq i} b_\alpha \frac{\partial u}{\partial x_\alpha} \Big) K(x_i, \gamma) \, \mathrm{d}x_i$$
$$+ \int_a^b \Big(a_{ii} \frac{\partial^2 u}{\partial x_i^2} + 2 \sum_{\beta \neq i} a_{i\beta} \frac{\partial^2 u}{\partial x_i \, \partial x_\beta} + b_i \frac{\partial u}{\partial x_i} + cu \Big) K(x_i, \gamma) \, \mathrm{d}x_i = \bar{f}\,, \tag{4}$$

where $\bar{f}$ is the integral transform of the free term f. Let us find sufficient conditions for this integral relationship to be transformed into a differential equation with respect to the integral transform

$$\bar{u} = \int_a^b uK(x_i, \gamma) \, \mathrm{d}x_i$$

of the unknown function u.

If the coefficients a_{jk} and b_j for $j, k \neq i$ do not depend on the variable of transformation x_i, the first integral on the left side of eq. (4) will take the form

$$\sum_{\alpha,\beta \neq i} a_{\alpha\beta} \frac{\partial^2 \bar{u}}{\partial x_\alpha \, \partial x_\beta} + \sum_{\alpha \neq i} b_\alpha \frac{\partial \bar{u}}{\partial x_\alpha}\,.$$

In this case, for example,

$$\int_a^b a_{jk} \frac{\partial^2 u}{\partial x_j \, \partial x_k} K(x_i, \gamma) \, \mathrm{d}x_i = a_{jk} \frac{\partial^2}{\partial x_j \, \partial x_k} \int_a^b uK(x_i, \gamma) \, \mathrm{d}x_i \equiv a_{jk} \frac{\partial^2 \bar{u}}{\partial x_j \, \partial x_k}\,.$$

We integrate the second integral on the left side of eq. (4) by parts:

$$\int_a^b \Big(a_{ii} \frac{\partial^2 u}{\partial x_i^2} + 2 \sum_{\beta \neq i} \frac{\partial^2 u}{\partial x_i \, \partial x_\beta} + b_i \frac{\partial u}{\partial x_i} + cu \Big) K(x_i, \gamma) \, \mathrm{d}x_i$$
$$= \left\{ \left[a_{ii} \frac{\partial u}{\partial x_i} + \Big(b_i - \frac{\partial a_{ii}}{\partial x_i} \Big) u + 2 \sum_{\beta \neq i} a_{i\beta} \frac{\partial u}{\partial x_\beta} \right] K(x_i, \gamma) - u a_{ii} \frac{\partial K}{\partial x_i} \right\} \Bigg|_a^b$$
$$+ \int_a^b \Big(\frac{\partial^2 a_{ii} K}{\partial x_i^2} - \frac{\partial b_i K}{\partial x_i} + cK \Big) u \, \mathrm{d}x_i - 2 \sum_{\beta \neq i} \int_a^b \frac{\partial u}{\partial x_\beta} \frac{\partial a_{i\beta} K}{\partial x_i} \, \mathrm{d}x_i\,. \tag{5}$$

The transformed equations will not contain integral terms if

$$\sum_{\beta \neq i} \int_a^b \frac{\partial u}{\partial x_\beta} \frac{\partial a_{i\beta} K}{\partial x_\beta} \, dx_i = 0 \, , \tag{6}$$

$$\frac{\partial^2 a_{ii} K}{\partial x_i^2} - \frac{\partial b_i K}{\partial x_i} + cK = -\lambda^2 K \, , \tag{7}$$

where λ^2 is a quantity that does not depend on x_i *.

The first of these relationships can be satisfied if

$$a_{ij} = 0 \quad \text{for} \quad j \neq i \, ; \tag{8}$$

that is, if the differential expression (3) is of the form

$$\mathcal{M} u = \mathcal{M}_i u + \mathcal{M}' u \, , \tag{9}$$

where

$$\mathcal{M}_i u = a_{ii} \frac{\partial^2 u}{\partial x_i^2} + b_i \frac{\partial u}{\partial x_i} + cu \, , \tag{10}$$

and $\mathcal{M}'u$ is an expression containing no derivatives with respect to x_i.

If the second relationship is satisfied, then

$$\int_a^b \left(\frac{\partial^2 a_{ii} K}{\partial x_i^2} - \frac{\partial b_i K}{\partial x_i} + cK \right) u \, dx_i = -\lambda^2 \int_a^b uK \, dx_i = -\lambda^2 \bar{u} \, .$$

If, in addition, the coefficients a_{ii}, b_i, and c do not depend on the variables x_j (where $j \neq i$), the quantity λ^2 will not depend on x_j. Then, the relationship (7) can be regarded as the equation for defining the kernel $K(x_i, \gamma)$.

When we satisfy the relationships (8), the integrand on the right side of eq. (5) can be written in the form

$$\left[\left(a_{ii} \frac{\partial u}{\partial x_i} + b_i u \right) K - u \frac{\partial a_{ii} K}{\partial x_i} \right] \Bigg|_a^b \, . \tag{11}$$

This expression contains the values of the function u and its derivative $\partial u / \partial x_i$ at $x_i = a$ and $x_i = b$. For us to calculate it, it is obviously necessary that the conditions of the problem (boundary, initial, and so on) admit an expression of the conditions with respect to the variable x_i in terms of known functions of the variables x_j (where $j \neq i$). Otherwise, the transformed equation would contain not only $\bar{u}$ but other unknown functions. However, this requirement alone is not sufficient, since the values of u and $\partial u / \partial x_i$ cannot be given arbitrarily for two different values of x_i and, consequently, they cannot appear simultaneously in the conditions of the problem. This does not allow us to take an arbitrary solution of eq. (7) for the kernel of the operator. Nonetheless, if we subject the kernel to additional requirements that we shall state in the following sections, the expression (11) can be computed. This makes possible the exclusion of differential and integral

* We write λ^2 and not λ, for example, because we shall need to use the square roots of these quantities and thus the notation that we have adopted is more convenient.

operations with respect to the variable x_i from the equation of the problem.

Let us consider the transformation of the conditions of the problem. The conditions of the problem for the original equation (3) apply to the entire set of variables x_j. The transformed equation does not contain differential operations with respect to x_i and, consequently, the conditions for the transformation of the problem must apply only to the variables x_j for $j \neq i$. According to our assumption, the conditions with respect to the variable x_i can be expressed in terms of known functions of the variables x_j (for $j \neq i$). Therefore, the conditions with respect to the variables x_j do not depend on the conditions with respect to x_i. If, in addition, the conditions with respect to x_j (for $j \neq i$) contain neither coefficients depending on x_i nor derivatives of u with respect to x_i, then when we transform these conditions using the integral operator with the chosen kernel, we can obtain the conditions for the transformed function $\bar{u}$. For example, the boundary condition

$$\alpha \frac{\partial u}{\partial x_j} + \beta u = \varphi \quad \text{for} \quad x_j = a_j$$

(under the assumption that change of order of differentiation and integration is valid) is obviously transformed to the form

$$\alpha \frac{\partial \bar{u}}{\partial x_j} + \beta \bar{u} = \bar{\varphi} \quad \text{for} \quad x_j = a_j \ ,$$

provided the coefficients α and β do not depend on x_i.

In conclusion, let us state the conditions which are sufficient to allow us to eliminate differential and integral operations with respect to one of the variables by use of an integral transformation.

1. The left side of the equation must be the sum of two expressions, one containing derivatives only with respect to the variable of transformation, and the other containing neither derivatives with respect to that variable nor coefficients dependent on it.

2. The supplementary conditions of the problem (boundary and initial conditions, etc.) must fall into two groups, one expressing the condition with respect to the variable of transformation (that is, for $x_i = a$ and $x_i = b$) in terms of given functions of the other variables, and the other containing no derivatives with respect to the variable of transformation and no coefficients that are dependent on it.

3. The limits of integration in the operator must coincide with the limits of variation of the variable of transformation, and the kernel of the operator must be a solution of eq. (7) and must satisfy the supplementary conditions (11).

In the next two sections, we shall make more precise the requirements of the kernel of the operator and shall give both the form of the transformed equation and formulae for the inverse operator.

3. *Finite integral transformations*

In order to give the formulae for the direct and inverse operators used

for a variable of transformation that changes within finite limits, we represent the kernel of the direct operator in the form

$$K(x_i, \gamma) = \frac{1}{C_\gamma}\, \rho(x_i)\, \overline{K}(x_i, \gamma) \ , \tag{12}$$

where C_γ, $\rho(x_i)$ and $\overline{K}(x_i, \gamma)$ are functions (to be determined later) of the indicated arguments. We shall call the function C_γ of the argument γ the *normalizing divisor*. When we substitute expression (12) into eq. (7), we obtain

$$a_{ii}\rho\, \frac{\partial^2 \overline{K}}{\partial x_i{}^2} + \left(2\, \frac{\mathrm{d}a_{ii}\rho}{\mathrm{d}x_i} - b_i\rho\right) \frac{\partial \overline{K}}{\partial x_i} - q\overline{K} = -\lambda^2 \rho \overline{K} \ , \tag{13}$$

where

$$q \equiv -\left(\frac{\mathrm{d}^2 a_{ii}\rho}{\mathrm{d}x_i{}^2} - \frac{\mathrm{d}b_i\rho}{\mathrm{d}x_i} + c\rho\right) . \tag{14}$$

We determine the function ρ by the condition

$$\mathrm{d}a_{ii}\rho/\mathrm{d}x_i = b_i\rho \ , \tag{15}$$

from which,

$$a_{ii}\, \frac{\mathrm{d}\rho}{\mathrm{d}x_i} + \left(\frac{\mathrm{d}a_{ii}}{\mathrm{d}x_i} - b_i\right) \rho = 0 \ .$$

It is easy to see that the solution of this equation will be the function

$$\rho(x_i) = \exp\left[-\int \frac{1}{a_{ii}} \left(\frac{\mathrm{d}a_{ii}}{\mathrm{d}x_i} - b_i\right) \mathrm{d}x_i\right] . \tag{16}$$

It follows from condition (15) that eq. (13) is of the form

$$\frac{\partial}{\partial x_i}\, p\, \frac{\partial \overline{K}}{\partial x_i} - q\overline{K} + \lambda^2 \rho \overline{K} = 0 \ , \tag{17}$$

where

$$p = a_{ii}\rho \ , \qquad q = -c\rho \ . \tag{18}$$

An equation of the form (17) is often encountered in mathematical physics and has been intensively studied in connection with the classical Sturm-Liouville problem of finding those values of the parameter λ^2 (the eigenvalues of the problem) for which eq. (17) has a solution not identically equal to zero (the eigenfunctions of the problem) satisfying the homogeneous boundary conditions

$$\left[\alpha_a\, \frac{\partial \overline{K}}{\partial x_i} - \beta_a \overline{K}\right]_{x_i=a} = 0 \ , \qquad \left[\alpha_b\, \frac{\partial \overline{K}}{\partial x_i} + \beta_b \overline{K}\right]_{x_i=b} = 0 \ . \tag{19}$$

We have already encountered a particular case of the Sturm-Liouville problem in Chapter VIII when we considered the vibrations of a string.

We shall use certain results of the Sturm-Liouville theory without proof, since proving them would require a complete development of the theory of integral equations.

Let us suppose

(1) that the functions p, dp/dx_i, q, and ρ are continuous on the interval $a \leqslant x_i \leqslant b$ and $p \geqslant 0$, $q \geqslant 0$, $\rho_0 \leqslant \rho \leqslant \rho_1$, where ρ_0 and ρ_1 are positive constants,

(2) that the integral

$$\int_a^b \frac{dx_i}{p}$$

converges, and

(3) that the numbers α_a and β_a and also α_b and β_b are non-negative and that the members of each pair are not simultaneously equal to zero.

Then,

(1) The eigenvalues λ^2 of the Sturm-Liouville problem form an infinite increasing sequence of positive numbers $\lambda_1{}^2 < \lambda_2{}^2 < \lambda_3{}^2 < \dots$.

(2) To each eigenvalue $\lambda_\gamma{}^2$ there corresponds one eigenfunction $\overline{K}_\gamma(x_i)$, that is, one function that reduces eq. (17) to an identity and satisfies the boundary conditions (19).

(3) The system of eigenfunctions $\overline{K}_\gamma$ ($\gamma = 1,2,3,\dots$) is complete in the sense of mean convergence; that is, an arbitrary square-integrable function $f(x_i)$ in the interval (a, b) can be represented in the form of a series

$$f(x_i) = \sum_{\gamma=1}^{\infty} a_\gamma \overline{K}_\gamma , \tag{20}$$

which converges in mean to the function $f(x_i)$. We note that, in accordance with a familiar property of series that converge in mean, the series (20) can be termwise integrated if we first multiply it by any square-integrable function.

(4) The eigenfunctions $\overline{K}_\alpha$ and $\overline{K}_\beta$ are pairwise orthogonal with weight $\rho(x)$; that is,

$$\int_a^b \rho(x_i)\, \overline{K}_\alpha \overline{K}_\beta \, dx_i = \begin{cases} 0 & \text{when } \alpha \neq \beta , \\ C_\alpha \neq 0 & \text{when } \alpha = \beta . \end{cases} \tag{21}$$

If the integral

$$\int_a^b \frac{dx_i}{p}$$

does not converge but the integrals

$$\int_a^{a+\epsilon} \frac{x_i - a}{p} dx_i , \quad \int_{a+\epsilon}^{b-\epsilon} \frac{dx_i}{p} \quad \text{and} \quad \int_{b-\epsilon}^{b} \frac{b - x_i}{p} dx_i ,$$

do converge for some positive ϵ, the same results hold, but the boundary conditions of the form (19) must be replaced by other conditions at the limit a or b at which $p(x_i) = 0$ (or at both limits if $p(a) = p(b) = 0$). Ordinarily, these are the conditions for finiteness of the eigenfunctions.

Even if the function $\rho = 0$ at one of the limits a, b, these results can

remain valid. (Specifically, this is the case with Bessel's equation, Legendre's equation, etc.).

When the coefficients in the Sturm-Liouville equation have the same values at the end points of the interval (a, b) and, instead of the homogeneous boundary condition (19), we have the periodicity condition

$$\left[\alpha \frac{\partial \overline{K}}{\partial x_i} + \beta \overline{K}\right]_{x_i=a} = \left[\alpha \frac{\partial \overline{K}}{\partial x_i} + \beta \overline{K}\right]_{x_i=b}, \tag{22}$$

then the first three of the results given above will still remain valid (except that to each eigenvalue γ there will ordinarily correspond not one but two linearly independent eigenfunctions, which can be extended as periodic functions beyond the end points of the interval (a, b)). Let us renumber these in such a way that the eigenvalue λ_α^2 corresponds to the eigenfunctions $\overline{K}_{2\alpha-1}(x_i)$ and $\overline{K}_{2\alpha}(x_i)$. As in the case of boundary conditions of the form (19), the eigenfunctions corresponding to different eigenvalues are mutually orthogonal. The functions $\overline{K}_{2\alpha-1}(x_i)$ and $\overline{K}_{2\alpha}(x_i)$, corresponding to the same eigenvalue can, because of their linear independence, always be chosen in such a way that they are mutually orthogonal *. Then, eq. (21) will be satisfied, with the result that the analogy for this variant of the boundary conditions will be more complete.

Here, we use the formulation of the Sturm-Liouville theory in terms of convergence in mean and not, for example, in terms of uniform convergence. That this formulation is sufficient for studying the problems of mathematical physics follows from the material in the last chapter of this book. In that chapter, we shall present the concept of *generalized solutions.* Here, we only note that because we are using the concept of convergence in mean, the method developed below will, generally speaking, include the generalized solutions. This means that the solutions of the problems of mathematical physics that we have found by means of integral operators (in those cases in which there is no classical solution representing a twice-differentiable function) can be generalized solutions; that is, they can, for example, fail to have first and second derivatives at certain points of the region in question, etc.

Generalized solutions have the same physical meaning as do the classical. Therefore, if a problem does not have a classical solution but has a generalized solution and if this solution is unique and depends continuously on the given conditions, the statement of the problem remains correct. We note also that if a problem has a unique classical solution, it must necessarily have a generalized solution, which then coincides with the classical.

Let us return to eq. (12) and set

$$\overline{K}(x_i, \gamma) = \overline{K}_\gamma(x_i) \ . \tag{23}$$

Then, the kernel $K(x_i, \gamma)$ of the direct operator will be a solution of eq. (7), because under the condition (12), eqs. (7) and (17) are equivalent. To show the feasibility of using an integral operator, we need to show that expression (11) can be computed.

* See V. I. Smirnov [1]. Vol. 2. p. 156. and also section 3 of Chapter XXV.

Recalling that, in accordance with relationship (15) and (18), $b_i\rho = da_{ii}\rho/dx_i$ and $\rho a_{ii} = p$, we reduce this expression to the form

$$\frac{1}{C_\gamma} p\left(\bar{K}\frac{\partial u}{\partial x_i} - \frac{\partial \bar{K}}{\partial x_i} u\right)\Big|_a^b . \tag{24}$$

Let us consider the boundary conditions of the general form:

$$\left[\alpha_a \frac{\partial u}{\partial x_i} - \beta_a u\right]_{x_i=a} = \varphi_a , \qquad \left[\alpha_b \frac{\partial u}{\partial x_i} + \beta_b u\right]_{x_i=b} = \varphi_b . \tag{25}$$

If $\alpha_a \neq 0$ and $\alpha_b \neq 0$, we choose (for constructing the kernel of the integral operator) a system of eigenfunctions that satisfy the homogeneous boundary conditions

$$\left[\alpha_a \frac{\partial \bar{K}_\gamma}{\partial x_i} - \beta_a \bar{K}_\gamma\right]_{x_i=a} = 0 , \qquad \left[\alpha_b \frac{\partial \bar{K}_\gamma}{\partial x_i} + \beta_b \bar{K}_\gamma\right]_{x_i=b} = 0 . \tag{26}$$

Then, expression (24) takes the form

$$\frac{1}{C_\gamma}\left[\frac{1}{\alpha_b} p(b)\ \varphi_b\ \bar{K}_\gamma(b) - \frac{1}{\alpha_a} p(a)\ \varphi_a\ \bar{K}_\gamma(a)\right] \tag{27}$$

and can be computed, since there are only known functions in it. If $\alpha_a = 0$, $\alpha_b \neq 0$ and $\beta_a \neq 0$, the first of the boundary conditions (25) can, without loss of generality, be written in the form $-u = \varphi_a$. In constructing the system of eigenfunctions, we set

$$\bar{K}_\gamma|_{x_i=a} = 0 , \tag{28}$$

so that expression (24) takes the form

$$\frac{1}{C_\gamma}\left[\frac{1}{a_b} p(b)\ \varphi_b\ \bar{K}_\gamma(b) - p(a)\ \varphi_a \left(\frac{\partial \bar{K}_\gamma}{\partial x_i}\right)_{x_i=a}\right] ; \tag{29}$$

this can also be computed *. In an analogous manner, we could examine the cases of $\alpha_b = 0$ and $\alpha_a = \alpha_b = 0$. The expressions that we would then obtain are obvious and can be calculated from the given boundary conditions of the problem. Thus, the required integral transformation can be carried out.

Let us now suppose that the solution of the problem in question is periodic with respect to the coordinate x_i with period equal to the interval (a, b). If the functions $\bar{K}_\gamma(x_i)$ also satisfy this periodicity condition, the expression (24) will vanish and, consequently, will be known.

From these results, we see that when it is possible to carry out an integral transformation with respect to the variable x_i which varies in a finite interval (a, b), the equation $\mathfrak{M}u = f$ is transformed to

$$\mathfrak{M}'\bar{u} - \lambda_\gamma \bar{u} = \bar{f} + \bar{N}_a - \bar{N}_b , \tag{30}$$

where $\bar{u}$ and $\bar{f}$ are the integral transforms of the functions u and f, and $\bar{N}_a$ and $\bar{N}_b$ are functions whose form is determined by the conditions given for the coordinate x_i at the end points of the interval (a, b). When boundary conditions of the form (25) are given,

* It is assumed that the derivative $\partial\bar{K}_\gamma/\partial x_i$ is bounded for $x_i = a$.

$$\begin{aligned}
\bar{N}_a &= \begin{cases} \dfrac{1}{C_\gamma \alpha_a} p(a)\, \bar{K}_\gamma(a)\, \varphi_a & \text{for} \quad \alpha_a \neq 0, \\ \dfrac{1}{C_\gamma} p(a) \left.\dfrac{\partial \bar{K}_\gamma}{\partial x_i}\right|_{x_i=a} \varphi_a & \text{for} \quad \alpha_a = 0,\ \beta_a = -1; \end{cases} \\
\bar{N}_b &= \begin{cases} \dfrac{1}{C_\gamma \alpha_b} p(b)\, \bar{K}_\gamma(b)\, \varphi_b & \text{for} \quad \alpha_b \neq 0, \\ \dfrac{1}{C_\gamma} p(b) \left.\dfrac{\partial \bar{K}_\gamma}{\partial x_i}\right|_{x_i=b} \varphi_b & \text{for} \quad \alpha_b = 0,\ \beta_b = 1. \end{cases}
\end{aligned} \tag{31}$$

Under the condition of periodicity with respect to the coordinate x_i, we have $\bar{N}_a - \bar{N}_b = 0$.

Finally, let us determine the original function u from the transformed function $\bar{u}$; that is, let us try to perform the inverse operation. To do this, we expand the function u in a series

$$u = \sum_{\gamma=1}^{\infty} a_\gamma \bar{K}_\gamma \tag{32}$$

in terms of a complete system of eigenfunctions $\bar{K}_\gamma$ of the Sturm-Liouville problem. When we multiply the series (32) by $\rho \bar{K}_\gamma$ and integrate it termwise, we obtain, because of the orthogonality conditions (21),

$$a_\gamma = \frac{1}{C_\gamma} \int_a^b \rho \bar{K}_\gamma u \, dx_i \ ,$$

where C_γ is a function of the parameter γ, defined by the formula (21). If we use the function C_γ as a normalizing divisor in eq. (12) (which is always possible because C_γ does not vanish), and if we note that the quantity $(1/C_\gamma)\rho\bar{K}_\gamma$ is then the kernel $K(x_i, \gamma)$ of the direct operator, we obtain

$$a_\gamma = \int_a^b K(x_i, \gamma)\, u(x_i)\, dx_i \equiv \bar{u}(\gamma) \ . \tag{33}$$

It now follows from formula (32) that

$$u = \sum_{\gamma=1}^{\infty} \bar{u}(\gamma)\, \bar{K}_\gamma \ . \tag{34}$$

This is the desired formula expressing the original function u in terms of its integral transform (33).

4. *Integral transformations in infinite intervals*

From analysis, the reader is familiar with Fourier's integral formula *

* See. for example, Smirnov [1)], Vol. 2, p. 160.

$$\frac{1}{\pi}\int_0^{\infty} d\zeta \int_{-\infty}^{\infty} f(\xi) \cos \zeta(\xi - x)\, d\xi = \tfrac{1}{2}[f(x+0)+f(x-0)] , \tag{35}$$

where $f(x)$ is a function of the one-dimensional argument x, defined on the entire real axis and satisfying certain additional conditions. For example, it is sufficient to require that the integral

$$\int_{-\infty}^{\infty} |f(\xi)|\, d\xi$$

converge. Then, the Fourier formula is valid for all values of x in whose neighbourhood the function $f(x)$ is of bounded variation. Furthermore, if the function $f(x)$ is continuous at the point x, the right side of the Fourier formula is equal to $f(x)$.

Fourier's integral formula can be obtained from a Fourier series

$$f(x) = \tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty} \left(a_\nu \cos \frac{\pi\nu}{l} x + b_\nu \sin \frac{\pi\nu}{l} x\right)$$

by passing to the limit as the interval $(-l, l)$ increases without bound. When we note that the Fourier series represents an expansion in eigenfunctions of the particular Sturm-Liouville problem

$$u'' + \lambda^2 u = 0 , \qquad u|_{x=-l} = u|_{x=l} ,$$

it is natural to assume that, in the general case, an expansion over an infinite interval is possible. This supposition is valid under certain conditions. Integral formulae are then obtained which make it possible to perform integral transformations and to exclude differential operations with respect to variables whose domains are infinite intervals.

We shall give a number of such integral formulae beginning with the Fourier integral formula (35).

Let us suppose that the function $f(x)$ is defined in the infinite interval $0 \leq x < \infty$. It is then possible to extend the definition to the interval $-\infty < x < 0$. For example, we can set $f(-x) = f(x)$ or $f(-x) = -f(x)$. In the first case,

$$\int_{-\infty}^{0} f(\xi) \cos \zeta(\xi - x)\, d\xi = \int_0^{\infty} f(\xi) \cos \zeta(\xi + x)\, d\xi ,$$

so that

$$\int_{-\infty}^{\infty} f(\xi) \cos \zeta(\xi - x)\, d\xi = \int_0^{\infty} f(\xi)\, [\cos \zeta(\xi - x) + \cos \zeta(\xi + x)]\, d\xi$$

$$= 2\, (\cos \zeta x) \int_0^{\infty} f(\xi) \cos \zeta\xi\, d\xi,$$

and Fourier's formula (35) can be written in the form

$$\frac{2}{\pi}\int_0^\infty \cos \zeta x \, d\zeta \int_0^\infty f(\xi) \cos \zeta\xi \, d\xi = \tfrac{1}{2}[f(x+0)+f(x-0)] . \tag{36}$$

In the second case, when $f(-x) = -f(x)$, we can, in a similar manner, obtain the formula

$$\frac{2}{\pi}\int_0^\infty \sin \zeta x \, d\zeta \int_0^\infty f(\xi) \sin \zeta\xi \, d\xi = \tfrac{1}{2}[f(x+0)+f(x-0)] . \tag{37}$$

Both these formulae are valid throughout the entire interval $0 \leqslant x < \infty$ of definition of $f(x)$ and give an even or odd extension, respectively, of the function $f(x)$ into the negative half of the real axis.

Let us now use the identity

$$\cos \zeta(\xi - x) = \tfrac{1}{2}\, e^{i\zeta(\xi-x)} + \tfrac{1}{2}\, e^{-i\zeta(\xi-x)}$$

to transform formula (35) to the form

$$\frac{1}{2\pi}\int_{-\infty}^\infty e^{i\zeta x} \, d\zeta \int_{-\infty}^\infty f(\xi)\, e^{-i\zeta\xi} \, d\xi = \tfrac{1}{2}[f(x+0)+f(x-0)] . \tag{38}$$

We again assume that the function $f(x)$ is defined only for non-negative x and let us extend its definition to include all negative values of x by setting $f(x) = 0$ for negative x. Consider the functions $f(x)\, e^{-\eta x}$, where η is a positive number. The integral

$$\int_{-\infty}^\infty |f(\xi)\, e^{-\eta\xi}| \, d\xi$$

converges even when the function $f(\xi)$ increases exponentially (provided the number η is sufficiently great). In formula (38), let us replace the function $f(x)$ with the function $f(x)\, e^{-\eta x}$. Multiplying both sides by $e^{\eta x}$, and recalling that $f(x) = 0$ for negative values of x, we obtain

$$\frac{1}{2\pi}\int_{-\infty}^\infty e^{(\eta-i\zeta)x} \, d\zeta \int_0^\infty f(\xi)\, e^{-(\eta-i\zeta)\xi} \, d\xi = \tfrac{1}{2}[f(x+0)+f(x-0)] \qquad (x \geqslant 0) .$$

If we make the substitution

$$(\eta - i\zeta) \equiv \gamma , \qquad d\zeta = -d\gamma/i ,$$

we obtain

$$\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{\gamma x} \, d\gamma \int_0^\infty f(\xi)\, e^{-\gamma\xi} \, d\xi = \tfrac{1}{2}[f(x+0)+f(x-0)] , \tag{39}$$

which is called the Laplace-Mellin integral formula. If, in this formula, we make the substitution

$$\gamma = -\gamma^*\,,\qquad x = \ln x^*\,,\qquad \xi = \ln \xi^*\,,\qquad \eta = -\eta^*\,,\qquad f(x) = f^*(x^*)$$

and then drop the asterisks, we obtain the formula

$$\frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty}\frac{d\gamma}{x^\gamma}\int_0^\infty f(\xi)\,\xi^{\gamma-1}\,d\xi = \tfrac{1}{2}[f(x+0)+f(x-0)] \qquad (x \geqslant 0)\,, \tag{40}$$

which is called Mellin's integral formula.

Let us now consider the function $e^{in\theta} f(x)$, where n is some number and x and θ are polar coordinates in the plane. We denote by $g(x_1, x_2)$ the expressions for this function in Cartesian coordinates x_1 and x_2. Assuming, for simplicity in calculation, that the function $g(x_1, x_2)$ is continuous, let us apply to it the Fourier integral formula, taking the coordinate x_1 as the variable of integration. We then obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\zeta_1 x_1}\,d\zeta_1 \int_{-\infty}^{\infty} g(\xi_1, x_2)\,e^{-\zeta_1\xi_1}\,d\xi_1 = g(x_1, x_2)\,.$$

Let us now apply the Fourier integral formula with respect to the coordinate x_2. After some simple manipulations, we obtain

$$\left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\zeta_1 x_1+\zeta_2 x_2)}\,d\zeta_1\,d\zeta_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi_1,\xi_2)\,e^{-i(\zeta_1\xi_1+\zeta_2\xi_2)}\,d\xi_1\,d\xi_2 = g(x_1,x_2)\,.$$

By means of the relationships $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $\xi_1 = \xi\cos\theta$, $\xi_2 = \xi\sin\theta'$, $\zeta_1 = \zeta\cos\theta''$, $\zeta_2 = \zeta\sin\theta''$, we introduce polar coordinates. Recalling that $d\xi_1\,d\xi_2 = \xi\,d\xi\,d\theta'$, $d\zeta_1\,d\zeta_2 = \zeta\,d\zeta\,d\theta''$, and that, in polar coordinates r, θ, the function $g(x_1, x_2)$ is given by the expression $e^{in\theta} f(x)$, we transform the formula in question to

$$\left(\frac{1}{2\pi}\right)^2 \int_0^\infty \zeta\,d\zeta \int_{-\pi}^{\pi} d\theta''\,\exp\left[i(r\zeta\cos(\theta-\theta') - n\theta)\right]$$
$$\times \int_0^\infty \xi\,d\xi \int_{-\pi}^{\pi} d\theta'\,f(\xi)\,\exp\left[-i(\xi\zeta\cos(\theta'-\theta'') - n\theta')\right] = f(r).$$

We note that, in the general case (that is, without assuming that the function $g(x_1, x_2)$ is continuous), we obtain (on the right side of this formula) the expression $\frac{1}{2}[f(r+0)+f(r-0)]$, which we shall use in the subsequent discussion Let us set

$$\theta' - \theta'' = \tfrac{1}{2}\pi + \varphi\,.$$

Then,

$$\int_{-\pi}^{\pi} \exp\left[-\mathrm{i}(\xi\zeta\cos(\theta'-\theta'') - n\theta')\right] \mathrm{d}\theta'$$
$$= \int_{-\frac{3}{2}\pi-\theta''}^{\frac{1}{2}\pi-\theta''} \exp\left[\mathrm{i}(n\varphi - \xi\zeta\sin\varphi)\right] \exp\left[\mathrm{i}n(\theta''+\tfrac{1}{2}n\pi)\right] \mathrm{d}\varphi$$
$$= \exp\left[\mathrm{i}n(\theta''+\tfrac{1}{2}\pi)\right] \int_{-\pi}^{\pi} \exp\left[\mathrm{i}(n\varphi - \xi\zeta\sin\varphi)\right] \mathrm{d}\varphi \ .$$

Since the integrand is periodic with period 2π, we again arrive at the limits of integration $(-\pi, \pi)$ in the last integral. If we now set

$$\theta'' - \theta \equiv \psi - \tfrac{1}{2}\pi \ ,$$

we obtain

$$\exp\left[\tfrac{1}{2}\mathrm{i}n\pi\right] \int_{-\pi}^{\pi} \exp\left[\mathrm{i}(r\zeta\cos(\theta-\theta'') - n\theta) + \mathrm{i}n\theta''\right] \mathrm{d}\theta'' = \int_{-\pi}^{\pi} \exp\left[\mathrm{i}(n\psi - r\zeta\sin\psi)\right] \mathrm{d}\psi \ .$$

If we substitute the transformed integrals into the general formula and remember that, for n greater than $-\frac{1}{2}$, in accordance with formula (58) of Chapter XII,

$$\int_{-\pi}^{\pi} \exp\left[\mathrm{i}(n\tau - z\sin\tau)\right] \mathrm{d}\tau = 2\pi J_n(z) \ ,$$

where $J_n(z)$ is Bessel's function of the first kind of order n, we obtain the Fourier-Bessel integral formula:

$$\int_0^{\infty} J_n(x\zeta)\ \zeta\ \mathrm{d}\zeta \int_0^{\infty} f(\xi)\ J_n(\xi\zeta)\ \xi\ \mathrm{d}\xi = \tfrac{1}{2}\left[f(x+0) + f(x-0)\right] \tag{41}$$
$$(x \geqslant 0,\ n > -\tfrac{1}{2}) \ .$$

The conditions given in connection with the Fourier integral formula are sufficient for the applicability of the Fourier-Bessel formula.

We shall consider the inner integrals in formulae (36) - (41) as integral transforms of the function $f(x)$ appearing in the integrand. The outer integral then represents the inverse operator, which is also an integral operator, for an infinite interval of variation of the variable.

Thus, we have at our disposal the following set of direct and inverse integral operators, which we now present with their commonly used names and with a statement of the conditions of their applicability:

1. The Fourier transform:

$$\bar{u}(\gamma) = \int_{-\infty}^{\infty} u(\xi)\ \mathrm{e}^{-\mathrm{i}\gamma\xi}\ \mathrm{d}\xi \ , \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{u}(\gamma)\ \mathrm{e}^{\mathrm{i}\gamma x}\ \mathrm{d}\gamma = u(x) \ . \tag{42}$$

Sufficient conditions for its applicability are that the integral

$$\int_{-\infty}^{\infty} |u(x)| \; dx$$

converge and that the functions $u(x)$ be piecewise-continuous and of bounded variation in an arbitrary finite interval.

2. The Fourier cosine transform:

$$\begin{gathered} \bar{u}(\gamma) = \int_0^{\infty} u(\xi) \cos \gamma\xi \; d\xi \,, \\ \frac{2}{\pi} \int_0^{\pi} \bar{u}(\gamma) \cos \gamma x \; d\gamma = \begin{cases} u(x) & \text{for} \quad x > 0 \,, \\ u(-x) & \text{for} \quad x < 0 \,. \end{cases} \end{gathered} \tag{43}$$

Sufficient conditions for its applicability are that the integral

$$\int_0^{\infty} |u(x)| \; dx$$

converge and that for non-negative x the functions $u(x)$ be piecewise-continuous and of bounded variation in an arbitrary finite interval.

3. The Fourier sine transform:

$$\begin{gathered} u(\gamma) = \int_0^{\infty} u(\xi) \sin \gamma\xi \; d\xi \,, \\ \frac{2}{\pi} \int_0^{\infty} \bar{u}(\gamma) \sin \gamma x \; d\gamma = \begin{cases} u(x) & \text{for} \quad x > 0 \,, \\ -u(-x) & \text{for} \quad x < 0 \,. \end{cases} \end{gathered} \tag{44}$$

The conditions for applicability are the same as for the Fourier cosine transform.

4. The Laplace transform:

$$\begin{gathered} \bar{u}(\gamma) = \int_0^{\infty} u(\xi) \; e^{-\gamma\xi} \; d\xi \,, \\ \frac{1}{2\pi i} \int_{\eta - i\infty}^{\eta + i\infty} \bar{u}(\gamma) \; e^{\gamma x} \; d\gamma = \begin{cases} u(x) & \text{for} \quad x > 0 \,, \\ 0 & \text{for} \quad x < 0 \,. \end{cases} \end{gathered} \tag{45}$$

Sufficient conditions for its applicability are that the function $u(x)$ for positive x be piecewise-continuous in an arbitrary finite interval and that there exist a number a such that the product $u(x) \; e^{-ax}$ remain bounded as x increases without bound. Here, we need to take η greater than a for the inverse operator.

5. The Mellin transform (for positive x):

$$\bar{u}(\gamma) = \int_0^{\infty} u(\xi) \; \xi^{\gamma - 1} \; d\xi \,, \qquad \frac{1}{2\pi i} \int_{\eta - i\infty}^{\eta + i\infty} \bar{u}(\gamma) \frac{d\gamma}{x^{\gamma}} = u(x) \quad \text{for} \quad x > 0 \,. \tag{46}$$

Sufficient conditions for its applicability are that the function $u(x)$ be piecewise-continuous and of bounded variation on an arbitrary finite interval of positive values of x and that there exist numbers $a > 0$, γ_1, and γ_2 (with $\gamma_1 < \gamma_2$) such that the integrals

$$\int_0^a |u(\xi)\ \xi^{\gamma_1 - 1}|\ d\xi \quad \text{and} \quad \int_a^\infty |u(\xi)\ \xi^{\gamma_2 - 1}|\ d\xi$$

converge. Here, for the inverse operator, we need to take $\gamma_1 < \eta < \gamma_2$. Other sufficient conditions for the applicability of the Mellin transform are that there exist numbers η_1 and η_2 with $\eta_1 < \eta_2$ such that for $\eta_1 < \eta < \eta_2$, the integral

$$\int_{\eta - i\infty}^{\eta + i\infty} |\bar{u}(\gamma)|\ d\gamma$$

converges and that the function $\bar{u}(\eta + i\chi)$ be analytic and approaches zero as x approaches $\pm\infty$.

6. The Hankel transform (for $x > 0$, $n > -\frac{1}{2}$):

$$\bar{u}(\gamma) = \int_0^\infty u(\xi)\ J_n(\xi\gamma)\ \xi\ d\xi\ , \qquad \int_0^\infty \bar{u}(\gamma)\ J_n(\gamma x)\ \gamma\ d\gamma = u(x) \quad \text{for} \quad x > 0\ . \tag{47}$$

The conditions for its applicability are the same as for the Fourier transform.

We note that in this list of operators we consistently wrote $u(x)$ for brevity instead of $\frac{1}{2}[u(x+0) + u(x-0)]$. These operators have found wide practical applications but, of course, we have not exhausted the list of integral operators that can be useful in some particular case or other.

With an infinite interval, the choice of the integral operator applicable for solving a given problem is conditioned by the same considerations as in the case of a finite interval. In particular, the kernel of the operator must be a solution of eq. (7) such that expression (11) can be calculated. If this kernel is one of the kernels that appear in the transformations (42) - (47), we can perform the direct and inverse transformations by the methods that we have given.

Let us represent the kernel $K(x_i, \gamma)$ of the direct integral operator with respect to the variable x_i in the form of the product

$$\frac{1}{C_\gamma}\, \rho(x_i)\ \bar{K}(x_i, \gamma)\ ,$$

where $\rho(x_i)$ is the function defined by eq. (16) and C_γ is a normalizing divisor. Here, as in the case of a finite integral operator, we obtain an equation of the form (17) for determining the function $\bar{K}(x_i, \gamma)$.

The variable of transformation can, in an infinite interval, be interpreted either as a space coordinate or as time. In the first case, the choice of the solution to eq. (17) is conditioned by the same considerations as in the case of finite integral operators.

Let us consider the second case. When the equation for which the problem is posed is of the hyperbolic type, the given conditions contain both values of the unknown function u and values of its derivative $\partial u/\partial t$ at the initial instant. Consequently, expression (11) can be calculated at the lower limit. With regard to the upper limit, the kernel of the operator can usually be chosen so that expression (11) will vanish at that point. Then, the transformation can be carried to completion. However, when the equation is of the parabolic type, the given conditions contain only the value of the unknown quantity u at the initial instant and not the values of its derivative. However, here, only the first derivative $\partial u/\partial t$ of the unknown function appears in the equation that we are studying (that is, $a_{ii} = 0$), so that expression (11) takes the form

$$b_i u K \Big|_{t=0}^{\infty}$$

and can again be calculated at the lower limit. As in the preceding case, we can usually arrange for expression (11) to vanish at the upper limit, as a result of the vanishing of the kernel of the operator.

5. *Summary of the results*

To make it easier to use the methods of integral operators, we give a summary of the basic results and formulae.

By the use of an integral operator, we can eliminate differential operations with respect to the variable x_i, which varies in a *finite* interval (a, b) when the following sufficient conditions are satisfied.

1. The differential equation of the problem can be represented in the form

$$\mathcal{M}_i u + \mathcal{M}' u = f, \tag{48}$$

where

$$\mathcal{M}_i u \equiv a_{ii} \frac{\partial^2 u}{dx_i^2} + b_i \frac{\partial u}{\partial x_i} + cu \tag{49}$$

is a differential expression with coefficients depending only on the variable x_i and $\mathcal{M}'u$ is a differential expression with coefficients not depending on x_i or on derivatives with respect to x_i; also, $a_{ii} \geqslant 0$, $c \leqslant 0$, a_{ii}, $\partial a_{ii}/\partial x_i$, and c are continuous in the interval (a, b); the integral

$$\rho(x_i) \equiv \exp\left[-\int \frac{1}{a_{ii}}\left(\frac{da_{ii}}{dx_i} - b_i\right) d\xi\right] \tag{50}$$

has a continuous derivative with respect to x_i in the interval (a, b); there is a non-negative number ϵ such that

$$\int_a^{a+\epsilon} \frac{x_i - a}{p(x_i)}\, dx_i, \quad \int_{a+\epsilon}^{b-\epsilon} \frac{dx_i}{p(x_i)}, \quad \int_{b-\epsilon}^{b} \frac{b - x_i}{p(x_i)}\, dx_i \quad (p(x_i) \equiv a_{ii}\rho) \tag{51}$$

converges.

2. If the boundary conditions are given in terms of the variable x_i of the transformation, they must, for $p(a) \neq 0$ and $p(b) \neq 0$, be representable in the form

$$\left[\alpha_a \frac{\partial u}{\partial x_i} - \beta_a u\right]_{x_i=a} = \varphi_a \,, \qquad \left[\alpha_b \frac{\partial u}{\partial x_i} + \beta_b u\right]_{x_i=b} = \varphi_b \,, \tag{52}$$

where the quantities α_a, α_b and α_a, α_b are non-negative and the terms in each pair not simultaneously equal to zero, and where φ_a and φ_b are known functions of the variables x_j (for $j \neq i$).

For $p(a) = 0$ ($p(b) = 0$), the boundary condition for $x_i = a$ ($x_i = b$) must satisfy the requirements of the theorem on expansion in eigenfunctions of the corresponding Sturm-Liouville problem. This usually means that u must be finite for $x_i = a$ ($x_i = b$).

3. The conditions of the problem in terms of the variables x_j (for $j \neq i$) contain neither derivatives $\partial u/\partial x_i$ with respect to the variable x_i of the transformation nor coefficients depending on x_i.

4. The kernel of the operator is equal to

$$K(x_i, \gamma) = \frac{1}{C_\gamma} \rho(x_i) \bar{K}_\gamma(x_i) \,,$$

where $K_\gamma(x_i)$ is the solution to the homogeneous differential equation

$$\frac{\partial}{\partial x_i} p \frac{\partial \bar{K}}{\partial x_i} - q\bar{K} + \lambda^2 \rho \bar{K} = 0 \qquad (q \equiv -c\rho) \,, \tag{53}$$

satisfying

(a) the homogeneous boundary conditions

$$\left[\alpha_a \frac{\partial \bar{K}}{\partial x_i} - \beta_a \bar{K}\right]_{x_i=a} = 0 \,, \qquad \left[\alpha_b \frac{\partial \bar{K}}{\partial x_i} + \beta_b \bar{K}\right]_{x_i=b} = 0 \tag{54}$$

if boundary conditions of the form (52) are given for the variable x_i;

(b) the condition of periodicity, if the condition of periodicity is given for the variable x_i.

For $p(a) = 0$ ($p(b) = 0$), the first (second) of conditions (54) must be replaced by the condition following from the requirement of the theorem of expansion in eigenfunctions of the corresponding Sturm-Liouville problem (see above).

When these conditions are satisfied,

1. Eq. (53) has solutions not identically equal to zero for values of λ^2 that form an infinite increasing sequence $\lambda_1^2, \lambda_2^2, \ldots, \lambda_\gamma^2, \ldots$ of positive numbers λ_γ^2 (the eigenvalues of the problem for eq. (53)). If the boundary conditions are given in terms of the variable x_i, then to each eigenvalue λ_γ^2 there corresponds only one linearly independent solution of eq. (53) $\bar{K}_\gamma(x_i)$; that is, for a fixed value of the variable x_i, there will be only one expression for the kernel

$$K(x_i, \gamma) = \frac{1}{C_\gamma} \rho(x_i) \bar{K}_\gamma(x_i) \,.$$

If the condition of periodicity with respect to the variable x_i is given,

then to each eigenvalue $\lambda_\eta{}^2$ there will ordinarily correspond two distinct linearly independent periodic solutions of eq. (53), which, by a linear transformation, can be made mutually orthogonal. It is convenient to denote these two solutions by $\bar{K}_{2\eta-1}(x_i)$ and $\bar{K}_{2\eta}(x_i)$. Then, the function $\bar{K}_\gamma(x_i)$ with integral arguments γ also appears in the kernel of the operator but to each pair of values of γ (equal to 2η and $2\eta - 1$ for $\eta = 1, 2, 3, \ldots$) there will correspond one eigenvalue $\lambda_\eta{}^2$.

2. The equation of the problem $\mathfrak{M}_i u + \mathfrak{M}' u = f$ can, by means of an integral operator with kernel

$$K(x_i, \gamma) = \frac{1}{C_\gamma} \rho(x_i) \, \bar{K}_\gamma(x_i) \tag{55}$$

(where the normalizing divisor, C_γ, is equal to

$$\int_a^b \rho(x_i) \, [K_\gamma(x_i)]^2 \, dx_i) \; , \tag{56}$$

be reduced to the form

$$\mathfrak{M}' \bar{u} - \lambda_\gamma{}^2 \bar{u} = \bar{f} + \bar{N}_a - \bar{N}_b \; , \tag{57}$$

where $\bar{u}$ and $\bar{f}$ are the integral transforms of the functions u and f with respect to the variable x_i, where $\lambda_\gamma{}^2$ is the eigenvalue with ordinal number γ of the boundary-value problem (53) - (54), and where the quantities $\bar{N}_a$ and $\bar{N}_b$ are defined as follows: When boundary conditions of the form (52) are given with respect to the variable x_i,

$$\bar{N}_a = \begin{cases} \dfrac{p(a) \, \bar{K}_\gamma(a)}{C_\gamma \alpha_a} \varphi_a & \text{for} \quad \alpha_a \neq 0, \\ \dfrac{1}{C_\gamma} p(a) \left. \dfrac{\partial \bar{K}_\gamma}{\partial x_i} \right|_{x_i = a} \varphi_a & \text{for} \quad \alpha_a = 0, \; \beta_a = -1, \end{cases} \tag{58}$$

$$\bar{N}_b = \begin{cases} \dfrac{p(b) \, \bar{K}_\gamma(b)}{C_\gamma \alpha_b} \varphi_b & \text{for} \quad \alpha_b \neq 0, \\ \dfrac{1}{C_\gamma} p(b) \left. \dfrac{\partial \bar{K}_\gamma}{\partial x_i} \right|_{x_i = b} \varphi_b & \text{for} \quad \alpha_b = 0, \; \beta_b = 1. \end{cases} \tag{59}$$

When the periodicity conditions are given with respect to the variable x_i,

$$\bar{N}_a - \bar{N}_b = 0 \; . \tag{60}$$

3. Additional conditions in terms of the variables x_j (for $j \neq i$), which are necessary for stating the problem for eq. (57), are obtained by replacing (in the corresponding conditions with respect to the variables x_j) the original problem for each of the functions of the variable x_i by its integral transform.

4. The solution u of the original problem is expressed in terms of the solution $\bar{u}_\gamma$ of the transformed problem by means of the series

$$u = \sum_{\gamma=1}^{\infty} \bar{u}(\gamma) \, \bar{K}_\gamma(x_i) \tag{61}$$

In the case of an integral transformation within finite limits, the various operations should be carried out in the following order:

(1) Establish the validity of using an operator in the problem in question.
(2) Calculate (from the values of the coefficients a_{ii}, b_i, and c of the differential expression $\mathcal{M}_i u$) the functions $\rho(x_i)$, $p(x_i)$, $q(x_i)$.
(3) Find the function $\overline{K}_\gamma(x_i)$.
(4) Find the kernel of the direct operator.
(5) Use the relationship (57) - (60) to write the transformed problem.
(6) Find the solution of the transformed problem. (Here, in particular, repeated application of integral transformations may be useful.)
(7) Write the solution of the original problem in the form of the series (61).

In the case in which it is necessary to eliminate differential operations with respect to a variable that varies within an infinite interval, these results are modified in the following respect:

1. The kernel of the direct integral operator must (up to a constant factor) be equal to the product

$$\rho(x_i)\ \overline{K}(x_i, \gamma)\ , \tag{62}$$

where $\overline{K}(x_i,\gamma)$ is a solution of the differential equation

$$\frac{\partial}{\partial x_i} p \frac{\partial \overline{K}}{\partial x_i} - q\overline{K} + \gamma^2 \rho \overline{K} = 0\ . \tag{63}$$

If the lower limit a of variation of the variable x_i is finite and the boundary condition is given with respect to the variable x_i, then the function $\overline{K}$ must, at that limit, satisfy the homogeneous condition corresponding to the condition of the problem (that is, the same condition as with respect to the variable x_i of the problem, but with the right side equal to zero). At the upper (infinite) limit, the function $\overline{K}$ must be such that the expression

$$p(x_i)\ \left(\overline{K}\ \frac{\partial u}{\partial x_i} - \frac{\partial \overline{K}}{\partial x_i} u\right) \tag{64}$$

vanishes. In particular, if it is known that the quantities u and $\partial u/\partial x_i$ vanish at infinity, the function $\overline{K}$ may be finite.

We note that in the case of integral operators within infinite limits the quantity γ takes on not a discrete sequence but a continuum of values.

2. The kernel of the direct integral operator $\rho\overline{K}$ must belong to the set of kernels for which an inverse operator exists. Formulae (42) - (47) summarize the most useful direct and inverse integral operators.

3. The equation of the problem (as a result of the transformation within infinite limits) is reduced to the form

$$\mathcal{M}'\overline{u} - \gamma^2\overline{u} = \overline{f} + \overline{N}_a\ ; \tag{65}$$

here, the function $\overline{N}_a$ is defined by eq. (58) is the lower limit a of variation of x_i is finite, and the boundary condition is given for $x_i = a$; it is equal to expression (64) for $x_i = a$ if the Cauchy conditions are given (initial conditions).

In the case of integral transformations within infinite intervals, the operations should be carried out in the following order:

(1) Show that there is nothing to make integral transformation of the problem impossible.
(2) Calculate the functions $\rho(x_i)$, $p(x_i)$, and $q(x_i)$.
(3) Find the general solution of eq. (63) and use formula (62) to determine the general form of the kernel of the direct operator.
(4) Beginning with the requirement relating to the transformation of the conditions of the problem in question, choose a kernel for the direct operator.
(5) Use formulae (42) - (47) to find the operator with this kernel.
(6) Use formulae (55), (58), and (64) to write the transformed problem.
(7) Find the solution of the transformed problem.
(8) Use the formula for the direct operator to write the solution of the original problem in integral form and calculate or simplify the integral relationship found.

Henceforth, in solving specific problems, we shall assume that the formulae given in this section are known to the reader and we shall not make specific references to them.

In deriving the relationships concerning integral transformations in sections 3 and 4, we made certain standard assumptions (for example, the assumption that the order of differentiation and integration could be reversed, and so on). Generally speaking, we cannot, therefore, assert that the solution found by means of integral transformations actually does satisfy the problem in question and (in doubtful cases) it makes verification of the solution necessary. Although such a verification may be tedious, this does not appreciably restrict the possibilities of the method, since (in all the properly stated problems of mathematical physics) the method leads at least to a generalized solution (see above).

Chapter XXXII

EXAMPLES OF THE APPLICATION OF FINITE INTEGRAL TRANSFORMATIONS

1. *Vibrations of a heavy thread*

Let us consider, as our first example of the application of finite integral transformations, Daniel Bernoulli's problem of the vibrations of a heavy thread.

The differential equation for small vibrations of a homogeneous nonstretchable heavy thread suspended at the upper end is of the form (Part I, Chapter XIII, section 2)

$$\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)+\frac{X}{gd}=\frac{1}{g}\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

where d is the linear density of the thread, $X = X(x, t)$ is the unit transverse load of the thread, and u is the displacement of the thread from its equilibrium position. The x-axis is assumed to be directed upward along the thread. The coordinate origin is at the point coinciding with the position of the lower end of the thread when in equilibrium. The boundary condition can be written in the form *

$$u|_{x=0}<\infty, \qquad u|_{x=l}=0, \tag{2}$$

where l is the length of the thread.

In the general case, the initial condition is of the form

$$u|_{t=0}=u_0(x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0}=u_1(x), \tag{3}$$

where $u_0(x)$ and $u_1(x)$ are known functions. Let us find an integral operator that will allow us to eliminate differentiation with respect to x. Let us define

* Since the motion of the thread at the free end is not restricted by any kind of external factors, the condition at $x = 0$ is not related to the physical aspect of the problem and hence it may be ignored. It expresses the general requirement (which applies not only to the boundary point but to the entire region in question) that the solution must be *bounded*. Statement of this problem by use of the condition at the point $x = 0$ is permissible, since this is a singular point of eq. (1), so that solutions of the equation exist that satisfy the necessary condition at $x = l$ but that increase without bound as x approaches zero. Here we gave the condition at a singular point of the equation in order to underscore the fact that we shall later extend the requirement that the solution be bounded at this point, to the kernel of the integral operator.

$$\mathcal{M}_x u \equiv \frac{\partial}{\partial x}\left(x\,\frac{\partial u}{\partial x}\right) .$$

For this expression *,

$$a_{xx} = x\ , \qquad b_x = 1\ , \qquad c = 0\ , \qquad p(x) = x\ , \qquad \rho(x) = 1\ , \qquad q(x) = 0\ .$$

Since $\rho = 1$, the kernel of the operator and the auxiliary function $K_\gamma(x)$ can differ only by a normalizing divisor. To find them, we seek the solution to the boundary-value problem.

$$\frac{\partial}{\partial x}\left(x\,\frac{\partial \overline{K}}{\partial x}\right) + \lambda^2 \overline{K} = 0\ , \tag{4}$$

$$\overline{K}|_{x=0} < \infty\ , \qquad \overline{K}|_{x=l} = 0\ . \tag{5}$$

This solution, considered as a function of the variable x and of the ordinal number γ of the eigenvalue $\lambda_\gamma{}^2$ is the desired kernel of the operator (up to a factor). By means of the substitution

$$z = 2x^{\frac{1}{2}}\ , \tag{6}$$

eq. (4) can be reduced to Bessel's equation of order zero:

$$\frac{\partial^2 \overline{K}}{\partial z^2} + \frac{1}{z}\,\frac{\partial \overline{K}}{\partial z} + \lambda^2 \overline{K} = 0\ .$$

Its solutions, bounded at $z = 0$, are the Bessel functions $J_0(\lambda z)$. Consequently, the functions $J_0(\mu_\gamma x^{\frac{1}{2}})$, where $\mu_\gamma = 2\lambda_\gamma$, are bounded solutions of eq. (4) at $x = 0$. To satisfy the second of the boundary conditions (5), we set

$$J_0(\mu_\gamma l^{\frac{1}{2}}) = 0\ . \tag{7}$$

The roots μ_γ of this equation form the set of eigenvalues of the problem (4) - (5).

We take $J_0(\mu_\gamma x^{\frac{1}{2}})$ for the function $\overline{K}_\gamma(x)$. Here, the kernel of the direct operator is the function

$$K(x,\gamma) = \frac{1}{C_\gamma}\,J_0(\mu_\gamma x^{\frac{1}{2}})\ , \tag{8}$$

where

$$C_\gamma = \int_0^a [J_0(\mu_\gamma x^{\frac{1}{2}})]^2\,dx\ .$$

To calculate C_γ, we use formula (38) of Chapter XII, according to which

$$\int_0^a z J_0{}^2(\lambda z)\,dz = \tfrac{1}{2}a^2 J_1{}^2(\lambda a)\ .$$

If we set $a = l$, $z = 2x^{\frac{1}{2}}$, and $\lambda = \frac{1}{2}\mu_\gamma$, we obtain

* All the notations correspond to the notations used in section 5 of the present chapter with obvious change in subscript.

$$C_\gamma = lJ_1{}^2(\mu l^{\frac{1}{2}}) .$$

When we perform the integral transformation with the kernel (8) and interval of integration $(0, l)$, remembering that $\lambda_\gamma = \frac{1}{2}\mu_\gamma$, we transform the problem (1) - (3) to the form *

$$\frac{\partial^2 \bar{u}}{\partial t^2} + \tfrac{1}{2}g\mu_\gamma \bar{u} = \frac{\bar{X}}{d} , \tag{9}$$

$$\bar{u}|_{t=0} = \bar{u}_0(\gamma) , \quad \left.\frac{\partial \bar{u}}{\partial t}\right|_{t=0} = \bar{u}_1(\gamma) , \tag{10}$$

where

$$\bar{u} = \bar{u}(\gamma, t) = \frac{1}{C_\gamma}\int_0^l u(x, t)\, J_0(\mu_\gamma x^{\frac{1}{2}})\, dx ,$$

$$\bar{X} = \bar{X}(\gamma, t) = \frac{1}{C_\gamma}\int_0^l X(x, t)\, J_0(\mu_\gamma x^{\frac{1}{2}})\, dx ,$$

$$\bar{u}_0(\gamma) = \frac{1}{C_\gamma}\int_0^l u_0(x)\, J_0(\mu_\gamma x^{\frac{1}{2}})\, dx , \quad \bar{u}_1(\gamma) = \frac{1}{C_\gamma}\int_0^l u_1(x)\, J_0(\mu_\gamma x^{\frac{1}{2}})\, dx .$$

We denote by $\bar{u}_\gamma$ the solution to the problem (9) - (10) for a given value of γ. The solution $u(x, t)$ to the problem (1) - (3) can then be written in the form of the series

$$u(x, t) = \sum_{\gamma=1}^{\infty} \bar{u}_\gamma J_0(\mu_\gamma x^{\frac{1}{2}}) .$$

The reader will find it instructive to calculate the expression for $\bar{u}_\gamma$ in various particular cases.

Problems

1. Show that the free vibrations ($X = 0$) of a heavy thread with the initial condition (3) can be represented in the form of the series

$$u(x, t) = \sum_{\gamma=1}^{\infty} J_0(\mu_\gamma x^{\frac{1}{2}}) \left(A_\gamma \cos \tfrac{1}{2}\mu_\gamma g^{\frac{1}{2}} t + B_\gamma \frac{2}{g^{\frac{1}{2}}\mu_\gamma} \sin \tfrac{1}{2}\mu_\gamma g^{\frac{1}{2}} t\right) ,$$

where

* The form of the transformed equation is given by formula (30) of Chapter XXXI. Therefore, we do not actually need to carry out all the calculations connected with the transformation, but can use that formula to write the transformed problem.

$$A_\gamma = \frac{1}{l}\,\frac{1}{J_1{}^2(\mu_\gamma l^{\frac{1}{2}})}\int_0^l u_1(\xi)\,J_0(\mu_\gamma \xi^{\frac{1}{2}})\,d\xi\ ,$$

$$B_\gamma = \frac{1}{l}\,\frac{1}{J_1{}^2(\mu_\gamma l^{\frac{1}{2}})}\int_0^l \dot{u}_1(\xi)\,J_0(\mu_\gamma \xi^{\frac{1}{2}})\,d\xi\ .$$

2. Show that the forced vibrations of a heavy thread under the influence of a force $F(t)$ concentrated at a point $x = a$ (where $0 \leqslant a \leqslant l$) with zero initial conditions can be represented in the form of the series

$$u(x,t) = \frac{1}{ldg^{\frac{1}{2}}}\sum_{\gamma=1}^{\infty}\frac{1}{\mu_\gamma}\,\frac{J_0(\mu_\gamma a^{\frac{1}{2}})}{J_1{}^2(\mu_\gamma l^{\frac{1}{2}})}\,J_0(\mu_\gamma x^{\frac{1}{2}})\int_{-\infty}^{t} F(\zeta)\sin\tfrac{1}{2}\mu_\gamma g^{\frac{1}{2}}\,(t-\zeta)\,d\zeta\ .$$

Method: The concentrated force should be regarded as the limiting case of a force that is uniformly distributed over a segment of the thread $(a-\eta, a+\eta)$. In taking the limit as η approaches zero, we should remember the relationship

$$\frac{d}{dz}J_1(z) = J_0(z) - \frac{1}{z}J_1(z)\ .$$

3. Show that under the conditions of the preceding problem, where $F(t) = F_0 \sin \omega t$,

$$u(x,t) = \frac{F_0}{dl\omega^2}\sum_{\gamma=1}^{\infty}\frac{1}{J_1{}^2(\mu_\gamma l^{\frac{1}{2}})}\,\frac{1}{(\mu_\gamma{}^2 g/4\omega^2)-1}\,J_0(\mu_\gamma x^{\frac{1}{2}})$$
$$\times\left(\sin\omega t - \frac{2\omega}{\mu_\gamma g^{\frac{1}{2}}}\sin\tfrac{1}{2}\mu_\gamma g^{\frac{1}{2}} t\right)\ .$$

Consider separately the resonance case in which, for some γ,

$$\omega = \tfrac{1}{2}\mu_\gamma g^{\frac{1}{2}}\ .$$

Show that in this case there is a term in the expansion of $u(x,t)$ that increases with the passage of time t as t^3.

2. *Vibrations of a membrane*

Let us consider the problems of small vibrations of a rectangular membrane fastened at the edges.

The differential equation of small vibrations of a plane homogeneous membrane is of the form (Chapter VI, section 1)

$$\frac{\partial^2 u}{\partial x_1{}^2} + \frac{\partial^2 u}{\partial x_2{}^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{Z(x_1, x_2, t)}{T}\ , \tag{11}$$

where u is the displacement of the membrane from its equilibrium position,

$Z(x_1, x_2, t)$ is the pressure on the membrane, and T is the tension in the membrane. The plane in which the membrane lies when at its equilibrium position is assumed to coincide with the plane x_1, x_2. We choose the coordinate origin at one of the corners of the membrane and we direct the x_1- and x_2-axes along the edges adjacent to this corner. We then have the boundary condition

$$u|_{x_1=a} = u|_{x_1=0} = u|_{x_2=b} = u|_{x_2=0} = 0 \,, \tag{12}$$

where a and b are the lengths of the sides of the membrane. We have the initial conditions in the general form

$$u|_{t=0} = u_0(x_1, x_2) \,, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1(x_1, x_2) \,. \tag{13}$$

We need to find the solution to the problem (11) - (13) in the rectangle

$$0 \leq x_1 \leq a \,, \qquad 0 \leq x_2 \leq b \,.$$

We use an integral operator twice to eliminate differentiation with respect to the coordinates x_1 and x_2. We define

$$\mathcal{M}_1 u \equiv \partial^2 u/\partial x_1^2 \,.$$

Since this differential expression is self-conjugate ($\rho(x_1) = 1$), the kernel of the operator and the function $K_\gamma(x_1)$ coincide except for a normalizing divisor. To find them, we consider the boundary problem:

$$\frac{\partial^2 \overline{K}}{\partial x_1^2} + \lambda^2 \overline{K} = 0 \,, \tag{14}$$

$$\overline{K}|_{x_1=0} = \overline{K}|_{x_1=a} = 0 \,, \tag{15}$$

whose solution is the function $\sin \lambda_\gamma x_1$ for $\lambda_\gamma = \pi\gamma/a$ (for $\gamma = 1, 2, 3, \ldots$). We take this function as $\overline{K}_\gamma(x_1)$. Then, the kernel of the direct operator requires the normalizing divisor

$$C_\gamma = \int_0^a \sin^2 \frac{\pi}{a} x_1 \, dx_1 = \pi \,.$$

By means of an integral transformation in the interval ($0 \leq x_1 \leq a$) with kernel $\frac{1}{\pi} \sin \frac{\pi}{a} \gamma x_1$, we transform the problem (11) - (13) to the form

$$\frac{\partial^2 \overline{u}}{\partial x_2^2} - \lambda_\gamma^2 \overline{u} = \frac{1}{c^2} \frac{\partial^2 \overline{u}}{\partial t^2} - \frac{\overline{Z}(\gamma, y, t)}{T} \,, \tag{16}$$

$$\overline{u}|_{x_2=0} = \overline{u}|_{x_2=b} = 0 \,, \tag{17}$$

$$\overline{u}|_{t=0} = \overline{u}_0(\gamma, x_2) \,, \qquad \left.\frac{\partial \overline{u}}{\partial t}\right|_{t=0} = \overline{u}_1(\gamma, x_2) \,, \tag{18}$$

where, according to the conventions that we have adopted, the macron over

a symbol denotes the mapping of the initial quantities to their integral transforms using the kernel in question.

To eliminate differentiation with respect to the variable x_2, we define

$$\mathcal{M}_2 u \equiv \partial^2 \bar{u}/\partial x_2^2 \ .$$

The problem of finding a suitable operator is completely analogous to the problem (14) - (15), as a result of which the function $\bar{K}_\eta(x_2)$ can be taken equal to $\sin \mu_\eta x_2$; then, the kernel of the direct operator will be of the form $(1/\pi)\sin \mu_\eta x_2$, and the eigenvalues μ_η will be determined by the expression $\mu_\eta = \pi\eta/b$ (for $\eta = 1, 2, 3, \ldots$). When we carry out the transformation with kernel $(1/\pi)\sin \mu_\eta x_2$ within the limits $0 \leqslant x_2 \leqslant b$, we reduce the problem (16) - (18) to the form

$$\frac{1}{c^2}\frac{d^2\tilde{u}}{dt^2} + (\lambda_\gamma{}^2 + \mu_\eta{}^2)\tilde{u} = \frac{\tilde{Z}(\gamma, \eta, t)}{T} \ , \tag{19}$$

$$\tilde{u}|_{t=0} = \tilde{u}_0(\gamma, \eta) \ , \qquad \left.\frac{d\tilde{u}}{dt}\right|_{t=0} = \tilde{u}_1(\gamma, \eta) \ , \tag{20}$$

where

$$\tilde{u} = \tilde{u}(\gamma, \eta, t) = \frac{1}{\pi^2}\int_0^a \int_0^b u(x_1, x_2, t)\sin\frac{\pi}{a}\gamma x_1 \sin\frac{\pi}{b}\eta x_2 \, dx_1\, dx_2 \ ,$$

$$\tilde{Z} = \tilde{Z}(\gamma, \eta, t) = \frac{1}{\pi^2}\int_0^a \int_0^b Z(x_1, x_2, t)\sin\frac{\pi}{a}\gamma x_1 \sin\frac{\pi}{b}\eta x_2 \, dx_1\, dx_2 \ ,$$

and the quantities $\tilde{u}_0$ and $\tilde{u}_1$ are determined by analogous formulae.

Thus, by applying two integral transformations to the problem (11) - (13), we have reduced this problem to the ordinary differential equation (19). If we denote by $\tilde{u}_{\gamma\eta}(t)$ the solution of this equation with the initial conditions (20), we can, on the basis of the general expression for an inverse transformation, represent the solution $\bar{u}_\gamma(x_2, t)$ of the problem (16) - (18) in the form of the series

$$\bar{u}_\gamma(x_2, t) = \sum_{\eta=1}^{\infty} \bar{u}_{\gamma\eta}(t)\, \tilde{K}(x_2\eta) = \sum_{\eta=1}^{\infty} \tilde{u}_{\gamma\eta}(t)\sin\frac{\pi}{b}\eta x_2 \ .$$

When we carry out the inverse transformation twice, we obtain the solution of the original problem (11) - (13):

$$u(x_1, x_2, t) = \sum_{\gamma=1}^{\infty} \bar{u}_\gamma(x_2, t)\, \bar{K}_\gamma(x_1) = \sum_{\gamma,\eta=1}^{\infty} \tilde{u}_{\gamma\eta}(t)\, \bar{K}_\gamma(x_1)\, \tilde{K}_\eta(x_2)$$

$$= \sum_{\gamma,\eta=1}^{\infty} \tilde{u}_{\gamma\eta}(t)\sin\frac{\pi}{a}\gamma x_1 \sin\frac{\pi}{b}\eta x_2 \ .$$

Problems

1. Show that small free vibrations of a rectangular homogeneous membrane can be represented in the form of the series

$$u(x_1,x_2,t) = \frac{1}{4ab} \sum_{\gamma=1}^{\infty} \sum_{\delta=1}^{\infty} \sin\frac{\pi}{a}\gamma x_1 \sin\frac{\pi}{b}\eta x_2 (A_{\gamma\delta}\cos\mu_{\gamma\delta}t + B_{\gamma\delta}\sin\mu_{\gamma\delta}t) \;,$$

where

$$\mu_{\gamma\delta} = \pi c\sqrt{\frac{\gamma^2}{a^2}+\frac{\delta^2}{b^2}} \,,$$

$$A_{\gamma\delta} = \int_0^a \int_0^b u_0(x_1,x_2) \sin\frac{\pi}{a}\gamma x_1 \sin\frac{\pi}{b}\delta x_2 \, dx_1\, dx_2 \;,$$

$$B_{\gamma\delta} = \int_0^a \int_0^b \dot{u}_0(x_1,x_2) \sin\frac{\pi}{a}\gamma x_1 \sin\frac{\pi}{b}\delta x_2 \, dx_1\, dx_2 \;.$$

2. Show that the small forced vibrations of a symmetrically loaded homogeneous circular membrane of radius a with fastened edges can be represented in polar coordinates in the form of the series

$$u(r,t) = \sum_{\gamma=1}^{\infty} \bar{u}(\gamma,t)\, J_0(\lambda_\gamma r) \;,$$

where $\bar{u}(\gamma,t)$ is the solution of the ordinary differential equation

$$\frac{1}{c^2}\frac{\partial^2\bar{u}}{\partial t^2} + \lambda_\gamma{}^2\bar{u} = \frac{\bar{Z}(\gamma,t)}{T}$$

with zero initial conditions, where the λ_γ are the positive roots of the equation

$$J_0(\lambda_\gamma a) = 0 \;,$$

numbered in order of magnitude, and where the macrons over the symbols indicate that the integral transformation with kernel

$$\frac{2}{a^2}\frac{1}{J_1{}^2(\lambda_\gamma a)}\, r J_0(\lambda_\gamma r)$$

has been performed on the respective quantities between the limits 0 and a. The quantities $Z(r,t)$ and T are, respectively, the pressure on the membrane and the tension in it.

3. *Heat-flow in a cylindrical rod*

Let us consider the cooling of a homogeneous cylindrical rod with circular cross section of radius a. We shall neglect the heat loss from the ends and we shall consider the initial distribution of the temperature at an

arbitrary cross section and the conditions of heat loss along the length of the rod as being uniform. Under these assumptions, the heat-flow is described in polar coordinates r and φ by the equation (Chapter XXX, section 3)

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} = \frac{1}{k}\frac{\partial T}{\partial t}, \tag{21}$$

where T is the temperature of the rod and k is the thermal conductivity *. The origin of the polar coordinates is assumed to be on the axis of the rod.

Let us suppose that energy is radiated from the surface of the rod into a medium whose temperature is zero. Then, the boundary condition with respect to r will be of the form **

$$T|_{r=0} < \infty, \quad \left[\frac{\partial T}{\partial r} + hT\right]_{r=a} = 0, \tag{22}$$

and the periodicity conditions must obviously hold for φ:

$$T|_{\varphi=0} = T|_{\varphi=2\pi}. \tag{23}$$

We take the initial condition in the form

$$T|_{t=0} = f(r, \varphi). \tag{24}$$

Let us apply integral transformations to eliminate differentiation with respect to φ and r.

We begin with the variable φ. Let us define

$$\mathcal{M}_\varphi T = \partial^2 T/\partial\varphi^2.$$

The function $\overline{K}_\gamma(\varphi)$ must satisfy the differential equation

$$\frac{\partial^2 \overline{K}}{\partial \varphi^2} + \mu^2 \overline{K} = 0, \tag{25}$$

and the periodicity condition

$$\overline{K}|_{\varphi=0} = \overline{K}|_{\varphi=2\pi}. \tag{26}$$

As we have remarked, the periodicity condition may imply double generation of eigenvalues; that is, to each eigenvalue there may correspond two linearly independent eigenfunctions. The functions $\cos\mu\varphi$ and $\sin\mu\varphi$ for $\mu = m = 0, 1, 2, \ldots$ are mutually orthogonal linearly independent solutions of the problem (25) - (26). Let us set

$$\overline{K}_{2m-1}(\varphi) = \sin m\varphi, \quad \overline{K}_{2m}(\varphi) = \cos m\varphi.$$

Since the expression $\mathcal{M}_\varphi T$ is self-conjugate, the kernel of the direct operator differs from the function $\overline{K}_\gamma(\varphi)$ only by the normalizing divisor

* In Chapter XXX, the thermal conductivity was denoted by a^2.

** The condition for $r = 0$ is not related to the physical nature of the problem, but to the fact that the point $r = 0$ is a singular point in a polar system of coordinates.

$$C_\gamma = \int_0^{2\pi} [\overline{K}_\gamma(\varphi)]^2 \, d\varphi = \int_0^{2\pi} \sin^2 m\varphi \, d\varphi = \int_0^{2\pi} \cos^2 m\varphi \, d\varphi = \pi ,$$

which, in the given case, does not depend on whether γ is even or odd. Consequently, the kernel of the direct operator is equal to

$$\frac{1}{\pi} \cos m\varphi \quad \text{for} \quad \gamma = 2m \; ; \qquad \frac{1}{\pi} \sin m\varphi \quad \text{for} \quad \gamma = 2m - 1 \, . \tag{27}$$

Let us perform the transformation in the interval $[0, 2\pi]$ with this kernel, remembering that the same eigenvalue m^2 corresponds to the value $\gamma = 2m$ and to the value $\gamma = 2m - 1$. This reduces the problem (21)-(24) to the form

$$\frac{\partial^2 \overline{T}}{\partial r^2} + \frac{1}{r} \frac{\partial \overline{T}}{\partial r} - \frac{m^2}{r^2} \overline{T} = \frac{1}{k} \frac{\partial \overline{T}}{\partial t} , \tag{28}$$

$$\overline{T}|_{r=0} < \infty , \qquad \left[\frac{\partial \overline{T}}{\partial r} + h\overline{T}\right]_{r=a} = 0 , \tag{29}$$

$$\overline{T}|_{t=0} = \begin{cases} \overline{f}_{2m}(r) , \\ \overline{f}_{2m-1}(r) , \end{cases} \tag{30}$$

where

$$\overline{T} = \int_0^{2\pi} T(r, \varphi, t) \, \overline{K}_\gamma(\varphi) \, d\varphi , \qquad f_\gamma(r) = \int_0^{2\pi} f(r, \varphi) \, \overline{K}_\gamma(\varphi) \, d\varphi , \qquad \gamma = \begin{cases} 2m \\ 2m - 1 \end{cases} .$$

Since there are two distinct initial conditions (30), to each value m there correspond two distinct solutions of eq. (28). We denote these solutions by $\overline{T}_{2m}$ and $\overline{T}_{2m-1}$, respectively.

To eliminate differentiation with respect to r, we set

$$\mathfrak{M}_r \overline{T} \equiv \frac{\partial^2 \overline{T}}{\partial r^2} + \frac{1}{r} \frac{\partial \overline{T}}{\partial r} - \frac{m^2}{r^2} \overline{T} .$$

In this differential expression,

$$a_{rr} = 1 , \qquad b_r = \frac{1}{r} , \qquad c = -\frac{m^2}{r^2} ,$$

so that

$$\rho(r) \equiv \exp\left[- \int \frac{1}{a_{rr}} \left(\frac{da_{rr}}{dr} - b_r\right) dr\right] = r ,$$

$$p(r) \equiv a_{rr}\rho(r) = r , \qquad q(r) \equiv c\rho(r) = m^2/r .$$

The auxiliary function $\widetilde{K}_\eta(r)$ must satisfy the equation

$$\frac{\partial}{\partial r} \left(r \frac{\partial \widetilde{K}}{\partial r}\right) - \frac{m^2}{r} \widetilde{K} + \lambda^2 r \widetilde{K} = 0 ,$$

which is reduced to Bessel's equation when we divide by r:

$$\frac{\partial^2 \tilde{K}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{K}}{\partial r} + \left(\lambda^2 - \frac{m^2}{r^2}\right)\tilde{K} = 0 , \tag{31}$$

and it must also satisfy the boundary conditions

$$\tilde{K}|_{r=0} < \infty , \qquad \left[\frac{\partial \tilde{K}}{\partial r} + h\tilde{K}\right]_{r=a} = 0 . \tag{32}$$

The solutions of eq. (31) that are bounded for $r = 0$ are the Bessel functions $J_m(\lambda r)$. When we substitute the function $J_m(\lambda r)$ in the second of the conditions (32), we obtain the equation

$$\lambda_{m\eta} J_m'(\lambda_{m\eta} a) + hJ_m(\lambda_{m\eta} a) = 0 ,$$

whose roots $\lambda_{m\eta}$ determine the eigenvalues $\lambda_{m\eta}{}^2$ of the problem (31) - (32).

We take $J_m(\lambda_{m\eta} r)$ for the function $\tilde{K}_\eta(r)$. Then, the kernel of the direct operator is expressed by the function $(1/C_{m\eta}) r J_m(\lambda_{m\eta} r)$, where $C_{m\eta}$ is a normalizing divisor. By means of formula (41) of Chapter XII, we find that

$$C_{m\eta} \equiv \int_0^a r[J_m(\lambda_{m\eta} r)]^2 \, dr = \frac{1}{2\lambda_{m\eta}{}^2} (a^2h^2 + a^2\lambda_{m\eta}{}^2 - m^2) \, J_m{}^2(\lambda_{m\eta} a) .$$

When we perform the integral transformation with kernel $(r/C_{m\eta}) J_m(\lambda_{m\eta} r)$ in the interval $[0, a]$, we reduce the problem (28) - (30) to the form

$$\frac{d\tilde{T}}{dt} + k\lambda_{m\eta}{}^2 \tilde{T} = 0 , \tag{33}$$

$$\tilde{T}|_{t=0} = \begin{cases} \bar{f}_{2m,\eta} \\ \bar{f}_{2m-1,\eta} \end{cases} , \tag{34}$$

where

$$\tilde{T} = \frac{1}{C_{m\eta}} \int_0^a \bar{T}(r, m) \, rJ_m(\lambda_{m\eta} r) \, dr ,$$

$$f_{m\eta} = \frac{1}{C_{m\eta}} \int_0^a f_\gamma(r) \, rJ_m(\lambda_{m\eta} r) \, dr , \qquad \gamma = \begin{cases} 2m \\ 2m - 1 \end{cases} .$$

The solution of the problem (33) - (34) is the function

$$\tilde{T}_{\gamma\eta} = f_{\gamma\eta} \, e^{-k\lambda_{m\eta}{}^2 t} , \qquad \gamma = \begin{cases} 2m \\ 2m - 1 \end{cases} .$$

When we carry out the inverse transformations, we obtain

$$\bar{T}_\gamma(r, t) = \sum_{\eta=1}^{\infty} f_{\gamma\eta} J_m(\lambda_{m\eta} r) \, e^{-k\lambda_{m\eta}{}^2 t} , \qquad \gamma = \begin{cases} 2m \\ 2m - 1 \end{cases} ,$$

$$T(r,\varphi,t) = \sum_{m=1}^{\infty} (\overline{T}_{2m}\overline{K}_{2m} + \overline{T}_{2m-1}\overline{K}_{2m-1})$$
$$= \sum_{m,\eta=1}^{\infty} J_m(\lambda_{m\eta} r)\, e^{-k\lambda_{m\eta}^2 t}\,(f_{2m,\eta}\cos m\varphi + f_{2m-1,\eta}\sin m\varphi)\ .$$

This is the desired solution of the problem (21) - (24) in the form of a double series.

Problems

1. Show that the solution of the problem discussed above (heat-flow in a rod) when there is symmetric initial temperature distribution ($T|_{t=0} = f(r)$) can be represented in the form of the series

$$T(r,t) = \sum_{\gamma=1}^{\infty} f_\gamma J_0(\lambda_\gamma r)\, e^{-k\lambda_\gamma^2 t}\ ,$$

where

$$f_\gamma \equiv \frac{2\lambda_\gamma^2}{a^2(h^2+\lambda_\gamma^2)}\,\frac{1}{J_0^2(\lambda_\gamma a)}\int_0^a f(r)\, rJ_0(\lambda_\gamma r)\,dr$$

and the λ_γ are the positive roots of the equation

$$hJ_0(\lambda a) - \lambda J_1(\lambda a) = 0\ ,$$

numbered in order of magnitude.

2. Show that if the surface of a rod is held at a constant temperature $T = 0$ and the other conditions are as in problem 1, the distribution of the temperature in the rod at an instant t is given by the formula

$$T(r,t) = \sum_{\gamma=1}^{\infty} f_\gamma J_0(\lambda_\gamma r)\, e^{-k\lambda_\gamma^2 t}\ ,$$

where

$$f_\gamma = \frac{2}{a^2}\,\frac{1}{J_1^2(\lambda_\gamma a)}\int_0^a f(r)\, rJ_0(\lambda_\gamma r)\,dr$$

and the λ_γ are the positive roots of the equation $J_0(\lambda a) = 0$, numbered in order of magnitude.

3. By making two integral transformations, solve the problem of heat-flow in the right parallelepiped $0 \leqslant x_1 \leqslant a$, $0 \leqslant x_2 \leqslant b$, $0 \leqslant x_3 \leqslant c$ with the initial condition

$$T|_{t=0} = f(x_1, x_2, x_3)\ ,$$

if, for positive values of t, the faces are kept at a constant temperature $T = 0$.

Method: Write the equation for thermal conductivity in rectangular Cartesian coordinates and eliminate successively differentiation with respect to x_1, x_2, and x_3 in a manner analogous to that employed in section 2.

Answer:

$$T = \frac{8}{abc} \sum_{m,n,s=1}^{\infty} f_{mns}\, e^{-k\mu_{mns}t} \sin\frac{\pi}{a} m x_1 \sin\frac{\pi}{b} n x_2 \sin\frac{\pi}{c} s x_3 ,$$

where

$$f_{mns} \equiv \int_0^a \int_0^b \int_0^c f(x_1, x_2, x_3) \sin\frac{\pi}{a} m x_1 \sin\frac{\pi}{b} n x_2 \sin\frac{\pi}{c} s x_3 \, dx_1\, dx_2\, dx_3 ,$$

$$\mu_{mns} \equiv \pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{s^2}{c^2}} .$$

4. *Heat-flow in a circular tube*

Let us now consider the problem of heat-flow in a circular tube, if the initial temperature in the tube is given and a temperature $T = 0$ is maintained on its internal and external walls. We shall consider the initial distribution of the temperature constant along the length of the tube and we shall neglect the heat losses at the ends. Under these assumptions, we have the problem

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} = \frac{1}{k}\frac{\partial T}{\partial t}, \tag{35}$$

$$T|_{r=a} = T|_{r=b} = 0 , \tag{36}$$

$$T|_{t=0} = f(r,\varphi) , \tag{37}$$

where a and b are the internal and external radii of the tube and $f(r,\varphi)$ is a given function. Let us eliminate successively the differentiations with respect to φ and r.

In eliminating derivatives with respect to φ, we have the same conditions as in the problem of the preceding section. When we apply the integral operator with kernel (27) in the interval $[0,2\pi]$, we reduce the problem (35)-(37) to the form

$$\frac{\partial^2 \bar{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}}{\partial r} - \frac{m^2}{r^2}\bar{T} = \frac{1}{k}\frac{\partial \bar{T}}{\partial t}, \tag{38}$$

$$\bar{T}|_{r=a} = \bar{T}|_{r=b} = 0 , \tag{39}$$

$$\bar{T}|_{t=0} = \bar{f}_\gamma(r) , \qquad \gamma = \begin{cases} 2m \\ 2m-1 \end{cases} , \tag{40}$$

where the functions $\bar{T}$ and $\bar{f}_\gamma(r)$ are determined by formulae analogous to the corresponding formulae of the preceding section.

Problem (38) - (40) differs from the problem of the preceding section (28) - (30) only by the boundary condition. Therefore, when we seek a transformation that will make it possible to eliminate differentiation with respect to r, we conclude that the auxiliary function $\tilde{K}_\eta(r)$ will satisfy Bessel's equation

$$\frac{\partial^2 \tilde{K}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{K}}{\partial r} + \left(\lambda^2 - \frac{m^2}{r^2}\right)\tilde{K} = 0 , \tag{41}$$

and the boundary conditions, according to (39), will be of the form

$$\tilde{K}|_{r=a} = \tilde{K}|_{r=b} = 0 . \tag{42}$$

When we impose the conditions (42) on the general solution $AJ_m(\lambda r) + BY_m(\lambda r)$ of eq. (41), we obtain

$$AJ_m(\lambda a) + BY_m(\lambda a) = 0 , \qquad AJ_m(\lambda b) + BY_m(\lambda b) = 0 . \tag{43}$$

For there to exist solutions other than the trivial solution $A = B = 0$, the determinant

$$\begin{vmatrix} J_m(\lambda a) & Y_m(\lambda a) \\ J_m(\lambda b) & Y_m(\lambda b) \end{vmatrix}$$

must be equal to zero. Then, for determining the eigenvalues $\lambda_{m\eta}{}^2$, we have the equation

$$J_m(\lambda a)\ Y_m(\lambda b) - Y_m(\lambda a)\ J_m(\lambda b) = 0 .$$

When we solve the system (43), we find that, up to an arbitrary constant factor,

$$A = Y_m(\lambda_{m\eta} b) , \qquad B = -J_m(\lambda_{m\eta} b) .$$

Thus, we may take

$$\tilde{K}_\eta(r) = Y_m(\lambda_{m\eta} b)\ J_m(\lambda_{m\eta} r) - J_m(\lambda_{m\eta} b)\ Y_m(\lambda_{m\eta} r) .$$

The kernel of the direct operator will then be of the form $(r/C_{m\eta})\tilde{K}_\eta(r)$, where $C_{m\eta}$ is a normalizing divisor. Using eq. (41) and the equation for the eigenvalues, we obtain

$$C_{m\eta} = \int_a^b [Y_m(\lambda_{m\eta} b)\ J_m(\lambda_{m\eta} r) - J_m(\lambda_{m\eta} b)\ Y_m(\lambda_{m\eta} r)]^2\ r\,\mathrm{d}r$$

$$= \frac{J_m{}^2(a\lambda_{m\eta}) - J_m{}^2(b\lambda_{m\eta})}{\pi\lambda_{m\eta}{}^2 J_m{}^2(\lambda_{m\eta} a)} .$$

When we perform the transformation with kernel $(r/C_{m\eta})\tilde{K}_\eta(r)$ in the interval $[a, b]$, we reduce the problem (38) - (40) to the form

$$\frac{\partial \tilde{T}}{\partial t} + k\lambda_{m\eta}{}^2\tilde{T} = 0 , \qquad \tilde{T}|_{t=0} = f_{\gamma\eta} , \qquad \gamma = \begin{cases} 2m \\ 2m - 1 \end{cases} ,$$

where

$$\tilde{T} = \frac{1}{C_{m\eta}} \int_a^b \bar{T}(r, m) \left[Y_m(\lambda_{m\eta} b) \, J_m(\lambda_{m\eta} r) - J_m(\lambda_{m\eta} b) \, Y_m(\lambda_{m\eta} r) \right] r \, dr \, ,$$

$$f_{\gamma\eta} = \frac{1}{C_{m\eta}} \int_a^b f_\gamma(r) \left[Y_m(\lambda_{m\eta} b) \, J_m(\lambda_{m\eta} r) - J_m(\lambda_{m\eta} b) \, Y_m(\lambda_{m\eta} r) \right] r \, dr \, .$$

Hence,

$$\tilde{T}_{\gamma\eta}(t) = f_{\gamma\eta} \, e^{-k\lambda_{m\eta}{}^2 t} \, , \qquad \gamma = \begin{cases} 2m \\ 2m - 1 \end{cases} .$$

When we carry out the inverse transformations, we obtain

$$T_\gamma(r, t) = \sum_{\eta=1}^{\infty} f_{\gamma\eta} \, e^{-k\lambda_{m\eta}{}^2 t} \left[Y_m(\lambda_{m\eta} b) \, J_m(\lambda_{m\eta} r) - J_m(\lambda_{m\eta} b) \, Y_m(\lambda_{m\eta} r) \right]$$

and

$$\begin{aligned} T(r, \varphi, t) &= \sum_{m=1}^{\infty} (\bar{T}_{2m} K_{2m} + \bar{T}_{2m-1} K_{2m-1}) \\ &= \sum_{m,\eta=1}^{\infty} (f_{2m,\eta} \cos m\varphi + f_{2m-1,\eta} \sin m\varphi) \left[Y_m(\lambda_{m\eta} b) \, J_m(\lambda_{m\eta} r) \right. \\ &\qquad \left. - J_m(\lambda_{m\eta} b) \, Y_m(\lambda_{m\eta} r) \right] e^{-k\lambda_{m\eta} t} \end{aligned}$$

This double series gives the solution to the problem (35) - (37).

Problem

Suppose that energy is being radiated from the external surface of a tube into a medium whose temperature is zero and that the other conditions of the problem just examined remain unchanged. Show that the eigenvalues $\lambda_{m\eta}{}^2$ of the problem are determined by the equation

$$\lambda \left[J_m(\lambda a) \, Y_m'(\lambda b) - Y_m(\lambda a) \, J_m'(\lambda b) \right] + h \left[J_m(\lambda a) \, Y_m(\lambda b) - Y_m(\lambda a) \, J_m(\lambda b) \right] = 0 \, .$$

5. *Heat-flow in a sphere*

The problem of heat-flow in a sphere leads to a transformation in which spherical functions serve as kernels.

Let us consider a homogeneous sphere of radius a whose surface is maintained at a constant temperature $T = 0$. Study of the process of heat build-up in the sphere leads us (in spherical coordinates r, θ, φ) to the problem of finding the solution of the thermal conductivity equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial T}{\partial \mu}\right] + \frac{1}{r^2(1-\mu^2)}\frac{\partial^2 T}{\partial \varphi^2} = \frac{1}{k}\frac{\partial T}{\partial t}, \quad \mu = \cos\theta \quad (44)$$

with the initial condition

$$T|_{t=0} = f(r, \theta, \varphi) \quad (45)$$

and the boundary condition

$$T|_{r=a} = 0 \quad (46)$$

Eliminating the derivatives with respect to the coordinate φ, we find that the function $\overline{K}_\gamma(\varphi)$ and the kernel of the operator must satisfy the equation

$$\frac{\partial^2 K}{\partial \varphi^2} + m^2 K = 0$$

and the periodicity condition

$$K|_{\varphi=0} = K|_{\varphi=2\pi} .$$

As in sections 3 and 4, we set

$$\overline{K}_\gamma(\varphi) = \cos m\varphi \quad \text{for} \quad \gamma = 2m , \qquad \sin m\varphi \quad \text{for} \quad \gamma = 2m-1 .$$

where m is a positive integer. The kernel of the direct operator then differs from the function $\overline{K}_\gamma(\varphi)$ by the factor $1/\pi$.

When we carry out the direct transformation, we reduce problem (44)-(46) to the form

$$\frac{\partial^2 \overline{T}}{\partial r^2} + \frac{2}{r}\frac{\partial \overline{T}}{\partial r} + \frac{1}{r^2}\left\{\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial \overline{T}}{\partial \mu}\right] - \frac{m^2}{1-\mu^2}\overline{T}\right\} = \frac{1}{k}\frac{\partial \overline{T}}{\partial t}, \quad (47)$$

$$\overline{T}|_{t=0} = \overline{f}_\gamma(\theta, r) , \qquad \gamma = \begin{cases} 2m \\ 2m-1 \end{cases}, \quad (48)$$

$$\overline{T}|_{r=0} < \infty , \qquad \overline{T}|_{r=a} = 0 , \quad (49)$$

where $\overline{T}$ and $\overline{f}_\gamma(\theta, r)$ are the integral transforms in the interval $[0, 2\pi]$ with kernel $(1/\pi)\overline{K}_\gamma(\varphi)$ of the functions T and $f(r, \theta, \varphi)$.

To eliminate the derivatives with respect to μ, we consider the differential expression

$$\mathfrak{M}_\mu \overline{T} \equiv \frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial \overline{T}}{\partial \mu}\right] - \frac{m^2}{1-\mu^2}\overline{T} ,$$

for which

$$a_{\mu\mu} = 1-\mu^2 , \qquad p(\mu) = 1-\mu^2 , \qquad q(\mu) = \frac{m^2}{1-\mu^2} , \qquad \rho = 1 .$$

The kernel of the operator that will make it possible to eliminate differentiation with respect to μ must satisfy Legendre's equation

$$\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial \widetilde{K}}{\partial \mu}\right] + \left(\lambda - \frac{m^2}{1-\mu^2}\right)\widetilde{K} = 0 ,$$

for which the points $\mu = \pm 1$ are singular. As we know from Chapter XXI, the

requirement that the solution of Legendre's equations be bounded at the singular point is satisfied when

$$\lambda = n(n+1) \qquad (n = 0, 1, 2, \dots) .$$

Here, the solutions that are bounded in the interval $[-1, 1]$ are the associated Legendre polynomials $P_{nm}(\mu)$. We take the function $P_{nm}(\mu)$ for the function $\widetilde{K}_m(\mu)$. The kernel of the direct operator will differ by a normalizing divisor C_{mn}. By means of formula (20) of Chapter XXI, we find that

$$C_{mn} \equiv \int_{-1}^{+1} [P_{nm}(\mu)]^2 \, d\mu = \frac{2\delta}{2n+1} \frac{(n+m)!}{(n-m)!} , \qquad \delta = \begin{cases} 2 & \text{for} \quad m = 0 , \\ 1 & \text{for} \quad m \neq 0 . \end{cases}$$

When we perform the transformation with kernel $(1/C_{mn})P_{nm}(\mu)$, in the interval $[-1, 1]$, we reduce the problem (47) - (49) to the form

$$\frac{\partial^2 \widetilde{T}}{\partial r^2} + \frac{2}{r} \frac{\partial \widetilde{T}}{\partial r} - \frac{n(n+1)}{r^2} \widetilde{T} = \frac{1}{k} \frac{\partial \widetilde{T}}{\partial t} , \tag{50}$$

$$\widetilde{T} \,|_{t=0} = \widetilde{f}_{\gamma n}(r) , \qquad \gamma = \begin{cases} 2m \\ 2m-1 \end{cases} , \tag{51}$$

$$\widetilde{T}|_{r>0} < \infty , \qquad \widetilde{T}|_{r=a} = 0 , \tag{52}$$

where $\widetilde{T}$ and $\widetilde{f}_{\gamma n}(r)$ are functions obtained by successive applications to the functions T and $f(r, \theta, \varphi)$ of the integral transformations with respect to φ and μ.

Finally, we eliminate differentiation with respect to r. For the differential expression on the left side of eq. (50),

$$a_{rr} = 1 , \qquad b_{rr} = \frac{2}{r} , \qquad \rho(r) = r , \qquad p(r) = r^2 .$$

When $p(r) = r^2$, the conditions of the theorem for series expansion in eigenfunctions of the Sturm-Liouville problem are not satisfied. Consequently, some doubt arises as to the possibility of applying an integral transformation to reduce the problem (50) - (52) to a yet simpler form. However, it is easy to avoid this difficulty. If we make the substitution

$$\widetilde{T} = \widetilde{v}/r^{\frac{1}{2}} ,$$

the problem (50) - (52) becomes

$$\frac{\partial^2 \widetilde{v}}{\partial r^2} + \frac{1}{r} \frac{\partial \widetilde{v}}{\partial r} - \frac{(n+\frac{1}{2})^2}{r^2} \widetilde{v} = \frac{1}{k} \frac{\partial \widetilde{v}}{t} , \tag{53}$$

$$\widetilde{v}|_{t=0} = r^{\frac{1}{2}} \widetilde{f}_{\gamma n}(r) , \qquad \gamma = \begin{cases} 2m \\ 2m-1 \end{cases} , \tag{54}$$

$$\widetilde{v}|_{r=0} < \infty , \qquad \widetilde{v}|_{r=a} = 0 . \tag{55}$$

Now, for the expression containing the derivatives with respect to r, we obtain

$$p(r) = r , \qquad \rho(r) = r .$$

We introduce the function $\hat{K}_s(r)$, which satisfies Bessel's equation of half-integral order

$$\frac{\partial^2 \hat{K}}{\partial r^2} + \frac{1}{r}\frac{\partial \hat{K}}{\partial r} + \left[\lambda^2 - \frac{(n+\frac{1}{2})^2}{r}\right]\hat{K} = 0 \tag{56}$$

and the boundary conditions

$$|\hat{K}|\,|_{r=0} < \infty\,, \qquad \hat{K}|_{r=a} = 0\,. \tag{57}$$

The solution to this equation that is bounded for $r = 0$ is the Bessel function $J_{n+\frac{1}{2}}(\lambda r)$. When we use the boundary condition for $r = a$, we obtain the equation

$$J_{n+\frac{1}{2}}(\lambda a) = 0\,,$$

whose roots λ_{ns}, when numbered in order of magnitude, determine the eigenvalues of the problem (56) - (57). We take the function $J_{n+\frac{1}{2}}(\lambda_{ns}r)$ for the function $\hat{K}_s(r)$. Then, the kernel of the direct operator will be equal to

$$\frac{r}{C_{ns}}\, J_{n+\frac{1}{2}}(\lambda_{ns}r)\,,$$

where

$$C_{ns} = \tfrac{1}{2}a\left[J'_{n+\frac{1}{2}}(\lambda_{ns}a)\right]^2\,.$$

When we perform the integral transformation with the above kernel in the interval $[0, a]$, we reduce the problem (53) - (55) to the form

$$\frac{\partial \hat{v}}{\partial t} + k\lambda_{ns}{}^2\hat{v} = 0\,, \tag{58}$$

$$\hat{v}|_{t=0} = f_{\gamma ns}\,, \qquad \gamma = \begin{cases} 2m \\ 2m-1 \end{cases}, \tag{59}$$

where $\hat{v}$ is the corresponding integral transform of the function $\tilde{v}$ and

$$f_{\gamma ns} = \frac{1}{C_{ns}}\int_0^a \tilde{f}_{\gamma n}(r)\, r^{\frac{3}{2}}\, J_{n+\frac{1}{2}}(\lambda_{ns}r)\, dr\,.$$

When we solve the problem (58) - (59), we obtain

$$\hat{v} = f_{\gamma ns}\, e^{-k\lambda_{ns}{}^2 t}\,.$$

When we carry out the inverse transformations, we obtain, one after the other,

$$v = \sum_{s=1}^{\infty} f_{\gamma ns}\, e^{-k\lambda_{ns}{}^2 t}\, J_{n+\frac{1}{2}}(\lambda_{ns}r)\,, \qquad \tilde{T} = \frac{1}{r^{\frac{1}{2}}}\sum_{s=1}^{\infty} f_{\gamma ns}\, e^{-k\lambda_{ns}{}^2 t}\, J_{n+\frac{1}{2}}(\lambda_{ns}r)\,,$$

$$T = \frac{1}{r^{\frac{1}{2}}}\sum_{n,s=1}^{\infty} f_{\gamma ns}\, e^{-k\lambda_{ns} t}\, J_{n+\frac{1}{2}}(\lambda_{ns}r)\, P_{nm}(\mu)\,,$$

$$T(r, \theta, \varphi, t) = \frac{1}{r^{\frac{1}{2}}} \sum_{m=0}^{\infty} \sum_{n,s=1}^{\infty} e^{-k\lambda_s t} J_{n+\frac{1}{2}}(\lambda_{ns} r) P_{nm}(\mu)$$

$$\times (f_{2m,ns} \cos m\varphi + f_{2m-1,ns} \sin m\varphi) .$$

The last of these series is the solution to the problem (44) - (46).

6. *Steady-state heat-flow in a parallelepiped*

Let us suppose that the face $x_3 = 0$, $0 \leqslant x_1 \leqslant a$, $0 \leqslant x_2 \leqslant b$ of a right parallelepiped $0 \leqslant x_1 \leqslant a$, $0 \leqslant x_2 \leqslant b$, $0 \leqslant x_3 \leqslant c$ is maintained at a temperature T_0, whereas energy is radiated from the remaining faces into a space where the temperature is equal to zero. Let us find the steady-state distribution of the temperature T in such a parallelepiped. Here, we arrive at Laplace's equation

$$\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} + \frac{\partial^2 T}{\partial x_3^2} = 0 \tag{60}$$

with the boundary conditions

$$\begin{gathered} \left[\frac{\partial T}{\partial x_1} - hT\right]_{x_1=0} = 0 , \quad \left[\frac{\partial T}{\partial x_1} + hT\right]_{x_1=a} = 0 , \\ \left[\frac{\partial T}{\partial x_2} - hT\right]_{x_2=0} = 0 , \quad \left[\frac{\partial T}{\partial x_2} + hT\right]_{x_2=b} = 0 , \\ T|_{x_3=0} = T_0 , \quad \left[\frac{\partial T}{\partial z} + hT\right]_{x_3=c} = 0 . \end{gathered} \tag{61}$$

Let us eliminate differentiation with respect to x_1 and x_2.

The kernel of the operator that will eliminate differentiation with respect to x_1 must be a solution of the boundary-value problem

$$\frac{\partial^2 \overline{K}}{\partial x_1^2} + \lambda^2 \overline{K} = 0 , \tag{62}$$

$$\left[\frac{\partial \overline{K}}{\partial x_1} - h\overline{K}\right]_{x_1=0} = 0 , \quad \left[\frac{\partial \overline{K}}{\partial x_1} + h\overline{K}\right]_{x_1=a} = 0 . \tag{63}$$

When we impose on the general solution $A \cos \lambda x_1 + B \sin \lambda x_1$ of eq. (62) the boundary conditions (63), we obtain the system of homogeneous equations

$$Ah - B\lambda = 0 , \quad A(h \cos \lambda a - \lambda \sin \lambda a) + B(\lambda \cos \lambda a + h \sin \lambda a) = 0 .$$

Solutions of this system (other than the trivial solution ($A = B = 0$)) exist only when its determinant is equal to zero. From this condition, we obtain, after some elementary transformations, the equation for determining the eigenvalues λ_m^2:

$$\tan \lambda_m a = \frac{2h\lambda_m}{\lambda_m^2 - h^2}.$$

When $\lambda = \lambda_m$ the coefficients A and B are, respectively, proportional to λ_m and h. Thus, up to a constant factor, the solutions of the boundary-value problem (62) - (63) are of the form

$$\cos \lambda_m x_1 + \frac{h}{\lambda_m} \sin \lambda_m x_1 . \tag{64}$$

We take this expression for the function $\bar{K}_m(x_1)$. Since the differential expression $\partial^2 T/\partial x_1^2$ is self-conjugate, the kernel of the direct operator will differ only by a normalizing factor

$$C_m = \int_0^a [\cos \lambda_m x_1 + \frac{h}{\lambda_m} \sin \lambda_m x_1]^2 \, dx_1$$
$$= \tfrac{1}{2}a\left(1 + \frac{h^2}{\lambda_m^2}\right) + \frac{1}{\lambda_m}\left(\cos^2 \lambda_m a + \frac{h}{\lambda_m} \sin^2 \lambda_m a\right).$$

When we carry out the transformation with this kernel in the interval $[0, a]$, we reduce the problem (60) - (61) to the form

$$\frac{\partial^2 \bar{T}}{\partial x_2^2} + \frac{\partial^2 \bar{T}}{\partial x_3^2} - \lambda_m^2 \bar{T} = 0 , \tag{65}$$

$$\left[\frac{\partial \bar{T}}{\partial x_2} - h\bar{T}\right]_{x_2=0} = 0 , \quad \left[\frac{\partial \bar{T}}{\partial x_2} + h\bar{T}\right]_{x_2=b} = 0 ,$$
$$\bar{T}|_{x_3=0} = \frac{aT_0}{C_m} , \quad \left[\frac{\partial \bar{T}}{\partial x_3} + h\bar{T}\right]_{x_3=c} = 0 . \tag{66}$$

In a completely analogous manner, we can eliminate differentiation with respect to the coordinate x_2. The problem (65) - (66) then takes the form

$$\frac{\partial^2 \tilde{T}}{\partial x_3^2} - (\mu_n^2 + \lambda_m^2)\tilde{T} = 0 , \tag{67}$$

$$\tilde{T}|_{x_3=0} = \frac{1}{C_m D_n} abT_0 , \quad \left[\frac{\partial \tilde{T}}{\partial x_3} + h\tilde{T}\right]_{x_3=c} = 0 , \tag{68}$$

where $\tilde{T}$ is the integral transform of the function $\bar{T}$ in the interval $[0, b]$ with kernel

$$\frac{1}{D_n} \cos (\mu_n x_2 + \frac{h}{\mu_n} \sin \mu_n x_2) ,$$

where

$$D_n = \tfrac{1}{2}b\left(1 + \frac{h^2}{\mu_n^2}\right) + \frac{1}{\mu_n}\left(\cos^2 \mu_n b + \frac{h}{\mu_n} \sin^2 \mu_n b\right) ,$$

and the μ_n are the roots of the equation

$$\tan \mu_n b = \frac{2h\mu_n}{\mu_n^2 - h^2},$$

numbered in order of magnitude.

When we impose the boundary condition (68) on the general solution of eq. (67), namely,

$$\tilde{T}_{mn}(x_3) \equiv A_{mn}\, e^{\nu_{mn}x_3} + B_{mn}\, e^{-\nu_{mn}x_3} \qquad (\nu_{mn} \equiv \sqrt{\lambda_m^2 + \mu_n^2}),$$

we obtain the following system of equations for determining the constants A_{mn} and B_{mn}

$$A_{mn} + B_{mn} = \frac{abT_0}{C_m D_n}, \qquad A_{mn}(h+\nu_{mn})\, e^{\nu_{mn}c} + B_{mn}(h+\nu_{mn})\, e^{-\nu_{mn}c} = 0,$$

so that

$$A_{mn} = \frac{(\nu_{mn} - h)\, e^{-\nu_{mn}c}}{(\nu_{mn}+h)\, e^{\nu_{mn}c} + (\nu_{mn} - h)\, e^{-\nu_{mn}c}} \frac{abT_0}{C_m D_n},$$

$$B_{mn} = \frac{(\nu_{mn}+h)\, e^{\nu_{mn}c}}{(\nu_{mn}+h)\, e^{\nu_{mn}c} + (\nu_{mn} - h)\, e^{-\nu_{mn}c}} \frac{abT_0}{C_m D_n},$$

$$T_{mn}(x_3) = a_{mn}\, e^{-\nu_{mn}(c-x_3)} + b_{mn}\, e^{\nu_{mn}(c-x_3)},$$

where

$$a_{mn} = \frac{(\nu_{mn} - h)}{(\nu_{mn}+h)\, e^{\nu_{mn}c} + (\nu_{mn} - h)\, e^{-\nu_{mn}c}} \frac{abT_0}{C_m D_n},$$

$$b_{mn} = \frac{(\nu_{mn} + h)}{(\nu_{mn}+h)\, e^{\nu_{mn}c} + (\nu_{mn} - h)\, e^{-\nu_{mn}c}} \frac{abT_0}{C_m D_n}.$$

When we perform the inverse transformation, we obtain the solution of the original problem (60) - (61) in the form of the double series

$$T(x_1, x_2, x_3) = \sum_{m,n=1}^{\infty} (\cos \lambda_m x_1 + \frac{h}{\lambda_m} \sin \lambda_m x_1)(\cos \mu_n x_2 + \frac{h}{\mu_n} \sin \mu_n x_2)$$
$$\times (a_{mn}\, e^{-\nu_{mn}(c-x_3)} + b_{mn}\, e^{\nu_{mn}(c-x_3)}).$$

Problem

Suppose that the face $x_3 = 0$ of the right parallelepiped $0 \leq x_1 \leq a$, $0 \leq x_2 \leq b$, $0 \leq x_3 \leq c$ is held at a constant temperature T_0 and the remaining faces are held at a temperature $T = 0$. Show that the steady-state dis-

tribution of the temperature can be represented in the form of the double series

$$T(x_1, x_2, x_3) = \frac{16T_0}{\pi^2} \sum_{r,s=0}^{\infty} \frac{1}{(2r+1)(2s+1)}$$

$$\times \sin \frac{(2r+1)\,\pi x_1}{a} \sin \frac{(2s+1)\,\pi x_2}{b} \frac{\sinh \mu x_3}{\sinh \mu c},$$

where

$$\mu = \pi \sqrt{\frac{(2r+1)^2}{a^2} + \frac{(2s+1)^2}{b^2}}\,.$$

Chapter XXXIII

EXAMPLES OF THE APPLICATION OF INTEGRAL TRANSFORMATIONS WITH INFINITE LIMITS

1. *The problem of the vibrations of an infinitely long string*

Let us begin the study of problems in which it is expedient to use integral operators with infinite limits with the problem of the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

in which x takes on all real values, with the boundary conditions

$$u|_{t=0} = u_0(x) , \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1(x) , \tag{2}$$

where $u_0(x)$ and $u_1(x)$ are given functions that vanish outside some finite region. We encounter such a problem when, for example, we examine the vibrations of an infinitely long string or the small one-dimensional vibrations of a gas under the action of a finite initial disturbance.

The kernel of the integral operator that will make it possible to eliminate differentiation with respect to x must satisfy the equation

$$\frac{\partial^2 K}{\partial x^2} + \gamma^2 K = 0 \tag{3}$$

and must be bounded for all real values of x. The last requirement is satisfied for all real values of γ^2; that is, the eigenvalues of the problem for eq. (3) have continuous spectrum. Then,

$$K = C\, e^{\pm i\gamma x},$$

from which it is clear that we should apply the Fourier transform.

When we apply the operator with the kernel $K = (1/2\pi)e^{-i\gamma x}$, we transform the problem (1) - (2) to the form *:

$$\frac{\partial^2 \bar{u}}{\partial t^2} + \gamma^2 c^2 \bar{u} = 0 , \tag{4}$$

$$\bar{u}|_{t=0} = \bar{u}_0(\gamma) , \qquad \left.\frac{\partial \bar{u}}{\partial t}\right|_{t=0} = \bar{u}_1(\gamma) , \tag{5}$$

* We recall that this result follows immediately on the basis of formula (65) in the résumé of formulae in Chapter XXXI.

where

$$\bar{u} = \bar{u}(t,\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)\ e^{-i\gamma x}\,dx\ , \tag{6}$$

$$\bar{u}_0(\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_0(x)\ e^{-i\gamma x}\,dx\ , \tag{7}$$

$$\bar{u}_1(\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_1(x)\ e^{-i\gamma x}\,dx\ . \tag{8}$$

The convergence of the integrals (6) - (8) is ensured by the fact that each of the functions u_0, u_1, and u vanish outside some finite interval. The function u has this property because in a finite interval of time the disturbance that is being propagated with velocity c covers only a finite distance. The solution of the problem (4) - (5) is the function

$$\bar{u} = \bar{u}_0 \cos\gamma ct + \frac{\bar{u}_1}{c}\sin\gamma ct\ .$$

If we express the functions $\cos\gamma ct$ and $\sin\gamma ct$ in exponential form and substitute the function $\bar{u}$ in the formula for the inverse Fourier transform, that is,

$$u(x,t) = \int_{-\infty}^{\infty} \bar{u}\ e^{i\gamma x}\,dx\ ,$$

we obtain

$$u(x,t) = \tfrac{1}{2}\int_{-\infty}^{\infty} \bar{u}_0(\gamma)\ e^{-i\gamma(x-ct)}\,d\gamma + \tfrac{1}{2}\int_{-\infty}^{\infty} \bar{u}_0(\gamma)\ e^{-i\gamma(x+ct)}\,d\gamma$$

$$+ \frac{1}{2i}\int_{-\infty}^{\infty}\frac{\bar{u}_1}{\gamma c}\ e^{-i\gamma(x-ct)}\,d\gamma - \frac{1}{2i}\int_{-\infty}^{\infty}\frac{\bar{u}_1}{\gamma c}\ e^{-i(x+ct)}\,d\gamma\ . \tag{9}$$

When we compare the first two integrals on the right side of this equation with the expression (7), we easily see that they are, respectively, equal to $\frac{1}{2}u_0(x-ct)$ and $\frac{1}{2}u_0(x+ct)$. To calculate the second pair of integrals, we note that

$$\frac{\partial}{\partial\xi}\left(\frac{1}{2i}\int_{-\infty}^{\infty}\frac{\bar{u}_1}{\gamma c}\ e^{-i\gamma\xi}\,d\gamma\right) = \frac{1}{2c}\int_{-\infty}^{\infty}\bar{u}_1\ e^{-i\gamma\xi}\,d\gamma = \frac{1}{2c}\,u_1(\xi)\ ,$$

from which it follows that

$$\frac{1}{2i}\int_{-\infty}^{\infty}\frac{\bar{u}_1}{\gamma c}\ e^{-i\gamma\xi}\,d\gamma = \frac{1}{2c}\int_a^{\xi} u_1(\zeta)\,d\zeta\ ,$$

where a is an arbitrary constant. If we now set ξ equal to $x-ct$ and $x+ct$, we see that the sum of the last two integrals on the right side of eq. (9) is equal to the integral

$$\frac{1}{2c}\int_{x-ct}^{x+ct} u_1(\zeta)\, d\zeta .$$

When we substitute the values that we have found into the right side of eq. (9), we obtain the solution to the original problem (1) - (2) in d'Alembert's form:

$$u(x, t) = \frac{u_0(x-ct) + u_0(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(\zeta)\, d\zeta .$$

We have already had occasion to deal with this expression in section 2 of Chapter III.

Problem

Use an integral transformation to derive the formulae determining the vibrations of a semi-infinite string fastened at the coordinate origin.

Method: Beginning with the fact that one end of the string is fastened, show that a Fourier sine transform should be used to eliminate differentiation with respect to x.

2. *Linear heat-flow in a semi-infinite rod*

Let us consider the problem of the distribution of temperature in a homogeneous semi-infinite rod with an insulated lateral surface. Suppose that its end is maintained at a temperature T_0 and that the initial temperature is everywhere zero. Let the positive end of the x-axis coincide with the rod. Our problem then becomes one of integrating the thermal conductivity equation

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{k}\frac{\partial T}{\partial t} \tag{10}$$

with initial condition

$$T|_{t=0} = 0 , \tag{11}$$

and boundary condition

$$T|_{x=0} = T_0 . \tag{12}$$

The kernel $K(x, \lambda)$ of the integral operator that will allow us to eliminate differentiation with respect to x must satisfy the requirement

$$\frac{\partial^2 K}{\partial x^2} + \lambda^2 K = 0 , \qquad K|_{x=0} = 0 , \qquad K|_{x=\infty} < \infty ,$$

from which it follows that the kernel K is, except for a constant factor, equal to $\sin \lambda x$, where λ is non-negative. Thus, we need to use the Fourier sine transform. Setting

$$K(x, \lambda) = \frac{2}{\pi} \sin \lambda x \ , \qquad K_1(x, \lambda) = \sin \lambda x \ ,$$

we transform the problem (10) - (12) to the form

$$\frac{\partial \overline{T}}{\partial t} + k\lambda^2 \overline{T} = \frac{2}{\pi} k\lambda T_0 \ , \qquad \overline{T}|_{t=0} = 0 \ ,$$

where

$$\overline{T} = \overline{T}(\lambda, t) = \frac{2}{\pi} \int_0^\infty T(x, t) \sin \lambda x \, dx \ .$$

Therefore,

$$\overline{T}(\lambda, t) = \frac{2}{\pi} \frac{T_0}{\lambda} (1 - e^{-k\lambda^2 t}) \ .$$

When we perform the inverse transformations, we obtain

$$T(x, t) = \int_0^\infty \overline{T}(\lambda, t) \sin \lambda x \, d\lambda = \frac{2}{\pi} T_0 \int_0^\infty \frac{1 - e^{-k\lambda^2 t}}{\lambda} \sin \lambda x \, d\lambda \ .$$

Recalling that

$$\int_0^\infty e^{-\zeta^2} \frac{\sin \zeta a}{\zeta} d\zeta = \sqrt{\pi} \int_0^{\frac{1}{2}a} e^{-\zeta^2} d\zeta \ ,$$

$$\sqrt{\pi} \int_0^\infty e^{-\zeta^2} d\zeta = \int_0^\infty \frac{\sin \zeta a}{\zeta} d\zeta \qquad (a > 0) \ ,$$

we transform the last relationship to the form

$$T(x, t) = \frac{2T_0}{\sqrt{\pi}} \int_{x/2\sqrt{kt}}^\infty e^{-\zeta^2} d\zeta \ .$$

This expression gives the solution to our problem. Noting that

$$\frac{2}{\pi} \int_0^\xi e^{-\zeta^2} d\zeta \equiv \Phi(\xi)$$

is the so-called probability integral, the values of which are tabulated, our solution can also be represented in the form

$$T(x, t) = T_0[1 - \Phi(x/2\sqrt{kt})] \ .$$

Problems

1. Show that the distribution of temperature in an infinite homogeneous rod with thermally insulated lateral surface is given by the formula

$$T(x,t) = \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} u_0(\zeta)\, e^{-(x-\zeta)^2/4kt}\, d\zeta ,$$

where

$$u_0(x) = T(x,0) .$$

Method: Take into oonsideration the fact that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-k\zeta^2 t - i\zeta x}\, d\zeta = \frac{1}{2\sqrt{\pi kt}}\, e^{-x^2/4kt}$$

and use the convolution theorem

$$\bar{f}(\gamma)\, \bar{g}(\gamma) = \int_{-\infty}^{x} f(\xi)\, g(x-\xi)\, d\xi ,$$

where $\bar{f}(\gamma)$ and $\bar{g}(\gamma)$ are the Fourier transforms of the functions $f(x)$ and $g(x)$.

2. Suppose that a semi-infinite rod has a thermally insulated lateral surface and that the temperature of the end ($x = 0$) is maintained at zero. Show that the temperature distribution is given by the formula

$$T(x,t) = \frac{1}{2\sqrt{\pi kt}} \int_{0}^{\infty} u_0(\zeta)\, [e^{-(x-\zeta)^2/4kt} - e^{-(x+\zeta)^2/4kt}]\, d\zeta ,$$

where $u_0(x) = T(x,0)$.

3. *The distribution of heat in a cylindrical rod whose surface is kept at two different temperatures*

Consider an infinite homogeneous cylindrical rod of circular cross sections. Suppose that the temperature of the rod is equal to zero at the initial instant of time. Suppose that the temperature of the surface of a segment of length $2l$ is then maintained at T_0, and that the temperature of the remaining portion is still kept at zero. Determine the temperature distribution in the rod.

We use cylindrical coordinates with the z-axis directed along the axis of the rod and with origin in the middle of the section that is kept at the temperature T_0. The temperature distribution then satisfies the equation

$$\frac{d^2 T}{dr^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k}\frac{\partial T}{\partial t} , \tag{13}$$

the initial condition

$$T|_{t=0} = 0 \tag{14}$$

and the boundary condition

$$T|_{r=a} = \begin{cases} T_0 & \text{for} \quad |z| < l \,, \\ 0 & \text{for} \quad |z| > l \,. \end{cases} \tag{15}$$

Obviously, the temperature distribution is symmetric about the plane $z = 0$. Therefore, it is sufficient to examine that part of the rod for which z is non-negative. Because of the symmetry,

$$\left.\frac{\partial T}{\partial z}\right|_{z=0} = 0 \,. \tag{16}$$

By applying integral transformations first to z and then to t, we reduce the problem (13) - (15) to an ordinary differential equation.

The kernel of the integral operator $K(z,\lambda)$ that will allow us to eliminate differentiation with respect to z must satisfy the equation

$$\frac{\partial^2 K}{\partial z^2} + \lambda^2 K = 0$$

and the boundary conditions

$$\left.\frac{\partial K}{\partial z}\right|_{z=0} = 0 \,, \qquad K|_{z=\infty} < 0 \,.$$

The first of these follows from eq. (16).

From this it is clear that the kernel K is, except for a constant factor, equal to $\cos \lambda z$; that is, we need to use the Fourier cosine transform. When we perform this transformation, we reduce the problem (13) - (15) to the form

$$\frac{\partial^2 \overline{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{T}}{\partial r} - \lambda^2 \overline{T} = \frac{1}{K}\frac{\partial \overline{T}}{\partial t} \,, \qquad \overline{T}|_{t=0} = 0 \,, \qquad \overline{T}|_{r=a} = \frac{2}{\pi} T_0 \frac{\sin \lambda l}{\lambda} \,,$$

where

$$\overline{T} = \overline{T}(r,\lambda,t) = \frac{2}{\pi}\int_0^\infty T(r,z,t) \cos \lambda z \, dz \,.$$

To eliminate differentiation with respect to t, we use the Laplace transform, whereupon we obtain the ordinary differential equation

$$\frac{d^2 \tilde{T}}{dr^2} + \frac{1}{r}\frac{d\tilde{T}}{dt} - \left(\lambda^2 + \frac{\gamma}{k}\right)\tilde{T} = 0 \,, \tag{17}$$

where

$$\tilde{T} = \tilde{T}(r,\lambda,\gamma) = \int_0^\infty \overline{T}(r,\lambda,t)\, e^{-\gamma t}\, dt \,,$$

and the boundary condition

$$\tilde{T}|_{r=a} = \frac{2}{\pi} T_0 \frac{\sin \lambda l}{\lambda \gamma} \,. \tag{18}$$

From section 7 of Chapter XII, the solution of eq. (17) that is bounded for $r = 0$ is the function $I_0(\mu r)$, where

$$\mu = \left(\lambda^2 + \frac{\gamma}{k}\right)^{\frac{1}{2}} . \tag{19}$$

By using the boundary condition (18), we obtain

$$\tilde{T} = \frac{2}{\pi} T_0 \frac{\sin \lambda l}{\lambda\gamma} \frac{I_0(\mu r)}{I_0(\mu a)} .$$

This function has no singularity anywhere in the complex plane except at the poles.

From the formula for the inverse Laplace transform, we then obtain

$$\bar{T} = \frac{1}{2\pi i} \frac{2T_0 \sin \lambda l}{\pi\lambda} \int_{b-i\infty}^{b+i\infty} e^{\gamma t} \frac{I_0(\mu r)}{I_0(\mu a)} \frac{d\gamma}{\gamma} , \tag{20}$$

provided the constants b can be chosen in such a way that all poles of the integrand lie to the left of the straight line $\operatorname{Re} \gamma = b$. The integrand has poles at $\gamma = 0$ and also at the values $\gamma = \gamma_m$ (for $m = 1, 2, 3, \ldots$) that satisfy the equation

$$I_0(\mu a) = 0 . \tag{21}$$

Then, from eq. (19), we have

$$\gamma_m = k(\mu_m{}^2 - \lambda^2) ,$$

where the μ_m are the roots of eq. (21). Since $I_0(ix) = iJ_0(x)$, the roots of eq. (21) are purely imaginary and are equal in absolute value to the roots of the Bessel function $J_0(\nu a)$, and we obtain

$$\gamma_m = -k(\nu_m{}^2 + \lambda^2) \qquad (m = 1, 2, 3, \ldots) ,$$

where the ν_m are the roots of the equations $J_0(\nu a) = 0$, numbered in order of magnitude.

Thus, all the poles of the integrand lie to the left of the half-plane and on the imaginary axis. Therefore, the number b can be chosen so that they are all situated to the left of the straight line $\operatorname{Re} \gamma = b$.

The Cauchy residue theorem * can now be used. It then follows that the integral on the right side of (20) is equal to $2\pi i$ multiplied by the sum of the residues of the integrand at the poles that are located to the left of the straight line $\operatorname{Re} \gamma = b$. At the point $\gamma = 0$, the residue of the integrand is equal to $I_0(\mu r)/I_0(\mu a)$. The residues at the point $\gamma = \gamma_m$ are equal to

$$\frac{e^{-k(\nu_m{}^2 + \lambda^2)t} J_0(\nu_m r)}{\gamma_m [(d/d\gamma) I_0(\mu a)]_{\gamma = \gamma_m}} .$$

Since

$$\frac{d}{d\gamma} I_0(\mu a) = \left[\frac{d}{d\mu} I_0(\mu a)\right] \frac{d\mu}{d\gamma} = \frac{a}{2k\mu} I_1(\mu a) , \qquad I_1(\mu_m a) = iJ_1(\nu_m a) ,$$

the denominator of the last expression can be written in the form

* See, for example, A.M.Efros and A.M.Danielevskii [39], Chapter I, or A.I.Lur'e [38].

$$\frac{a}{2\nu_m}(\nu_m{}^2+\lambda^2)\,J_1(\nu_m a)\ .$$

If we now take the sum of the residues and substitute its value into eq. (20), replacing the integral that appears there, we obtain

$$\overline{T}=\frac{2T_0}{\pi}\,\frac{\sin\lambda l}{\lambda}\left[\frac{I_0(\lambda r)}{I_0(\lambda a)}-\frac{2}{a}\sum_{m=1}^{\infty}\frac{\nu_m}{\nu_m{}^2+\lambda^2}\,\frac{J_0(\nu_m r)}{J_1(\nu_m a)}\,e^{-k(\nu_m{}^2+\lambda^2)t}\right].$$

When we perform the inverse Fourier cosine transform, we obtain the solution to our problem:

$$T=T_0\left[-\frac{4}{\pi a}\sum_{m=1}^{\infty}e^{-k\nu_m{}^2t}\,\frac{\nu_m J_0(\nu_m r)}{J_1(\nu_m a)}\int_0^{\infty}e^{-k\lambda^2t}\,\frac{\sin\lambda l\cos\lambda z}{\lambda(\nu_m{}^2+\lambda^2)}\,d\lambda\right.$$
$$\left.+\frac{2}{\pi}\int_0^{\infty}\frac{I_0(\lambda r)}{I_0(\lambda a)}\,\frac{\sin\lambda l\cos\lambda z}{\lambda}\,d\lambda\right].$$

Problem

Use the Laplace transform to find the temperature distribution in an infinite homogeneous circular rod if the initial temperature is everywhere equal to zero, and the temperature on the surface is subsequently raised to and maintained at a temperature T_0.
Answer:

$$T=T_0\left[1-\frac{2}{a}\sum_{m=1}^{\infty}e^{-k\mu_n{}^2t}\,\frac{J_0(\mu_m r)}{\mu_m J_1(\mu_m a)}\right],$$

where the μ_m are the roots of the equation $J_0(\mu a)=0$.

4. *The steady thermal state of an infinite wedge*

Consider an infinite wedge whose angle (at the edge) is $2\zeta<\pi$. Let us assume that the temperature of the lateral surface of the wedge is kept at zero except for two strips of width a bordering the edge of the wedge. Suppose that these strips are kept at a temperature T_0.

Let us find the steady-state temperature distribution in the wedge. We set up a polar coordinate system r, φ in a plane perpendicular to the edge of the wedge with origin on the edge. We then have Dirichlet's problem

$$\Delta T\equiv\frac{\partial^2T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2T}{\partial\varphi^2}=0\ , \tag{22}$$

$$T|_{\varphi=\pm\zeta}=\begin{cases}T_0 & \text{for } r<a\ ,\\ 0 & \text{for } r>a\ .\end{cases} \tag{23}$$

Let us eliminate differentiation with respect to r. First, we multiply Laplace's equation (22) by r^2 and obtain

$$r^2 \frac{\partial^2 T}{\partial r^2} + r \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial \varphi^2} = 0 .$$

Consider the differential expression

$$\mathcal{M}_r T \equiv r^2 \frac{\partial^2 T}{\partial r^2} + r \frac{\partial T}{\partial r} .$$

In this expression, $a_{rr} = r^2$, $b_r = r$, and $c = 0$. Therefore,

$$\rho(r) = \frac{1}{r}, \qquad p(r) = r , \qquad q(r) = 0 .$$

Consequently, the kernel of the integral operator that we may use to eliminate differentiation with respect to r must be equal to the product $r\overline{K}(r,\gamma)$, where $\overline{K}(r,\gamma)$ is a solution of the equation

$$\frac{\partial}{\partial r}\left(r \frac{\partial \overline{K}}{\partial r}\right) + \frac{\gamma^2}{r}\overline{K} = 0 .$$

This is an Euler-type equation. It is easy to see that it is satisfied by the function $\overline{K}(r,\gamma) = r^{\pm i\gamma}$, which leads to the expression $r^{\pm i\gamma}$ for the kernel of the desired operator. Setting $\gamma = i\mu$, where μ is a real number, we obtain the kernel $r^{\mu-1}$ of the Mellin transform.

When we perform the Mellin transformations in the interval $[0,\infty]$, we reduce the problem (22) - (23) to the form

$$\frac{\partial^2 \overline{T}}{\partial \varphi^2} + \mu^2 \overline{T} = 0 , \tag{24}$$

$$\overline{T}|_{\varphi=\pm\zeta} = \overline{T}_0 , \tag{25}$$

where

$$\overline{T} = \int_0^\infty T(r)\, r^{\mu-1}\, dr , \qquad \overline{T}_0 = \int_0^a T_0\, r^{\mu-1}\, dr = T_0 \frac{a^\mu}{\mu} .$$

The right side of eq. (24) is equal to zero because here $p(0) = 0$.

Of course, the following question arises: To what extent are the conditions which ensure the applicability of the Mellin transformation satisfied? However (in addition to the general considerations that may be mentioned with regard to the nature of the decrease of the function $T(r,\varphi)$ at infinity as a harmonic function), we can verify that the necessary conditions are satisfied from the expression for the inverse operator (see Chapter XXXI, section 4).

If we impose the boundary conditions (25) on the general solution

$$A_\mu \cos \mu\varphi + B_\mu \sin \mu\varphi$$

of eq. (24), we obtain

$$A_\mu = T_0 \frac{a^\mu \cos \mu\varphi}{\mu \cos \mu\zeta}\,, \qquad B_\mu = 0\,,$$

so that

$$\bar{T}(\mu,\varphi) = T_0 \frac{a^\mu \cos \mu\varphi}{\mu \cos \mu\zeta}\,.$$

We now apply the formula for the inverse operator. This gives us

$$T(r,\varphi) = \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} T_0(a/r)^\mu \frac{\cos \mu\varphi}{\cos \mu\zeta}\frac{d\mu}{\mu}\,. \tag{26}$$

The integrand has a pole at $\mu = 0$ and a subsequent pole at $\mu = \pi/2\zeta$. For $0 < \eta < \pi/2\zeta$, the integrand is analytic and approaches zero uniformly as $\mathrm{Im}\ \mu \to \pm\infty$, and the integral

$$\int_{\eta-i\infty}^{\eta+i\infty} |\bar{T}(\mu,\varphi)|\ d\mu$$

converges for all $|\varphi| < \zeta$. The last assertion follows when we calculate the value of $|\bar{T}|$ as $\mathrm{Im}\,\mu \to \pm\infty$. For if we set $\mu = \eta + i\eta'$, we have $\mathrm{Im}\,\mu = \eta'$. Then,

$$|\bar{T}(\mu,\varphi)| = |T_0| \left| \frac{a^{\eta+i\eta'}}{\eta + i\eta'} \frac{e^{i\eta\varphi-\eta'\varphi} + e^{-i\eta\varphi+\eta'\varphi}}{e^{i\eta\zeta-\eta'\zeta} + e^{i\eta\zeta+\eta'\zeta}} \right|\,.$$

If η' approaches ∞, the expression $\bar{T}(\eta + i\eta', \varphi)$ is of order $e^{\eta'(\varphi-\zeta)}$ and decreases exponentially, since the difference $\varphi - \zeta$ is negative because $|\varphi| < \zeta$. We obtain an analogous result as $\eta' \to -\infty$. This proves the convergence of the integral in question. Thus, the Mellin transform is meaningful throughout the entire interval $-\zeta < \varphi < \zeta$ in which we are interested.

In the integral (26), the path of integration can be shifted to the imaginary axis if we consider a small semicircle around the pole $\mu = 0$ (in the right half-plane). The integral over the semicircle will be equal to the residue of the integrand for $\mu = 0$ multiplied by πi, that is, $\frac{1}{2}T_0$. Except for the neighbourhood of the point $\mu = 0$, the integral along the imaginary axis will be equal to the principal value of the integral

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} T_0(a/r)^\mu \frac{\cos \mu\varphi}{\cos \mu\zeta}\frac{d\mu}{\mu} = \frac{T_0}{2\pi i}\int_{-\infty}^{\infty} (a/r)^{i\xi} \frac{\cosh \xi\varphi}{\cosh \xi\zeta}\frac{d\xi}{\xi}\,.$$

The integrand in the last integral can be simplified. Specifically, in the expression

$$(a/r)^{i\xi} = e^{i\xi \ln a/r} = \cos(\xi \ln a/r) + i \sin(\xi \ln a/r)\,,$$

it is sufficient to keep only the odd portion because the function

$$\cos(\xi \ln a/r) \frac{\cosh \xi\varphi}{\cosh \xi\zeta}\frac{1}{\xi}$$

is odd and the principal value of the integral of this part is equal to zero.

The integrand then is even and we integrate in the usual sense in a neighbourhood of the point $\xi = 0$. It will then be possible to carry out the integration only along the positive portion of the real axis.

Taking these facts into account, we rewrite formula (26) in the form

$$T(r,\varphi) = \frac{T_0}{2} + \frac{T_0}{\pi}\int_0^\infty \sin(\xi \ln a/r)\,\frac{\cosh \xi\varphi}{\cosh \xi\zeta}\,\frac{d\xi}{\xi}\,.$$

This formula is the solution to our problem.

Problem

Show that, for $-\zeta < \varphi < \zeta$, the function $T(r,\varphi)$ has the following properties:

$$T(a,\varphi) = \tfrac{1}{2}T_0\,, \qquad T(r,\varphi)\begin{cases} > \tfrac{1}{2}T_0 & \text{for} \quad 0 < r < a, \\ < \tfrac{1}{2}T_0 & \text{for} \quad r > a \end{cases},$$

$$\lim_{r\to\infty} T(r,\varphi) = 0\,.$$

Chapter XXXIV

MAXWELL'S EQUATIONS

1. *The system of Maxwell's equations*

The electromagnetic-field equations discovered by Maxwell provide a profound generalization of physical facts whose examination would carry us far beyond the scope of the present book. Therefore, we shall not deal with the physical basis of Maxwell's equations, but shall confine ourselves to an exposition of the information necessary for understanding them.

To describe an electromagnetic field, we introduce four vectors: $\mathcal{E}$ and $\mathcal{H}$, that is, the electric and magnetic field intensities or electric and magnetic vectors (we shall use the latter names) and the vectors $\mathcal{D}$ and $\mathcal{B}$, that is, the electric and magnetic inductions ($\mathcal{D}$ is also known as the electric displacement). These vectors satisfy Maxwell's equations, which, in the Gaussian system of electromagnetic units, can be written in the form

$$\operatorname{curl} \mathcal{E} = -\frac{1}{c}\frac{\partial \mathcal{B}}{\partial t}, \qquad \operatorname{curl} \mathcal{H} = \frac{1}{c}\frac{\partial \mathcal{D}}{\partial t} + \frac{4\pi}{c} i,$$

where i is the current density and c is the electrodynamic constant.

Rather than assume that the reader is familiar with vector analysis, we shall use scalar notation. Let us agree to denote the components of the vectors in rectangular Cartesian coordinates by the same letters as the vectors themselves, but with subscripts corresponding to the number of the component. Let us also agree that if three literal subscripts α, β, and γ appear in an equation, this equation is valid for values of α, β, and γ that are respectively equal to the numbers 1, 2, and 3, or to the numbers 2, 3, and 1 or to 3, 1, and 2. The last two combinations are obtained from the numbers 1, 2, and 3 by a cyclic permutation. If one literal subscript appears in an equation, we shall consider the equation true for all three values (1, 2, and 3) of that subscript. These conventions make it possible to write three scalar equations in the form of a single relationship and thus to retain (to a large extent) the compactness of vector notation.

In scalar form, Maxwell's equations can be written as

$$\frac{\partial \mathcal{E}_\gamma}{\partial x_\beta} - \frac{\partial \mathcal{E}_\beta}{\partial x_\gamma} = -\frac{1}{c}\frac{\partial \mathcal{B}_\alpha}{\partial t}, \tag{1}$$

$$\frac{\partial \mathcal{H}_\gamma}{\partial x_\beta} - \frac{\partial \mathcal{H}_\beta}{\partial x_\gamma} = \frac{4\pi}{c} i_\alpha + \frac{1}{c}\frac{\partial \mathcal{D}_\alpha}{\partial t}. \tag{2}$$

As indicated above, each of the relationships (1) - (2) represents three scalar equations (which can be obtained from one another by a cyclic permutation of the subscripts).

To Maxwell's equations, we add the law of conservation of electric charge (see problem 1):

$$\frac{\partial \rho}{\partial t} + \sum_{\alpha=1}^{3} \frac{\partial i_\alpha}{\partial x_\alpha} = 0 , \tag{3}$$

where ρ is the density of free electric charges. Remembering this law and applying the identity

$$\sum_{\circlearrowleft} \frac{\partial}{\partial x_\alpha}\left(\frac{\partial u_\gamma}{\partial x_\beta} - \frac{\partial u_\beta}{\partial x_\gamma}\right) \equiv 0 , \tag{4}$$

where u_α, u_β, and u_γ are arbitrary twice differentiable functions and the symbol $\circlearrowleft$ indicates that the summed terms are obtained from one another by a cyclic permutation of the subscripts α, β, and γ, we easily obtain

$$\frac{\partial}{\partial t} \sum_{\alpha=1}^{3} \frac{\partial \mathcal{B}_\alpha}{\partial x_\alpha} = 0 , \tag{5}$$

$$\frac{\partial}{\partial t}\left(\rho - \frac{1}{4\pi} \sum_{\alpha=1}^{3} \frac{\partial \mathcal{D}_\alpha}{\partial x_\alpha}\right) = 0 . \tag{6}$$

If we integrate these equations with respect to time, we obtain on the right side of each an arbitrary function of the coordinates. It is easy to show on the basis of physical considerations that this function can, without loss of generality, be taken equal to zero, so that we may write

$$\sum_{\alpha=1}^{3} \frac{\partial \mathcal{B}_\alpha}{\partial x_\alpha} = 0 , \qquad \sum_{\alpha=1}^{3} \frac{\partial \mathcal{D}_\alpha}{\partial x_\alpha} = 4\pi\rho .$$

The six Maxwell equations (1) - (2) relate fifteen quantities, namely, the components of the vectors $\mathcal{E}$, $\mathcal{H}$, $\mathcal{D}$, $\mathcal{B}$, and i. Consequently, to describe the field, we need to add nine more equations to those of Maxwell. These equations characterize the medium in which the electromagnetic phenomena take place. We shall confine ourselves to *homogeneous isotropic* media. In this case, the equations that supplement the Maxwell equations, thus forming a complete system, are of the form

$$\mathcal{D}_\alpha = \epsilon \mathcal{E}_\alpha , \qquad \mathcal{B}_\alpha = \mu \mathcal{H}_\alpha , \tag{7}$$

$$i_\alpha = \frac{\sigma}{4\pi} \mathcal{E}_\alpha , \tag{8}$$

where ϵ and μ are the dielectric and magnetic permeabilities of the medium and $\sigma/4\pi$ is its conductivity (the factor $1/4\pi$ in (8) is introduced for convenience in later calculations). We shall assume that the quantities ϵ, μ, and σ are independent of the values of $\mathcal{E}$ and $\mathcal{H}$. This assumption is justified in practice for a wide range of values of $\mathcal{E}$ and $\mathcal{H}$ for the great majority of media. Eq. (8) is Ohm's law.

In a number of problems, we shall assume that we can control the current in some finite region of space (in some device, antenna, etc.) and we shall concern ourselves with the field that results. We shall call this cur-

rent the external current and we shall denote its density by the symbol $i^{(e)}$. Ohm's law then takes the form

$$i_\alpha = i_\alpha^{(e)} + \frac{\sigma}{4\pi}\mathcal{E}_\alpha , \tag{9}$$

where the term $(\sigma/4\pi)\mathcal{E}_\alpha$ represents that part of a component of total current density that is induced by the field being considered and $i_\alpha^{(e)}$ represents that part whose value is regulated by the device.

We shall confine ourselves to a study of steady-state electromagnetic fields. As we shall see, an electromagnetic field is of a *wave* nature. Therefore, we shall apply the same method as in making the transition from the wave equation to the Helmholtz equation describing steady-state fields (Chapter XXV). Specifically, we shall seek solutions of Maxwell's equations that are purely periodic with respect to time by setting

$$\mathcal{E} = \mathrm{Re}\, \boldsymbol{E}\, e^{-i\omega t}, \qquad \mathcal{H} = \mathrm{Re}\, \boldsymbol{H}\, e^{-i\omega t} \tag{10}$$

and

$$i^{(e)} = \mathrm{Re}\, \boldsymbol{j}^{(e)}\, e^{-i\omega t}, \tag{11}$$

where $\boldsymbol{E}$, $\boldsymbol{H}$, and $\boldsymbol{j}^{(e)}$ are vectors with complex components and ω is the angular frequency of the vibrations.

The vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ may be called the complex electric and magnetic vectors and the vector $\boldsymbol{j}^{(e)}$ is the complex current density. In analogy with what we did in the study of a sound field, we shall omit the word "complex". This should lead to no confusion since we shall not be using the vectors $\mathcal{E}$ and $\mathcal{H}$.

When we substitute expressions (7), (9), (10), and (11) into the Maxwell equations (1) - (2), we obtain

$$\frac{\partial E_\gamma}{\partial x_\beta} - \frac{\partial E_\beta}{\partial x_\gamma} = \frac{i\omega\mu}{c} H_\alpha , \tag{12}$$

$$\frac{\partial H_\gamma}{\partial x_\beta} - \frac{\partial H_\beta}{\partial x_\gamma} = \frac{\sigma - i\omega\epsilon}{c} E_\alpha + \frac{4\pi}{c} j_\alpha^{(e)} . \tag{13}$$

We shall also call these equations Maxwell's equations. In view of the fact that we have made the substitutions (7) and (9), the number of unknown terms is equal to six, that is, to the number of equations (12) - (13).

Maxwell's equations (12) - (13) can be transformed into another form. From eqs. (13),

$$E_\alpha = \frac{c}{\sigma - i\omega\epsilon}\left(\frac{\partial H_\gamma}{\partial x_\beta} - \frac{\partial H_\beta}{\partial x_\gamma}\right) + \frac{4\pi}{i\omega\epsilon - \sigma} j_\alpha^{(e)} .$$

When we differentiate these equations with respect to x_α and then sum them with respect to α, we obtain

$$\sum_{\alpha=1}^{3} \frac{\partial E_\alpha}{\partial x_\alpha} = \frac{4\pi}{i\omega\epsilon - \sigma} \sum_{\alpha=1}^{3} \frac{\partial j_\alpha^{(e)}}{\partial x_\alpha} . \tag{14a}$$

In an analogous manner, we obtain

$$\sum_{\alpha=1}^{3} \frac{\partial H_\alpha}{\partial x_\alpha} = 0 . \tag{14b}$$

On the other hand, when we substitute the expression for E_α in the relations (12), we obtain

$$\frac{i\omega\mu}{c} H_\alpha = \frac{c}{\sigma - i\omega\epsilon}\left(\frac{\partial^2 H_\beta}{\partial x_\alpha \partial x_\beta} - \frac{\partial^2 H_\alpha}{\partial x_\beta{}^2} + \frac{\partial^2 H_\alpha}{\partial x_\gamma{}^2} + \frac{\partial^2 H_\gamma}{\partial x_\alpha \partial x_\gamma}\right) + \frac{4\pi}{i\omega\epsilon - \sigma}\left(\frac{\partial j_\gamma{}^{(e)}}{\partial x_\beta} - \frac{\partial j_\beta{}^{(e)}}{\partial x_\gamma}\right) .$$

When we differentiate (14b) with respect to x_α, we obtain

$$\frac{\partial^2 H_\beta}{\partial x_\alpha \partial x_\beta} + \frac{\partial^2 H_\gamma}{\partial x_\alpha \partial x_\gamma} = -\frac{\partial^2 H_\alpha}{\partial x_\alpha{}^2} .$$

Substituting this relationship into the preceding, we obtain the inhomogeneous Helmholtz equations for the components H_α:

$$\frac{i\omega\mu}{c} H_\alpha = -\frac{c}{\sigma - i\omega\epsilon}\Delta H_\alpha - \frac{4\pi}{\sigma - i\omega\epsilon}\left(\frac{\partial j_\gamma{}^{(e)}}{\partial x_\beta} - \frac{\partial j_\beta{}^{(e)}}{\partial x_\gamma}\right) .$$

If we define

$$\frac{\omega^2\epsilon\mu + i\omega\mu\sigma}{c^2} \equiv k^2 , \tag{15}$$

we transform these equations to the form

$$\Delta H_\alpha + k^2 H_\alpha = -\frac{4\pi}{c}\left(\frac{\partial j_\gamma{}^{(e)}}{\partial x_\beta} - \frac{\partial j_\beta{}^{(e)}}{\partial x_\gamma}\right) . \tag{16a}$$

Analogously, by using eqs. (13), we obtain

$$\Delta E_\alpha + k^2 E_\alpha = -\frac{4\pi}{c} j_\alpha{}^{(e)} - \frac{4\pi}{\sigma - i\omega\epsilon}\frac{\partial}{\partial x_\alpha}\left(\frac{\partial j_1{}^{(e)}}{\partial x_1} + \frac{\partial j_2{}^{(e)}}{\partial x_2} + \frac{\partial j_3{}^{(e)}}{\partial x_3}\right) . \tag{16b}$$

Eqs. (14a) - (14b) and (16a) - (16b) constitute a system that is equivalent to the Maxwell equations (12) - (13).

When there are no external currents, the Helmholtz' equations (16a)-(16b) become homogeneous. If, in addition, the conductivity of the medium $\sigma = 0$, the number $k^2 = \omega^2\epsilon\mu/c^2$ will be real. In this case, as we know from section 5 of Chapter XXV, there will be solutions to Helmholtz' equation in the form of travelling waves. The quantity $c_1 = \omega/k$ represents their phase velocity. Consequently, the phase velocity of travelling electromagnetic waves is

$$c_1 = c/\sqrt{\epsilon\mu} .$$

In a vacuum, $\epsilon = \mu = 1$; that is, the quantity c represents the phase velocity of electromagnetic waves in a vacuum.

Problems

1. Starting with the law of conservation of charge in the form

$$\frac{\partial}{\partial t}\iiint\limits_V \rho \, \mathrm{d}V = -\iint\limits_{\mathcal{F}V} i_n \, \mathrm{d}S ,$$

where i_n is the component of the current density vector in the direction of the outward normal to the boundary of an arbitrary region V, derive eq. (3).
Method: Use the Ostrogradskii-Gauss formula.

2. Show that in cylindrical coordinates (r, φ, z) Maxwell's equations (12) - (13) are of the form

$$\frac{1}{r}\frac{\partial E_z}{\partial \varphi} - \frac{\partial E_\varphi}{\partial z} = \frac{\mathrm{i}\omega\mu}{c} H_r , \qquad \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = \frac{\mathrm{i}\omega\epsilon}{c} H_\varphi ,$$

$$\frac{1}{r}\frac{\partial}{\partial r} rE_\varphi - \frac{1}{r}\frac{\partial E_r}{\partial \varphi} = \frac{\mathrm{i}\omega\epsilon}{c} H_z , \qquad \frac{1}{r}\frac{\partial H_z}{\partial \varphi} - \frac{\partial E_\varphi}{\partial z} = \frac{\sigma - \mathrm{i}\omega\epsilon}{c} E_r + \frac{4\pi}{c} j_r^{(\mathrm{e})} ,$$

$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = \frac{\sigma - \mathrm{i}\omega\epsilon}{c} E_\varphi + \frac{4\pi}{c} j_\varphi^{(\mathrm{e})} ,$$

$$\frac{1}{r}\frac{\partial}{\partial r} rH_\varphi - \frac{1}{r}\frac{\partial H_r}{\partial \varphi} = \frac{\sigma - \mathrm{i}\omega\epsilon}{c} E_z + \frac{4\pi}{c} j_z^{(\mathrm{e})} .$$

3. Show that in spherical coordinates (r, θ, φ) Maxwell's equations are of the form

$$\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta} E_\varphi \sin\theta - \frac{\partial E_\theta}{\partial\varphi}\right) = \frac{\mathrm{i}\omega\mu}{c} H_r ,$$

$$\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\varphi} - \frac{\partial}{\partial r} rE_\varphi\right) = \frac{\mathrm{i}\omega\mu}{c} H_\theta , \qquad \frac{1}{r}\left(\frac{\partial}{\partial r} rE_\theta - \frac{\partial E_r}{\partial\theta}\right) = \frac{\mathrm{i}\omega\mu}{c} H_\varphi ,$$

$$\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta} H_\varphi \sin\theta - \frac{\partial H_\theta}{\partial\varphi}\right) = \frac{\sigma - \mathrm{i}\omega\epsilon}{c} E_r + \frac{4\pi}{c} j_r^{(\mathrm{e})} ,$$

$$\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\varphi} - \frac{\partial}{\partial r} rH_\varphi\right) = \frac{\sigma - \mathrm{i}\omega\epsilon}{c} E_\theta + \frac{4\pi}{c} j_\theta^{(\mathrm{e})} ,$$

$$\frac{1}{r}\left(\frac{\partial}{\partial r} rH_\theta - \frac{\partial H_r}{\partial\theta}\right) = \frac{\sigma - \mathrm{i}\omega\epsilon}{c} E_\varphi + \frac{4\pi}{c} j_\varphi^{(\mathrm{e})} .$$

2. *Electromagnetic field potentials*

The electromagnetic field equations can be reduced to a form in which the number of equations determining the field is less than the number of Maxwell equations. One such method of reduction is provided by the introduction of field potentials.

We introduce the vector $\boldsymbol{A}$ satisfying the conditions

$$H_\alpha = \frac{1}{\mu}\left(\frac{\partial A_\gamma}{\partial x_\beta} - \frac{\partial A_\beta}{\partial x_\gamma}\right). \tag{17}$$

We shall call this vector the *vector potential*. Eq. (14b) is now satisfied identically. On the other hand, eq. (14b), as is known, implies the existence of the vector $\boldsymbol{A}$.

Eq. (17) does not define the vector potential uniquely. In addition to the vector $\boldsymbol{A}$, the vector with component

$$A_\alpha + \partial\psi/\partial x_\alpha \quad ,$$

where $\psi = \psi(x_1, x_2, x_3)$ is an arbitrary function of the coordinates that is sufficiently many times differentiable, also satisfies eq. (17).

Substituting eq. (17) into eqs. (12), we obtain

$$\frac{\partial E_\gamma}{\partial x_\beta} - \frac{\partial E_\beta}{\partial x_\gamma} = \frac{\mathrm{i}\omega}{c}\left(\frac{\partial A_\gamma}{\partial x_\beta} - \frac{\partial A_\beta}{\partial x_\gamma}\right)$$

or

$$\frac{\partial}{\partial x_\beta}\left(\frac{\mathrm{i}\omega}{c} A_\gamma - E_\gamma\right) = \frac{\partial}{\partial x_\gamma}\left(\frac{\mathrm{i}\omega}{c} A_\beta - E_\beta\right).$$

From the latter equation, it follows * that the quantities $[(\mathrm{i}\omega/c)A_\alpha - E_\alpha]$ are the partial derivatives with respect to x of some function φ, that is, that

$$E_\alpha = \frac{\mathrm{i}\omega}{c} A_\alpha - \frac{\partial\varphi}{\partial x_\alpha}. \tag{18}$$

The function φ is called the scalar potential. Since the components A_α are defined up to the derivative $\partial\psi/\partial x_\alpha$ of an arbitrary function ψ, the scalar potential φ may be arbitrary. However, the vector potential is then uniquely defined by the requirement that eq. (18) be satisfied.

Substituting (17) into (13) and introducing, as in the preceding section, the notation

$$k^2 = \frac{\omega^2\epsilon\mu - \mathrm{i}\omega\mu\sigma}{c^2},$$

we see that

$$E_\alpha = -\frac{\mathrm{i}\omega}{ck^2}\left(\Delta A_\alpha - \frac{\partial}{\partial x_\alpha}\sum_{\beta=1}^{3}\frac{\partial A_\beta}{\partial x_\beta} + \frac{4\pi\mu}{c} j_\alpha^{(\mathrm{e})}\right). \tag{19}$$

Equating the right sides of eqs. (18) and (19), we obtain

$$\Delta A_\alpha + k^2 A_\alpha = -\frac{4\pi\mu}{c} j_\alpha^{(\mathrm{e})} + \frac{\partial}{\partial x_\alpha}\left(\sum_{\beta=1}^{3}\frac{\partial A_\beta}{\partial x_\beta} + \frac{ck^2}{\mathrm{i}\omega}\varphi\right).$$

We now define the scalar potential φ by the relationship

$$\varphi = -\frac{\mathrm{i}\omega}{ck^2}\sum_{\alpha=1}^{3}\frac{\partial A_\alpha}{\partial x_\alpha}. \tag{20}$$

* See, for example, V. I. Smirnov [1], Vol. 2, p. 110.

Then the equations for determining the components of the vector potential are of the form

$$\Delta A_\alpha + k^2 A_\alpha = -\frac{4\pi\mu}{c} j_\alpha{}^{(e)} . \tag{21}$$

If the three components A_α of the vector potential are known, the magnetic vector can be found by using formulae (17). In determining the components of the electric vector from eqs. (19) and (21), we obtain the formulae

$$E_\alpha = \frac{i\omega}{c} A_\alpha + \frac{i\omega}{ck^2} \frac{\partial}{\partial x_\alpha} \sum_{\beta=1}^{3} \frac{\partial A_\beta}{\partial x_\beta} . \tag{22}$$

Suppose that $\boldsymbol{E}$ and $\boldsymbol{H}$ are vectors satisfying Maxwell's equations and that there are no external currents. Then, the vectors

$$\boldsymbol{E}' = -\frac{\mu^2\omega^2}{c^2 k^2} \boldsymbol{H} \quad \text{and} \quad \boldsymbol{H}' = \boldsymbol{E} \tag{23}$$

also satisfy Maxwell's equations. This makes it possible, in the absence of external currents, to introduce two potentials, the so-called Hertz vectors. These potentials are in a mutually symmetric relationship, analogous to that given by formulae (23). The vector with components

$$\Pi_\alpha \equiv \frac{i\omega}{ck^2} A_\alpha \tag{24}$$

is one of the Hertz vectors. Just as in the case of the vector potential when there are no external currents, the components of this Hertz vector satisfy Helmholtz' equations

$$\Delta \Pi_\alpha + k^2 \Pi_\alpha = 0 . \tag{25}$$

On the basis of eqs. (17) and (22), the field vectors can be expressed in terms of this Hertz vector by means of the formulae

$$E_\alpha = k^2 \Pi_\alpha + \frac{\partial}{\partial x_\alpha} \sum_{\beta=1}^{3} \frac{\partial \Pi_\beta}{\partial x_\beta} , \tag{26}$$

$$H_\alpha = \frac{ck^2}{i\omega\mu} \left(\frac{\partial \Pi_\gamma}{\partial x_\beta} - \frac{\partial \Pi_\beta}{\partial x_\gamma} \right) \tag{27}$$

But, on the basis of eqs. (23), showing the mutual relationships, the field vectors determined by the formulae

$$E_\alpha = \frac{i\omega\mu}{c} \left(\frac{\partial \Pi_\gamma}{\partial x_\beta} - \frac{\partial \Pi_\beta}{\partial x_\gamma} \right) , \tag{28}$$

$$H_\alpha = k^2 \Pi_\alpha + \frac{\partial}{\partial x_\alpha} \sum_{\beta=1}^{3} \frac{\partial \Pi_\beta}{\partial x_\beta} \tag{29}$$

also satisfy Maxwell's equations when there are no external currents.

Suppose now that $\boldsymbol{\Pi}$ and $\boldsymbol{\Pi}^*$ are two vectors whose components Π_α and Π_α^* satisfy Helmholtz' equations (25). For the first of these, let us express the field vectors by means of formulae (26) - (27) and for the second let us express them by means of formulae (28) - (29). If we add the resulting expressions, we obtain

$$E_\alpha = k^2\Pi_\alpha + \frac{\partial}{\partial x_\alpha}\sum_{\beta=1}^{3}\frac{\partial \Pi_\beta}{\partial x_\beta} + \frac{i\omega\mu}{c}\left(\frac{\partial \Pi_\gamma^*}{\partial x_\beta} - \frac{\partial \Pi_\beta^*}{\partial x_\gamma}\right), \tag{30}$$

$$H_\alpha = \frac{ck^2}{i\omega\mu}\left(\frac{\partial \Pi_\gamma}{\partial x_\beta} - \frac{\partial \Pi_\beta}{\partial x_\gamma}\right) + k^2\Pi_\alpha + \frac{\partial}{\partial x_\alpha}\sum_{\beta=1}^{3}\frac{\partial \Pi_\beta^*}{\partial x_\beta}, \tag{31}$$

which, in the absence of external currents, are also solutions of Maxwell's equations. The vector $\boldsymbol{\Pi}^*$ is the other Hertz vector.

The vectors used for forming the field vectors from formulae (26) - (27) and from formulae (28) - (29) are called, respectively, the electric and magnetic Hertz vectors. Depending on the boundary conditions of a problem, it may be expedient to use either one or the other of the Hertz vectors or possibly both of them.

Problems

1. Show that when there are no external currents, the scalar potential φ satisfies Helmholtz' equation $\Delta\varphi + k^2\varphi = 0$.

2. Find the electromagnetic field produced in an unbounded space for a given system of steady-state currents $\boldsymbol{j}$.
Method: We want to determine the vector potential of the field. Particular solutions of Helmholtz' equations that satisfy the radiation conditions at infinity are oscillational potentials (section 6, Chapter XXV). Consequently, one of the solutions of our problem may be represented in the form of an oscillational potential. This solution is unique because if there were two such, their difference would satisfy the homogeneous Helmholtz equation. But the solutions of the homogeneous Helmholtz equation describe free fields that do not contain sources. Therefore, there can be only one solution describing the field of forced oscillations. The components of the vector potential describing this field are determined by the formulae

$$A_\alpha = \frac{\mu}{c}\iiint\limits_V j_\alpha^{(e)}\frac{e^{ikr}}{r}\,dV.$$

3. *Boundary conditions*

As we have stated, the solutions of the system of Maxwell equations are of a wave nature under certain conditions. Consequently, we may expect that each of the components of the vectors of the field caused by

sources located in a bounded portion of space must satisfy, at infinity, a radiation condition of the form (62) of Chapter XXV. It was shown in section 6 that satisfaction of the radiation condition under properly chosen boundary conditions ensures uniqueness of the solution of the system of Maxwell equations. Thus, there is no need for any other conditions to be satisfied at infinity.

Let us consider the boundary conditions on surfaces separating media with different properties.

We shall consider a discontinuity in the properties at the boundary between two media as being the limiting case of a continuous transition under which the properties of one medium change gradually into the properties of the other in some small region adjacent to the dividing surface. The surface of the division itself we shall consider piecewise-smooth. Under these assumptions, the Ostrogradskii-Gauss and Stokes formulae can be used for finding the boundary conditions.

Suppose that S is the surface separating two media, which we shall refer as the medium e and the medium i. Suppose that ξ is a point on S and that $\boldsymbol{n}$ is the outward normal to S at the point ξ. We shall take as positive the direction of the normal $\boldsymbol{n}$ from the medium i to the medium e.

Let us construct a cylindrical surface C with axis coinciding with the normal $\boldsymbol{n}$ and radius ah, where a is a pure number and h is an arbitrarily chosen unit of length. Suppose also that S' and S'' are the surfaces obtained by displacing the surface S a distance a^2h in the positive and negative directions, respectively, of the normal $\boldsymbol{n}$. The surfaces C, S', and S'' define a closed region V_ϵ of the point ξ with boundary formed by the elements C_ϵ, S'_ϵ, and S''_ϵ of the surfaces constructed (fig. 67).

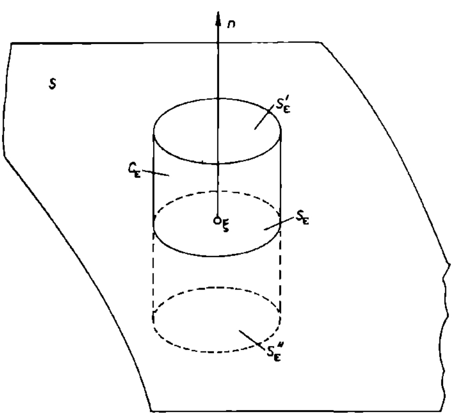

Fig. 67.

If we apply the Ostrogradskii-Gauss formula to the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ in the neighbourhood V_ϵ, we obtain, on the basis of eqs. (12) and (13),

$$\iint\limits_{C_\epsilon+S'_\epsilon+S''_\epsilon} [(i\omega\epsilon - \sigma)E_n - 4\pi j_n^{(e)}]\, dS = 0 \ , \qquad \iint\limits_{C_\epsilon+S'_\epsilon+S''_\epsilon} \mu H_n\, dS = 0 \ .$$

The symbols E_n, H_n, and $j_n^{(e)}$ denote the projections of the vectors $\boldsymbol{E}$, $\boldsymbol{H}$, and $\boldsymbol{j}^{(e)}$ on the normal to the element dS of the surfaces C_ϵ, S'_ϵ, and S''_ϵ. As we let a approach zero, we obtain for the areas c_ϵ, s'_ϵ, and s''_ϵ of the regions C_ϵ, S'_ϵ, and S''_ϵ the following:

$$c_\epsilon \approx (2\pi ah)2a^2h^2 = 4\pi a^3h^3 \ , \qquad s'_\epsilon = s''_\epsilon = s_\epsilon \geqslant \pi a^2h^2 \ .$$

If we apply the mean-value theorem to the integral of

$$[(i\omega\epsilon - \sigma)E_n - 4\pi j_n^{(e)}]$$

we obtain

$$\iint\limits_{C_\epsilon+S'_\epsilon+S''_\epsilon} [(i\omega\epsilon - \sigma)E_n - 4\pi j_n^{(e)}]\, dS = c_\epsilon[(i\omega\epsilon - \sigma)E_n - 4\pi j_n^{(e)}]_{\mathrm{av}C_\epsilon}$$

$$+ s'_\epsilon[(i\omega\epsilon - \sigma)E_n - 4\pi j_n^{(e)}]_{\mathrm{av}S'_\epsilon} + s''_\epsilon[(i\omega\epsilon - \sigma)E_n - 4\pi j_n^{(e)}]_{\mathrm{av}S''_\epsilon} \ , \tag{32}$$

where the subscripts $\mathrm{av}C_\epsilon$, $\mathrm{av}S'_\epsilon$, $\mathrm{av}S''_\epsilon$ indicate that the expression in the square brackets is to be evaluated between its maximum and minimum values on C_ϵ, S'_ϵ, and S''_ϵ. If we let a approach zero and use the approximations that we have obtained,

$$\lim_{a\to 0} \frac{1}{s_\epsilon} \iint\limits_{C_\epsilon+S'_\epsilon+S''_\epsilon} [(i\omega\epsilon - \sigma)E_n - 4\pi j_n^{(e)}]\, dS$$

$$= [(i\omega\epsilon_e - \sigma_e)E_{ne} - 4\pi j_{ne}^{(e)}] - [(i\omega\epsilon_i - \sigma_i)E_{ni} - 4\pi j_{ni}^{(e)}] \ , \tag{33}$$

where the symbols e and i denote the limiting values of the quantities as the point ξ is approached, respectively, from the side of the medium e and the medium i. The minus sign is present in front of $[(i\omega\epsilon_i - \sigma_i)E_{ni} - 4\pi j_{ni}^{(e)}]$, because the outward normal to the section S''_ϵ is directed oppositely to the outward normal to S. An analogous relationship is obtained for the normal components of the vector $\boldsymbol{H}$ with one difference, the quantity $(i\omega\epsilon - \sigma)$ is replaced by μ and there is no term depending on the current density.

Thus, we obtain the following boundary conditions for the normal component of the field vector:

$$\begin{gathered} (i\omega\epsilon_e - \sigma_e)E_{ne} - (i\omega\epsilon_i - \sigma_i)E_{ni} = 4\pi(j_{ne}^{(e)} - j_{ni}^{(e)}) \ , \\ \mu_e H_{ne} - \mu_i H_{ni} = 0 \ . \end{gathered} \tag{34}$$

Let us consider the first of these. On its right side is the difference between the normal components of the current density; this is equal to the difference between the flow of charges into the surface separating the media and the flow out from that surface. Since each of the quantities being considered represents the complex amplitude of oscillations that take place with frequency ω, a difference between inflow and outflow means that there

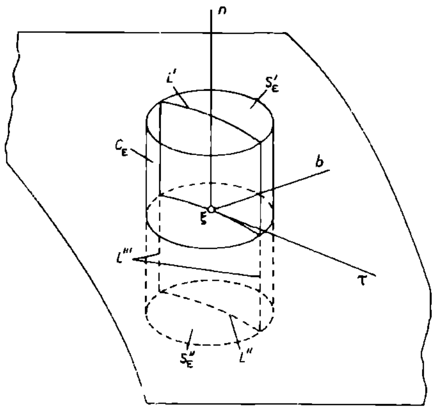

Fig. 68.

is a periodic oscillation of charge on the boundary surface; that is, a simple vibrating electric layer is located on this surface.

To find the boundary conditions for the tangential component of the electric and magnetic vectors, let us consider the curve L formed by the intersections of an arbitrary plane Σ passing through the normal n at the point ξ with the regions C_ϵ, S_ϵ', and S_ϵ'' (fig. 68). Suppose that Σ_L is that portion of the plane bounded by the contour L. For simplicity in later calculations, we introduce a rectangular Cartesian coordinate system with origin at the point ξ and with axes n, τ, and b, where n is the outward normal referred to, τ is the axis directed tangentially to the dividing surface in the plane Σ, and the axis b is directed in such a way that the system n, τ, b will be a right-handed system.

If we make the substitutions $\alpha = b$, $\beta = n$, and $\gamma = \tau$ in Maxwell's equations (12) and (13), we obtain

$$\frac{\partial E_\tau}{\partial n} - \frac{\partial E_n}{\partial \tau} = \frac{\mathrm{i}\omega\mu}{c} H_b , \tag{35}$$

$$\frac{\partial H_\tau}{\partial n} - \frac{\partial H_n}{\partial \tau} = \frac{\sigma - \mathrm{i}\omega\epsilon}{c} E_b + \frac{4\pi}{c} j_b^{(e)} . \tag{36}$$

If we integrate the first of these equations over the surface Σ_L, we obtain

$$\iint_{\Sigma_L} \left(\frac{\partial H_\tau}{\partial n} - \frac{\partial H_n}{\partial \tau}\right) \mathrm{d}\Sigma = \iint_{\Sigma_L} \left(\frac{\sigma - \mathrm{i}\omega\epsilon}{c} E_b + \frac{4\pi}{c} j_b^{(e)}\right) \mathrm{d}\Sigma . \tag{37}$$

Since the τ-axis is perpendicular to the surface Σ_L, we may use Stokes' formula for a plane region * to transform the left side of eq. (37). This gives

* See V. I. Smirnov 1), Vol. 2, p. 79.

$$\iint_{\Sigma_L} \left(\frac{\partial H_\tau}{\partial n} - \frac{\partial H_n}{\partial \tau}\right) d\Sigma = \int_L [H_n \cos(l,n) + H_\tau \cos(l,\tau)]\, dL \,. \qquad (38)$$

Let us partition the curve L into segments L', L'', and L''' formed by the intersection of the plane Σ with the regions S'_ϵ, S''_ϵ and the cylindrical surface C_ϵ (fig. 68). We partition the surface Σ_L into two portions $\Sigma_{L\mathrm{e}}$ and $\Sigma_{L\mathrm{i}}$, one of which is in the medium e and the other in the medium i. Since S'_ϵ and S''_ϵ are at a distance a^2h from the surface separating the two media, if we let the radius ah of the cylinder in question approach zero, we obtain

$$\sigma_{L\mathrm{e}} = \sigma_{L\mathrm{i}} = 2a^3h^2 \,, \qquad \overline{L'} = \overline{L''} \approx 2ah \,, \qquad \overline{L'''} = 4a^2h \,,$$

where $\sigma_{L\mathrm{e}}$ and $\sigma_{L\mathrm{i}}$ are the areas of the regions $\Sigma_{L\mathrm{e}}$ and $\Sigma_{L\mathrm{i}}$, and where $\overline{L'}$, $\overline{L''}$, and $\overline{L'''}$ are the length of the segments L', L'', and L'''. On L' and L'', $\cos(l,n) \approx 0$. Furthermore, $\cos(l,b)$ is identically equal to zero on L''. If we apply the mean-value theorem to eqs. (37) and (38) and use the results just obtained, we find

$$a^2h\left[\frac{\sigma - \mathrm{i}\omega\epsilon}{c} E_b + \frac{4\pi}{c} j_b^{(\mathrm{e})}\right]_{\mathrm{av}\Sigma_{L\mathrm{i}}} + a^2h\left[\frac{\sigma - \mathrm{i}\omega\epsilon}{c} E_b + \frac{4\pi}{c} j_b^{(\mathrm{e})}\right]_{\mathrm{av}\Sigma_{L\mathrm{e}}}$$

$$= [H_\tau \cos(l,\tau)]_{\mathrm{av}L'} + [H_\tau \cos(l,\tau)]_{\mathrm{av}L''} + 2ah[H_n \cos(l,n)]_{\mathrm{av}L'''} \,, \qquad (39)$$

where, as above, the notations $\mathrm{av}\Sigma_{L\mathrm{i}}$, $\mathrm{av}\Sigma_{L\mathrm{e}}$, $\mathrm{av}L'$, $\mathrm{av}L''$, and $\mathrm{av}L'''$ indicate that the expression within the square brackets is to be evaluated between the maximum and minimum values on the corresponding segment.

If the conductivity σ of both media is bounded, the volume density of current $j_b^{(\mathrm{e})}$ will also be bounded. Therefore, if we take the limit in eq. (39), we obtain

$$\lim_{a\to 0} \{[H_\tau \cos(l,\tau)]_{\mathrm{av}L'} + [H_\tau \cos(l,\tau)]_{\mathrm{av}L''}\} = 0 \,. \qquad (40)$$

Since

$$\lim_{a\to 0} \cos(l,\tau) = \begin{cases} -1 & \text{on } L' \,, \\ +1 & \text{on } L'' \,, \end{cases} \qquad \lim_{a\to 0} H_\tau = \begin{cases} H_{\tau\mathrm{e}} & \text{on } L' \,, \\ H_{\tau\mathrm{i}} & \text{on } L'' \,, \end{cases}$$

we finally obtain

$$H_{\tau\mathrm{e}} = H_{\tau\mathrm{i}} \,.$$

In a completely analogous way, we can find the boundary conditions for the tangential components of the electric vector:

$$E_{\tau\mathrm{e}} = E_{\tau\mathrm{i}} \,.$$

Thus, we have the four boundary conditions:

$$\left.\begin{aligned} (\mathrm{i}\omega\epsilon_\mathrm{e} - \sigma_\mathrm{e})E_{n\mathrm{e}} - (\mathrm{i}\omega\epsilon_\mathrm{i} - \sigma_\mathrm{i})E_{n\mathrm{i}} &= 4\pi(j_{n\mathrm{e}}^{(\mathrm{e})} - j_{n\mathrm{i}}^{(\mathrm{e})}) \,, \\ \mu_\mathrm{e}H_{n\mathrm{e}} &= \mu_\mathrm{i}H_{n\mathrm{i}} \,, \end{aligned}\right\} \qquad (41)$$

$$E_{\tau\mathrm{e}} = E_{\tau\mathrm{i}} \,, \qquad H_{\tau\mathrm{e}} = H_{\tau\mathrm{i}} \,. \qquad (42)$$

However, it is easy to see that satisfaction of the last two conditions for the tangential component automatically implies satisfaction of the first two conditions for the normal component.

To show this, let us use the local system of coordinates n , τ and b that was introduced above (fig. 68), where n is the normal to the boundary separating the media, and the τ- and b-axis are in the plane tangent to the boundary. If we set $\alpha = n$, $\beta = \tau$, and $\gamma = b$ in Maxwell's equations (12) - (13), we obtain

$$\begin{gathered}\frac{\partial E_b}{\partial \tau} - \frac{\partial E_\tau}{\partial b} = \frac{i\omega\mu}{c} H_n , \\ \frac{\partial H_b}{\partial \tau} - \frac{\partial H_\tau}{\partial b} = \frac{\sigma - i\omega\epsilon}{c} E_n + \frac{4\pi}{c} j_n^{(e)} .\end{gathered} \tag{43}$$

Since the tangential components E_τ, E_b, H_τ, and H_b are continuous, the right sides of these equations are also continuous at the boundary, from which conditions (41) follow.

Since many media have quite high conductivities, the concept of a conductor of infinite conductivity is very useful. Such a conductor we shall call an ideal conductor.

By using Maxwell's equations, it is easy to show that the total electromagnetic field in a conductor decreases exponentially as one goes deeper inside the conductor. Here, the index of damping is proportional to the conductivity (see, for example, the problem at the end of this section). Therefore, the field must vanish at the boundary of an ideal conductor. This is often expressed by saying that an electromagnetic field does not penetrate into an ideal conductor.

Let us use this fact to find the conditions at the boundary of an ideal conductor. Consider the integral relationship (37). The evaluations of the terms in this equation, on the basis of which we obtained the boundary condition for the tangential components of the magnetic vector, depended on the boundedness requirement for the field vectors and the current density. For an ideal conductor, the last requirement is not met and we must modify our estimate for the terms depending on the current density.

According to Ohm's law, the component of the total current density along the b-axis is equal to

$$j_b = \frac{\sigma}{4\pi} E_b + j_b^{(e)} ,$$

so that

$$\iint_{\Sigma_{Le}} \left(\frac{\sigma}{c} E_b + \frac{4\pi}{c} j_b^{(e)}\right) dS = \frac{4\pi}{c} \iint_{\Sigma_{Le}} j_b \, dS .$$

The integral $\iint_{\Sigma_{Le}} j_b \, dS$ represents the total current flowing through that element Σ_{Le} of the area Σ_L situated in the medium e. From the above, we see that the electromagnetic field and the total current that flows through an ideal conductor are concentrated on its surface. Consequently, as $\sigma \to \infty$, the integral that we are considering approaches the line integral

$$\int_{\tilde{L}} j_b' \, \mathrm{d}L \ ,$$

where $\tilde{L}$ is that portion (of the boundary of the area Σ_{Le}) belonging to the surface separating the media (fig. 68), and j_b' is the component of the surface current density $\boldsymbol{j}'$ along the b-axis. As a approaches zero, the integral $\int_{\tilde{L}} j_b' \, \mathrm{d}L$ is of the order of $2ahj_b'$. Therefore, as is easy to see, instead of eq. (40), we have

$$\lim_{a\to 0} \{[H_\tau \cos(l,\tau)]_{\mathrm{av}L'} + [H_\tau \cos(l,\tau)]_{\mathrm{av}L''}\} = \frac{4\pi}{c} j' \ .$$

Since the field vectors are equal to zero in the medium e, recalling eq. (39), we finally obtain

$$H_{\tau\mathrm{i}} = \frac{4\pi}{c} j_b' \ . \tag{44}$$

This equation imposes no restrictions on the solution, since the function on the right side does not appear in Maxwell's equation. On the other hand, when a solution to Maxwell's equation is given, eq. (44) makes it possible to determine the surface current.

The derivation of the boundary conditions for the tangential components of the electric vector begins with eqs. (35), which do not depend on the current density. Consequently, we have $E_{\tau\mathrm{e}} = E_{\tau\mathrm{i}}$ for an ideal conductor, just as for an arbitrary medium. But since the field in an ideal conductor is equal to zero, we have $E_{\tau\mathrm{e}} = 0$ and hence

$$E_{\tau\mathrm{i}} = 0 \ . \tag{45}$$

Thus, on the boundary of an ideal conductor, we have only the one condition requiring the vanishing of the tangential component of the electric vector.

Problem

The half-space $x_1 > 0$ is filled with a medium of conductivity σ. Suppose that the value of the electric vector is equal to a constant vector $\boldsymbol{E}_0$ on the boundary $x_1 = 0$. Show that the field in the medium decreases exponentially with increasing x_1 and that the damping factor is proportional to the conductivity σ.

Method: The desired electric vector satisfies the equation

$$\frac{\mathrm{d}^2\boldsymbol{E}}{\mathrm{d}x_1^2} + k^2\boldsymbol{E} = 0 \ ,$$

where

$$k^2 = \frac{\epsilon\mu}{c^2}\omega^2 \left(1 + \mathrm{i}\frac{\sigma}{\epsilon\omega}\right) .$$

4. *Representation of an electromagnetic field by means of two scalar functions*

It is easy to show by means of the formulae of section 7 of Chapter XVII that Maxwell's equations (12)-(13) in arbitrary orthogonal curvilinear coordinates ζ_1, ζ_2, ζ_3 are of the form

$$\frac{\partial}{\partial\zeta_\alpha} h_\beta E_\beta - \frac{\partial}{\partial\zeta_\beta} h_\alpha E_\alpha = \frac{i\omega\mu}{c} h_\alpha h_\beta H_\gamma \ , \tag{46}$$

$$\frac{\partial}{\partial\zeta_\alpha} h_\beta H_\beta - \frac{\partial}{\partial\zeta_\beta} h_\alpha H_\alpha = \frac{4\pi}{c} h_\alpha h_\beta j_\gamma{}^{(e)} + \frac{\sigma - i\omega\epsilon}{c} h_\alpha h_\beta E_\gamma \ , \tag{47}$$

where h_α, h_β, and h_γ are the Lamé coordinate parameters.

Let us assume that there are no external currents, that is, that $j_\alpha{}^{(e)} = 0$ (for $\alpha = 1, 2, 3$). For brevity, we use the notations

$$\frac{i\omega\mu}{c} = k_H \ , \qquad \frac{\sigma - i\omega\epsilon}{c} = k_E \ . \tag{48}$$

Eqs. (46) - (47) then take the form

$$\frac{\partial}{\partial\zeta_\alpha} h_\beta E_\beta - \frac{\partial}{\partial\zeta_\beta} h_\alpha E_\alpha = k_H h_\alpha h_\beta H_\gamma \ , \tag{49}$$

$$\frac{\partial}{\partial\zeta_\alpha} h_\beta H_\beta - \frac{\partial}{\partial\zeta_\beta} h_\alpha H_\alpha = k_E h_\alpha h_\beta E_\gamma \ . \tag{50}$$

Let us suppose that some problem posed for eqs. (49) - (50) has solutions such that

$$E_l \neq 0 \ , \qquad H_l = 0 \ , \tag{51}$$

or

$$E_l = 0 \ , \qquad H_l \neq 0 \ , \tag{52}$$

where l is an index representing one of the numbers 1, 2, 3. We shall call solutions satisfying eqs. (51) solutions of the electric type and those that satisfy eqs. (52) we shall call solutions of the magnetic type.

For solutions of the electric type, it follows from eqs. (49) that, for $\gamma = l$,

$$\frac{\partial}{\partial\zeta_j} h_k E_k = \frac{\partial}{\partial\zeta_k} h_j E_j \ , \tag{53}$$

where the subscripts j and k and the subscript l form an even permutation j, k, l of the indices 1, 2, 3. In contrast with the Greek indices α, β, and γ, which may denote an *arbitrary* even permutation of the indices 1, 2, 3, the three Roman indices j, k, and l are assumed fixed.

It follows from eq. (53) that the components E_j and E_k of the electric vector can be represented in the form

$$E_j = \frac{1}{h_j}\frac{\partial u^*}{\partial\zeta_j} \ , \qquad E_k = \frac{1}{h_k}\frac{\partial u^*}{\partial\zeta_k} \ , \tag{54}$$

where u^* is some function. If we substitute these expressions into eqs. (50) for $\gamma = j, k$, then, recalling that $H_l = 0$, we obtain

$$\frac{\partial}{\partial \zeta_l} h_k H_k = -k_E \frac{h_k h_l}{h_j} \frac{\partial u^*}{\partial \zeta_j}, \qquad \frac{\partial}{\partial \zeta_l} h_j H_j = k_E \frac{h_l h_j}{h_k} \frac{\partial u^*}{\partial \zeta_k}. \tag{55}$$

Let us suppose that the coefficients h_1, h_2, and h_3 can be represented in the form

$$h_j = \psi_j(\zeta_j, \zeta_l)\, \psi(\zeta_k), \qquad h_k = \psi_k(\zeta_j, \zeta_k)\, \psi(\zeta_l), \qquad h_l = 1. \tag{56}$$

Then, if we set

$$u^* = \partial u / \partial \zeta_l,$$
$$H_k = -\frac{k_E}{h_j} \frac{\partial u}{\partial \zeta_j}, \qquad H_j = \frac{k_E}{h_k} \frac{\partial u}{\partial \zeta_k}, \tag{57}$$

where u is some function, we satisfy eqs. (55) identically. If we substitute the expressions given by eqs. (57) for H_k and H_j into eqs. (50), we obtain, for $\gamma = l$,

$$E_l = -\frac{1}{h_j h_k} \left[\frac{\partial}{\partial \zeta_j} \left(\frac{h_k}{h_j} \frac{\partial u}{\partial \zeta_j} \right) + \frac{\partial}{\partial \zeta_k} \left(\frac{h_j}{h_k} \frac{\partial u}{\partial \zeta_k} \right) \right]. \tag{58}$$

Finally, on the basis of eqs. (54), we obtain

$$E_j = \frac{1}{h_j} \frac{\partial^2 u}{\partial \zeta_j \, \partial \zeta_l}, \qquad E_k = \frac{1}{h_k} \frac{\partial^2 u}{\partial \zeta_k \, \partial \zeta_l}. \tag{59}$$

Thus, all the components of the field vectors for a solution of the electric type can be expressed in terms of some function of u by means of four of the six equations (49) - (50).

If we substitute the expressions found above for the components of the field vectors into those of eqs. (49), for which $\gamma = j, k$, we obtain

$$\frac{\partial}{\partial \zeta_k} \left\{ \frac{1}{h_j h_k} \left[\frac{\partial}{\partial \zeta_j} \left(\frac{h_k}{h_j} \frac{\partial u}{\partial \zeta_j} \right) + \frac{\partial}{\partial \zeta_k} \left(\frac{h_j}{h_k} \frac{\partial u}{\partial \zeta_k} \right) \right] + \frac{\partial^2 u}{\partial \zeta_l^2} \right\} = -k_H k_E \frac{\partial u}{\partial \zeta_k},$$

$$\frac{\partial}{\partial \zeta_j} \left\{ \frac{\partial^2 u}{\partial \zeta_l^2} + \frac{1}{h_j h_k} \left[\frac{\partial}{\partial \zeta_j} \left(\frac{h_k}{h_j} \frac{\partial u}{\partial \zeta_j} \right) + \frac{\partial}{\partial \zeta_k} \left(\frac{h_j}{h_k} \frac{\partial u}{\partial \zeta_k} \right) \right] \right\} = -k_H k_E \frac{\partial u}{\partial \zeta_j}.$$

Both these equations can be obtained by differentiating the equation

$$\frac{1}{h_j h_k} \left[\frac{\partial}{\partial \zeta_j} \left(\frac{h_k}{h_j} \frac{\partial u}{\partial \zeta_j} \right) + \frac{\partial}{\partial \zeta_k} \left(\frac{h_j}{h_k} \frac{\partial u}{\partial \zeta_k} \right) \right] + \frac{\partial^2 u}{\partial \zeta_l^2} + k^2 u = 0, \tag{60}$$

where

$$k^2 \equiv k_H k_E = \frac{\omega^2 \epsilon \mu + i \omega \mu \sigma}{c^2}. \tag{61}$$

Consequently, they will be satisfied if the function u is a solution of eq. (60). We also see that by a suitable choice of the function u, we can write expressions for the field vectors that satisfy all the Maxwell equations.

If we combine expressions (57) - (59) for the components of the field vectors for solutions of the electric type, and if we take eq. (60) into consideration, we may write

$$E_j = \frac{1}{h_j}\frac{\partial^2 u}{\partial\zeta_j\,\partial\zeta_l}\,, \qquad E_k = \frac{1}{h_k}\frac{\partial^2 u}{\partial\zeta_k\,\partial\zeta_l}\,, \qquad E_l = \frac{\partial^2 u}{\partial\zeta_l{}^2} + k^2 u\,, \tag{62}$$

$$H_j = \frac{\sigma - i\omega\epsilon}{c}\,\frac{1}{h_k}\,\frac{\partial u}{\partial\zeta_k}\,, \qquad H_k = -\,\frac{\sigma - i\omega\epsilon}{c}\,\frac{1}{h_j}\,\frac{\partial u}{\partial\zeta_j}\,, \qquad H_l = 0\,. \tag{63}$$

If we were to examine the solutions of the magnetic type in the same way, we would obtain the equations

$$H_j = \frac{1}{h_j}\frac{\partial^2 v}{\partial\zeta_j\,\partial\zeta_l}\,, \qquad H_k = \frac{1}{h_k}\frac{\partial^2 v}{\partial\zeta_k\,\partial\zeta_l}\,, \qquad H_l = \frac{\partial^2 v}{\partial\zeta_l{}^2} + k^2 v\,; \tag{64}$$

$$E_j = \frac{i\omega\mu}{c}\,\frac{1}{h_k}\,\frac{\partial v}{\partial\zeta_k}\,, \qquad E_k = -\,\frac{i\omega\mu}{c}\,\frac{1}{h_j}\,\frac{\partial v}{\partial\zeta_j}\,, \qquad E_l = 0\,, \tag{65}$$

where v is also a function satisfying eq. (60). Thus, if the solution to the problem in question for Maxwell's equation can be represented as a sum of solutions of the electric and magnetic types, all we need to do to find each of these is to find one scalar function that is a solution of eq. (60) with boundary conditions ensuring that the boundary conditions for the field vectors are satisfied. Once such a function is found, the field vectors can be determined from formulae (62) - (63) or (64) - (65).

Satisfaction of eqs. (56) is a necessary condition for the existence of solutions of the electric and magnetic types. It is possible to show that these conditions, in orthogonal coordinate systems, are satisfied for any coordinate ζ_l for which either parallel planes or concentric spheres serve as coordinate surfaces. However, these types of coordinates exhaust all the cases in which eqs. (56) are satisfied.

Examples of coordinates that satisfy eqs. (56) are orthogonal Cartesian coordinates (where any one of the coordinates can be chosen for ζ_l), cylindrical coordinates r, φ, z for $\zeta_l = z$, and spherical coordinates r, θ, φ with $\zeta_l = r$.

For example, let us consider spherical coordinates. Here,

$$h_r = 1\,, \qquad h_\theta = r\,, \qquad h_\varphi = r\sin\theta$$

and eqs. (56) are satisfied with $\zeta_l = r$. Let us set

$$r = \zeta_l\,, \qquad \theta = \zeta_j\,, \qquad \varphi = \zeta_k\,.$$

Here, eq. (60) takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2 u}{\partial\varphi^2} + k^2 u = 0\,.$$

By means of the substitution

$$u \equiv r\bar{u}\,,$$

it can be reduced to the Helmholtz equation

$$\frac{\partial^2\bar{u}}{\partial r^2} + \frac{2}{r}\,\frac{\partial\bar{u}}{\partial r} + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\bar{u}}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2\bar{u}}{\partial\varphi^2} + k^2\bar{u} = 0\,.$$

Knowing solutions $\bar{u}$ and $\bar{v}$ of this last equation (that meet the conditions en-

suring satisfaction of the boundary conditions for the solutions of the electric and magnetic types, respectively), we can find all the fields by means of formulae that are obvious consequences of formulae (62) and (65). The functions $\bar{u}$ and $\bar{v}$ are called the Debye potentials.

Problems

1. Assume that the Lamé parameters h_j, h_k, and h_l of the coordinate system ζ_j, ζ_k, and ζ_l do not depend on ζ_l. Examine an electromagnetic field in a vacuum, where the field is also independent of the coordinate ζ_l. Show that the field vectors can be computed from the formulae

$$E_j = -\frac{c}{i\omega}\frac{1}{h_k h_l}\frac{\partial u^*}{\partial \zeta_k}, \qquad E_k = \frac{c}{i\omega}\frac{1}{h_j h_l}\frac{\partial u^*}{\partial \zeta_j}, \qquad E_l = \frac{1}{h_l}v^*,$$

$$H_j = \frac{c}{i\omega}\frac{1}{h_k h_l}\frac{\partial v^*}{\partial \zeta_k}, \qquad H_k = -\frac{c}{i\omega}\frac{1}{h_j h_l}\frac{\partial v^*}{\partial \zeta_j}, \qquad H_l = \frac{1}{h_l}u^*,$$

where u^* and v^* are functions satisfying the equation

$$\frac{\partial}{\partial \zeta_j}\left(\frac{h_k}{h_j h_l}\frac{\partial u}{\partial \zeta_j}\right) + \frac{\partial}{\partial \zeta_k}\left(\frac{h_j}{h_k h_l}\frac{\partial u}{\partial \zeta_k}\right) + \frac{\omega^2}{c^2}\frac{h_j h_k}{h_l}u = 0 .$$

Remark: The functions u^* and v^* are called Abraham's potentials. The case considered in this problem is the only one besides that examined above in which the electromagnetic field can be represented by means of two scalar functions.

2. Show that systems with symmetry of rotation about one of the coordinate axes (for example, cylindrical or spherical coordinates) belong to the set of coordinate systems examined in the preceding problem.

3. Show that in Cartesian and cylindrical coordinates the solutions of Maxwell's equations that are of the electric and magnetic types can be represented, respectively, by the electric and magnetic Hertz vectors only one component of which, Π_3 and Π_3^* or Π_z and Π_z^*, is different from zero. Show also that we may set Π_3 (or Π_z) $= u$ and Π_3^* (or Π_z^*) $= v$, where u and v are the functions defined earlier in the present section.

5*. *A uniqueness theorem*

A somewhat more significant question from a theoretical standpoint is the matter of uniqueness of the solutions of the system of Maxwell's equation.

Suppose that S is a closed surface separating two media i and e occupying, respectively, a finite region V_i (situated inside S) and an infinite region V_e (situated outside S). On the boundary S, we shall assume that the conditions for conjugacy of the tangential component of the field vectors are satisfied:

$$E_{\tau i} - E_{\tau e} = 0 \,, \qquad H_{\tau i} - H_{\tau e} = 0 \,. \tag{66}$$

We shall also assume that the radiation condition is satisfied as r increases without bound:

$$r\left(\frac{\partial E_\alpha}{\partial r} - ik_e E_\alpha\right) \to 0 \,, \quad r\left(\frac{\partial H_\alpha}{\partial r} - ik_e H_\alpha\right) \to 0 \,, \quad E_\alpha \to 0 \,, \quad H_\alpha \to 0 \,, \tag{67}$$

$$\alpha = 1, 2, 3 \,,$$

where k_e is the square root of the expression

$$k_e{}^2 = \frac{\omega^2 \epsilon_e \mu_e - i\omega\mu_e\sigma_e}{c^2}$$

with positive real part. The quantities appearing in this last expression are given the subscript e because an infinitely distant point belongs to the medium e. Henceforth, we shall write the subscript only when it is necessary to emphasize that a given quantity refers to a particular medium.

Finally, we shall assume that the conductivity of the medium i

$$\sigma_i \neq 0 \,. \tag{68}$$

Under these rather broad conditions, we have the following theorem.

UNIQUENESS THEOREM. The solution of the system of Maxwell equations (12) - (13) satisfying the boundary conditions (66), the radiation conditions (67), and the condition (68) is unique.

Proof: Let us suppose that there are two solutions, that is, that there are two systems of field vectors $\boldsymbol{E}^{(1)}$ and $\boldsymbol{H}^{(1)}$, on the one hand, and $\boldsymbol{E}^{(2)}$ and $\boldsymbol{H}^{(2)}$, on the other, both satisfying the requirements of the theorem. In this case, the differences

$$\boldsymbol{E} = \boldsymbol{E}^{(1)} - \boldsymbol{E}^{(2)} \,, \qquad \boldsymbol{H} = \boldsymbol{H}^{(1)} - \boldsymbol{H}^{(2)} \,,$$

will also satisfy the homogeneous system of Maxwell's equations

$$\frac{\partial E_\gamma}{\partial x_\beta} - \frac{\partial E_\beta}{\partial x_\gamma} = \frac{i\omega\mu}{c} H_\alpha \,, \tag{69}$$

$$\frac{\partial H_\gamma}{\partial x_\beta} - \frac{\partial H_\beta}{\partial x_\gamma} = \frac{\sigma - i\omega\epsilon}{c} E_\alpha \,, \tag{70}$$

the boundary conditions (66), and the radiation condition (67) at infinity. The theorem will be proven if we show that the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ satisfying the conditions listed are identically equal to zero throughout all space.

Consider the expression

$$m \equiv \frac{\sigma - i\omega\epsilon}{c} |\boldsymbol{E}|^2 + \frac{i\omega\mu}{c} |\boldsymbol{H}|^2 \tag{71}$$

and its complex conjugate

$$m^* \equiv \frac{\sigma + i\omega\epsilon}{c} |\boldsymbol{E}|^2 - \frac{i\omega\mu}{c} |\boldsymbol{H}|^2 \,. \tag{72}$$

The asterisk indicates the complex conjugate of the quantity without the asterisk. The sum

$$m + m^* = \frac{\sigma}{c} |\boldsymbol{E}|^2 \geqslant 0 . \tag{73}$$

Let us expand the expression for the squares of the moduli of the field vectors. By definition,

$$|\boldsymbol{E}|^2 = \sum_{\alpha=1}^{3} E_\alpha E_\alpha^* .$$

On the basis of eq. (70),

$$E_\alpha = \frac{c}{\sigma - i\omega\epsilon} \left(\frac{\partial H_\gamma}{\partial x_\beta} - \frac{\partial H_\beta}{\partial x_\gamma}\right) ,$$

so that

$$|\boldsymbol{E}|^2 = \frac{c}{\sigma - i\omega\epsilon} \sum_{\circlearrowleft} E_\alpha^* \left(\frac{\partial H_\gamma}{\partial x_\beta} - \frac{\partial H_\beta}{\partial x_\gamma}\right) = \frac{c}{\sigma + i\omega\epsilon} \sum_{\circlearrowleft} E_\alpha \left(\frac{\partial H_\gamma^*}{\partial x_\beta} - \frac{\partial H_\beta^*}{\partial x_\gamma}\right) ,$$

where the symbol $\circlearrowleft$ means that the summed terms are obtained one from the other by a cyclic permutation of the subscripts α, β, and γ. If we use eq. (70), we obtain by an analogous process

$$|\boldsymbol{H}|^2 = \frac{c}{i\omega\mu} \sum_{\circlearrowleft} H_\alpha^* \left(\frac{\partial E_\gamma}{\partial x_\beta} - \frac{\partial E_\beta}{\partial x_\gamma}\right) = - \frac{c}{i\omega\mu} \sum_{\circlearrowleft} H_\alpha \left(\frac{\partial E_\gamma^*}{\partial x_\beta} - \frac{\partial E_\beta^*}{\partial x_\gamma}\right) .$$

Substituting these expressions into eq. (72), we obtain

$$m^* = \sum_{\circlearrowleft} \left[E_\alpha \left(\frac{\partial H_\gamma^*}{\partial x_\beta} - \frac{\partial H_\beta^*}{\partial x_\gamma}\right) - H_\alpha^* \left(\frac{\partial E_\gamma}{\partial x_\beta} - \frac{\partial E_\beta}{\partial x_\gamma}\right)\right] = \sum_{\circlearrowleft} \frac{\partial}{\partial x_\alpha} (H_\beta^* E_\gamma - H_\gamma^* E_\beta) ,$$

from which it is clear that

$$m + m^* = \sum_{\circlearrowleft} \frac{\partial}{\partial x_\alpha} [(H_\beta E_\gamma^* - H_\gamma E_\beta^*) + (H_\beta^* E_\gamma - H_\gamma^* E_\beta)] .$$

Suppose that Σ is a spherical surface and that the region V_i is inside it. It is easy to see on the basis of the boundary conditions (66) that the normal component of the vector $\boldsymbol{T}$, with components

$$T_\alpha = (H_\beta E_\gamma^* - H_\gamma E_\beta^*) + (H_\beta^* E_\gamma - H_\gamma^* E_\beta) \tag{74}$$

is continuous at the interface S. To show this, suppose for simplicity that the x_α-axis is perpendicular to S. The normal component T_n of the vector $\boldsymbol{T}$ is then equal to T_α, and the quantities E_β, E_γ, H_β, and H_γ defining it are components of the tangential components of the field vectors and, consequently, are continuous. Therefore, the normal component, in which we are interested, is continuous.

Because of the continuity of the normal component of the vector $\boldsymbol{T}$ at the interface S, we may, throughout the entire region V_Σ lying within Σ, apply the Ostrogradskii-Gauss formula to the function

$$m + m^* = \sum_{\alpha=1}^{3} \frac{\partial T_\alpha}{\partial x_\alpha}.$$

On the basis of (73), this gives us

$$\iiint\limits_{V_\Sigma} \frac{\sigma}{c} |\boldsymbol{E}|^2 \, dV = \iint\limits_{\Sigma} T_n \, dS . \tag{75}$$

We introduce the spherical coordinates (r, θ, φ) with origin at the center of the spherical surface Σ. On Σ,

$$T_n = T_r = (H_\theta E_\varphi^* - H_\varphi E_\theta^*) + (H_\theta^* E_\varphi - H_\varphi^* E_\theta) . \tag{76}$$

Using Maxwell's equations in spherical coordinates (problem 3 of section 1), we obtain

$$H_\theta = \frac{c}{i\omega\mu}\left[\frac{1}{r \sin\theta}\frac{\partial E_r}{\partial\varphi} - \frac{1}{r}E_\varphi - \frac{\partial E_\varphi}{\partial r}\right]$$
$$\equiv \frac{c}{i\omega\mu}\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\varphi} - E_\varphi - r\left(\frac{\partial E_\varphi}{\partial r} - ikE_\varphi\right)\right] - \frac{ck}{\omega\mu}E_\varphi ,$$
$$H_\varphi = \frac{c}{i\omega\mu}\left[\frac{1}{r}E_\theta + \frac{\partial E_\theta}{\partial r} - \frac{1}{r}\frac{\partial E_r}{\partial\theta}\right]$$
$$\equiv \frac{c}{i\omega\mu}\frac{1}{r}\left[E_\theta - \frac{\partial E_r}{\partial\theta} + r\left(\frac{\partial E_\theta}{\partial r} - ikE_\theta\right)\right] + \frac{ck}{\omega\mu}E_\theta .$$

Let us evaluate the order of the term on the right side of these equations as r increases without bound. On the basis of the radiation condition (67), as r increases without bound,

$$E_r \to 0 , \quad E_\theta \to 0 , \quad E_\varphi \to 0 , \quad r\left(\frac{\partial E_\varphi}{\partial r} - ikE_\varphi\right) \to 0 , \quad r\left(\frac{\partial E_\theta}{\partial r} - ikE_\theta\right) \to 0 .$$

Therefore, the terms containing the factor $1/r$ approach zero more rapidly then does $1/r$. Therefore, it follows that

$$H_\theta = -\frac{ck}{\omega\mu}E_\varphi + o(1/r) , \qquad H_\varphi = \frac{ck}{\omega\mu}E_\theta + o(1/r) , \tag{77}$$

where $o(1/r)$ denotes the set of terms which approach zero more rapidly than $1/r$. Substituting these expressions into formula (76), we obtain

$$T_n = -\frac{ck}{\omega\mu}(E_\theta E_\theta^* + E_\varphi E_\varphi^*) - \frac{ck^*}{\omega\mu}(E_\theta E_\theta^* + E_\varphi E_\varphi^*) + o(1/r^2)$$
$$= -\frac{2c \operatorname{Re} k}{\omega\mu}(|E_\theta|^2 + |E_\varphi|^2) + o(1/r^2) ,$$

where $o(1/r^2)$ denotes the set of terms of higher order than $1/r^2$. It now follows from eq. (75) that

$$\iiint\limits_{V_i} \frac{\sigma_i}{c}|\boldsymbol{E}|^2 dV + \iiint\limits_{V_e} \frac{\sigma_e}{c}|\boldsymbol{E}|^2 dV + \frac{2c \operatorname{Re} k}{\omega\mu_e} \lim_{r\to\infty} \iint\limits_{\Sigma}(|E_\theta|^2 + |E_\varphi|^2)\, dS = 0 .$$

Since all terms of this equation are non-negative, each of them is equal to zero, and since the integrands are non-negative, we have

$$\frac{\sigma_i}{c}|\boldsymbol{E}|^2 = 0 \quad \text{in} \quad V_i\,, \qquad \frac{\sigma_e}{c}|\boldsymbol{E}|^2 = 0 \quad \text{in} \quad V_e\,,$$
$$\lim_{r\to\infty}\iint_{\Sigma}(|E_\theta|^2 + |E_\varphi|^2)\,d\Sigma = 0\,. \tag{78}$$

On the basis of eqs. (77); it then follows that

$$\lim_{r\to\infty}\iint_{\Sigma}(|H_\theta|^2 + |H_\varphi|^2)\,d\Sigma = 0\,.$$

Finally, if we use Maxwell's equations in spherical coordinates, we find

$$E_r = \frac{c}{\sigma - i\omega\epsilon}\,\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}H_\varphi\sin\theta - \frac{\partial H_\theta}{\partial\varphi}\right],$$
$$H_r = \frac{c}{i\omega\mu}\,\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}E_\varphi\sin\theta - \frac{\partial E_\theta}{\partial\varphi}\right],$$

from which it is clear that the radial components E_r and H_r do not exceed the angular components in order of magnitude, so that

$$\lim_{r\to\infty}\iint_{\Sigma}(|E_r|^2 + |H_r|^2)\,d\Sigma = 0\,,$$

and therefore, in general,

$$\lim_{r\to\infty}\iint_{\Sigma}|\boldsymbol{E}|^2\,d\Sigma = \lim_{r\to\infty}\iint_{\Sigma}|\boldsymbol{H}|^2\,d\Sigma = 0\,. \tag{79}$$

If $\sigma_i \neq 0$ and $\sigma_e \neq 0$, it follows from eqs. (78) that $\boldsymbol{E} = 0$ throughout all space and, on the basis of eqs. (69), $\boldsymbol{H}$ is also equal to zero. However, if $\sigma_e = 0$, the parameter

$$k = k_e = \frac{\omega\sqrt{\epsilon_e\mu_e}}{c}$$

will have a real value in the outside region. Therefore, remembering that each of the components of the field vectors satisfies Helmholtz' equations, we may use the fundamental lemma of the theory of Helmholtz' equations (section 7 of Chapter XXV), according to which satisfaction of condition (79) for a real value of k implies the vanishing of $\boldsymbol{E}$ and $\boldsymbol{H}$ throughout the entire external region. In the inner region, $\boldsymbol{E}$ and $\boldsymbol{H}$ are equal to zero because of the first of eqs. (78). This proves the theorem.

Chapter XXXV

EMISSION OF ELECTROMAGNETIC WAVES

1. *General remarks*

In studying the emission of electromagnetic waves, it is convenient to write the field equations in terms of the vector potential $\boldsymbol{A}$ (Chapter XXXIV, section 2). This is clear, for example, from what follows. As was shown in section 2 of Chapter XXXIV, the vector potential satisfies the system of Helmholtz' equations

$$\Delta A_\alpha + k^2 A_\alpha = -4\pi \frac{\mu}{c} j_\alpha^{(e)} , \tag{1}$$

$$k^2 = \frac{\omega^2 \mu\epsilon + i\omega\mu c}{c^2} , \tag{2}$$

where the $j_\alpha^{(e)}$ (for $\alpha = 1, 2, 3, \ldots$) are the components of the vector representing the external current density. If the external currents are parallel to some axis (as is the case in a number of types of antenna, for example), we can, by choosing this axis as one of the axes of a Cartesian coordinate system, see that system (1) will be satisfied if we set the components of the vector $\boldsymbol{A}$ along the other two axes equal to zero. Since the field vectors can be expressed in terms of the vector potential by means of formulae (17) and (19) of Chapter XXXIV, we may expect that the study of the field (in the above case) will be reduced to the case of a single scalar non-homogeneous Helmholtz equation relative to one of the components of the vector potential * that is not identically equal to zero.

Since, in addition to system (1), the vector potential must still satisfy the boundary conditions, the consideration that we have just made concerning the possibility of setting two components of the vector potential equal to zero does not of itself constitute a proof. However, in the most important cases, it is fully or partially justified and it does simplify the problem. We will consider in section 5 the matter of the conditions under which this simplification is possible. At the moment, we make the following observation. If, by introducing this simplification without further justification, we find a solution of the problem that satisfies the boundary condition, it will be the desired solution because of the uniqueness theorem for the system of Maxwell equations.

In solving the problem presented in this chapter, we shall use the

* Such a reduction can be attained, of course, without introducing the vector potential, but the most natural way of approaching the problem is in terms of the vector potential.

method of integral transformations. In order to make this method applicable without a detailed study of the convergence of the integrals, we shall assume, following Grinberg [27)], that the propagation of the radiation takes place in media that have at least *some* conductivity, which ensures exponential decrease in the field at infinitely distant points. The radiation condition can then be assumed to have the form (70) of Chapter XXV.

We note, in conclusion, that in presenting the material of this chapter we are primarily following Grinberg's monograph [27)]. Accordingly, we shall consider problems for the case in which there is only one plane boundary separating two homogeneous media (for example, the earth and the atmosphere). However, the method can, without significant changes, be extended to the case in which there are a number of parallel interfaces at which the properties of the media undergo discontinuities (laminar media). In such a case, only the number of boundary conditions to be given is increased. Finally, this method can be very easily generalized to the case in which the properties of the media depend on one of the coordinates. We shall only touch upon this, referring the reader to Grinberg's monograph for details.

As a rule, we shall consider the problem solved when a solution is found in the form of a definite integral containing only known functions. A vast literature has been devoted to investigating and finding numerical approximations of these integrals, but consideration of that problem would take us far beyond the scope of the present book.

Problems

1. Suppose that the dielectric constant and the conductivity depend on the coordinates. Show that the vector potential satisfies the equation (in Cartesian coordinates)

$$\Delta A_\alpha + k^2 A_\alpha - \left(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}\right)\frac{\partial \ln k^2}{\partial x_\alpha} = -4\pi \frac{\mu}{c} j_\alpha^{(e)} \quad (\alpha = 1,2,3)\ , \qquad (*)$$

$$k^2 = \frac{\omega^2 \mu\epsilon + i\omega\mu\sigma}{c^2}\ .$$

Method: Write the scalar potential in the usual form

$$\varphi = -\frac{i\omega}{ck^2}\left(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}\right)$$

and use Maxwell's equation. (We might note that the equation (*) is encountered primarily in the study of the propagation of radio waves in laminar media.)

2. Show that the components of the field vectors are related to the components of the vector potential by the following formulae in cylindrical coordinates (r, φ, z):

$$H_r = \frac{1}{\mu r}\left(\frac{\partial A_r}{\partial \varphi} - \frac{\partial r A_\varphi}{\partial z}\right), \quad H_\varphi = \frac{1}{\mu}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right), \quad H_z = \frac{1}{\mu r}\left(\frac{\partial r A_\varphi}{\partial r} - \frac{\partial A_r}{\partial \varphi}\right),$$

$$E_r = \frac{i\omega}{c} A_r + \frac{i\omega}{ck^2} \frac{\partial}{\partial r}\left[\frac{1}{r}\left(\frac{\partial r A_r}{\partial r} + \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial r A_z}{\partial z}\right)\right],$$

$$E_\varphi = \frac{i\omega}{c} A_\varphi + \frac{i\omega}{ck^2} \frac{\partial}{r\partial \varphi}\left[\frac{1}{r}\left(\frac{\partial r A_r}{\partial r} + \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial r A_z}{\partial z}\right)\right],$$

$$E_z = \frac{i\omega}{c} A_z + \frac{i\omega}{ck^2} \frac{\partial}{\partial z}\left[\frac{1}{r}\left(\frac{\partial r A_r}{\partial r} + \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial r A_z}{\partial z}\right)\right].$$

2. *A vertical emitter in a homogeneous medium over an ideally conducting plane*

Consider a system of vertical currents that is bounded in space and that is symmetric about a vertical axis. We shall call this a vertical emitter, and we shall call its axis of symmetry the axis of the emitter. We shall assume that the radiation takes place in a half-space occupied by a homogeneous dielectric (homogeneous atmosphere). We also assume that this half-space is bounded by a horizontal plane that is the boundary of an ideal conductor (the surface of the earth).

We introduce cylindrical coordinates (r, φ, z) with the z-axis directed along the axis of the emitter and with the origin on the horizontal plane. The equation of this plane will then be $z = 0$. In accordance with section 1, to find the electromagnetic field of the emitter, we need to solve the nonhomogeneous Helmholtz equation for the component A_z of the vector potential. Since the field is symmetric about the axis of the emitter, this equation in the coordinates (r, φ, z) will be of the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) + \frac{\partial^2 A_z}{\partial z^2} + k^2 A_z = -4\pi \frac{\mu}{c} j_z^{(e)}, \tag{3}$$

and the component A_z will not depend on φ.

By means of the formulae given in problem 2 of section 1, we can show that the components of the field vectors in the case in question are equal to

$$\begin{gathered} E_r = \frac{i\omega}{ck^2}\frac{\partial}{\partial r}\frac{\partial A_z}{\partial z}, \quad E_\varphi = 0, \quad E_z = \frac{i\omega}{c} A_z + \frac{i\omega}{ck^2}\frac{\partial^2 A_z}{\partial z^2}, \\ H_r = 0, \quad H_\varphi = -\frac{1}{\mu}\frac{\partial A_z}{\partial r}, \quad H_z = 0; \end{gathered} \tag{4}$$

that is, the field of a vertical emitter represents a system of waves with mutually perpendicular electric and magnetic vectors.

According to section 3 of Chapter XXXIV, the only boundary condition that is not identically satisfied on the boundary of an ideal conductor consists, on the basis of Maxwell's equations, in the fact that the tangential component of the electric vector must vanish on the boundary. From eqs. (4), this condition will be satisfied if we set

$$\left.\frac{\partial A_z}{\partial z}\right|_{z=0} = 0 \ . \tag{5}$$

Finally, the radiation condition must be satisfied at infinity:

$$\lim_{z\to\infty} A_z = 0 \ . \tag{6}$$

Let us eliminate differentiation with respect to the coordinate r from eq. (3) by means of an integral transformation. Consider the differential expression

$$\mathcal{M}_r A_r = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) , \tag{7}$$

for which $a_{rr} = 1$, $b_r = 1/r$, and $c = 0$. Then, from the formulae of section 5 of Chapter XXXI, we have

$$\rho(r) = r \ , \qquad p(r) = r \ , \qquad q = 0 \ . \tag{8}$$

Consequently, the kernel of the desired operator must be equal to the product $r\overline{K}(\gamma, r)$, where $\overline{K}(\gamma, r)$ is the solution of the equation

$$\frac{\partial}{\partial r}\left(r\frac{\partial \overline{K}}{\partial r}\right) + \gamma^2 r\overline{K} = 0 \ .$$

If we divide by r, this equation becomes Bessel's equation of order zero:

$$\frac{\partial^2\overline{K}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{K}}{\partial r} + \gamma^2\overline{K} = 0 \ , \tag{9}$$

whose bounded solutions are the Bessel functions $J_0(\gamma r)$. Thus, we need to apply the Hankel transform (Chapter XXXI, section 4) with kernel $rJ_0(\gamma r)$. When this transformation is carried out, the problem (3) - (5) is reduced to the form *

$$\frac{\partial^2\overline{A}_z}{\partial z^2} - (\gamma^2 - k^2)\overline{A}_z = -4\pi\frac{\mu}{c}\overline{j}_z{}^{(e)} \ , \tag{10}$$

$$\overline{A}_z|_{z=0} = 0 \ , \qquad \lim_{z\to\infty}\overline{A}_z = 0 \ . \tag{11}$$

The general solution of eq. (10) is equal to

$$\begin{aligned}\overline{A}_z(\gamma, z) &= -\frac{2\pi}{c}\frac{\mu}{q}\int_0^z \overline{j}_z{}^{(e)}(\gamma,\zeta)\left[e^{q(z-\zeta)} - e^{-q(z-\zeta)}\right]d\zeta + B_1\, e^{qz} + B_2\, e^{-qz} \\ &= \frac{2\pi}{c}\frac{\mu}{q}\left[e^{qz}\left(B_1^* - \int_0^z \overline{j}_z{}^{(e)}(\gamma,\ )\, e^{-q\zeta}\, d\zeta\right)\right. \\ &\qquad\left. + e^{-qz}\left(B_2^* + \int_0^z \overline{j}_z{}^{(e)}(\gamma,\zeta)\, e^{q\zeta}\, d\zeta\right)\right] ,\end{aligned} \tag{12}$$

* We recall that this result follows immediately from the formulae of section 5 of Chapter XXXI.

where B_1, B_2, B_1^*, and B_2^* are arbitrary constants and q denotes that root of the expression $(\gamma^2 - k^2)$ whose real part is positive. Since the system of currents of the emitter is, by hypothesis, bounded in space (so that, beginning with some z, the function $\bar{j}_z^{(e)}(\gamma, z) = j_z^{(e)}(r, z) = 0$), the integrals of the expressions containing this function as a factor are bounded. Therefore, it is sufficient to set

$$B_1^* = \int_0^\infty \bar{j}_z^{(e)}(\gamma, \zeta)\, e^{-q\zeta}\, d\zeta \tag{13}$$

in order to ensure the vanishing of the function $\bar{A}_z(\gamma, z)$ as z increases without bound. As one can easily see, the boundary condition (11) gives

$$B_2^* = B_1^* \, . \tag{14}$$

When we substitute these values of the constants into solution (12) and extend the function $j_z^{(e)}(r, z)$ for negative values of z as an even function by setting

$$j_z^{(e)}(r, -z) \equiv j_z^{(e)}(r, z) \, ,$$

we finally obtain

$$\bar{A}_z(\gamma, z) = \frac{2\pi}{c}\frac{\mu}{q}\left(e^{qz}\int_z^\infty \bar{j}_z^{(e)}(\gamma, \zeta)\, e^{-q\zeta}\, d\zeta + e^{-qz}\int_{-z}^\infty \bar{j}_z^{(e)}(\gamma, \zeta)\, e^{-q\zeta}\, d\zeta\right). \tag{15}$$

The inverse Hankel transform

$$A(r, z) = \int_0^\infty \bar{A}_z(\gamma, z)\, J_0(\gamma r)\, \gamma\, d\gamma \tag{16}$$

gives the solution of the problem posed in the form of a definite integral, which we shall not write out in detail.

Let us consider some very important particular cases.

If the emitter is a cylindrical rod of radius r_0 located somewhere in the region $z_0 \leq z \leq z_1$, and if the current density through a cross section of the rod is given by the formula

$$j_z^{(e)}(r, z) = \frac{j_0}{\sqrt{r_0^2 - r^2}} \qquad (r \leq r_0,\ z_0 \leq z \leq z_1) \, ,$$

where j_0 is a constant, then

$$\bar{j}_z^{(e)}(\gamma, z) = j_0 \int_0^\infty \frac{j_0}{\sqrt{r_0^2 - \zeta^2}}\, J_0(\gamma\zeta)\, \zeta\, d\zeta = \begin{cases} j_0(\sin \gamma r_0)/\gamma & \text{for } z_0 \leq z \leq z_1, \\ 0 & \text{for } z < z_0,\ z > z_1. \end{cases} \tag{17}$$

We note that the total current flowing through a cross section of the rod is

$$I = 2\pi j_0 \int_0^{r_0} \frac{r\, dr}{\sqrt{r_0^2 - r^2}} = 2\pi j_0 r_0 \, . \tag{18}$$

If we let r_0 approach zero and at the same time increase the current density in such a way that the total current remains the same, we obtain

$$\lim_{r_0\to 0} \bar{j}_z^{(e)}(\gamma, z) = \begin{cases} I/2\pi & \text{for} \quad z_0 \leqslant z \leqslant z_1, \\ 0 & \text{for} \quad z < z_0,\ z > z_1 . \end{cases} \tag{19}$$

This case corresponds to a vertical segment of a conductor with negligibly small diameter (a vertical linear current).

When we substitute expression (19) into eq. (15), we obtain, for values of z within the interval (z_0, z_1),

$$\bar{A}_z(\gamma, z) = \begin{cases} I\dfrac{\mu}{cq}(e^{qz} + e^{-qz})\dfrac{e^{-qz_0} - e^{-qz_1}}{q} & \text{for} \quad z < z_0 , \\ I\dfrac{\mu}{cq}e^{-qz}\dfrac{e^{qz_1} - e^{qz_0} + e^{-qz_0} - e^{-qz_1}}{q} & \text{for} \quad z > z_1 . \end{cases} \tag{20}$$

If we let the point z_1 approach the point z_0, by using a series expansion in terms of the small distances $(z_1 - z_0)$, we obtain

$$\frac{1}{q}(e^{-qz_0} - e^{-qz_1}) = \frac{e^{-qz_0}}{q}(1 - e^{q(z_0 - z_1)}) \to e^{-qz_0}(z_1 - z_0) ,$$

$$\frac{1}{q}(e^{qz_1} - e^{qz_0}) = \frac{e^{qz_0}}{q}(e^{q(z_1 - z_0)} - 1) \to e^{qz_0}(z_1 - z_0)$$

As we let the difference $z_1 - z_0$ decrease, let us increase I in such a way that the product

$$P = I(z_1 - z_0)$$

will remain unchanged. The limiting case as $z_1 - z_0$ approaches zero represents an oscillating electric dipole with vertical axis and moment P located at the point $r = 0$, $z = z_0$. The relationship (20) then takes the form

$$\bar{A}_z(\gamma, z) = \begin{cases} \dfrac{P\mu}{cq}e^{-qz_0}(e^{qz} + e^{-qz}) & \text{for} \quad z < z_0 , \\ \dfrac{P\mu}{cq}e^{-qz}(e^{qz_0} + e^{-qz_0}) & \text{for} \quad z > z_0 . \end{cases} \tag{21}$$

The expressions for $z < z_0$ and $z > z_0$ can be combined into a single expression:

$$\bar{A}_z(\gamma, z) = \frac{P\mu}{cq}(e^{-q|z-z_0|} + e^{-q|z+z_0|}) , \tag{22}$$

which is valid for all positive values of z. If we substitute this expression into formula (16) and remember that $q = \sqrt{(\gamma^2 - k^2)}$, we obtain

$$A_z(r, z) = \frac{P\mu}{c}\int_0^\infty \frac{\exp[-|z-z_0|\sqrt{\gamma^2 - k^2}] + \exp[-|z+z_0|\sqrt{\gamma^2 - k^2}]}{\sqrt{\gamma^2 - k^2}} J_0(\gamma r)\,\gamma\, d\gamma .$$

This integral can be tabulated by means of a formula from the theory of Bessel functions *:

$$\int_0^\infty \frac{\exp\left[-x\sqrt{\gamma^2 - k^2}\right]}{\sqrt{\gamma^2 - k^2}} J_0(\gamma y)\, \gamma\, d\gamma = \frac{\exp\left[ik\sqrt{x^2 + y^2}\right]}{\sqrt{x^2 + y^2}}, \tag{23}$$

so that

$$A_z(r, z) = \frac{P\mu}{c}\left[\frac{\exp\left[ik\sqrt{(z-z_0)^2 + r^2}\right]}{\sqrt{(z-z_0)^2 + r^2}} + \frac{\exp\left[ik\sqrt{(z+z_0)^2 + r^2}\right]}{\sqrt{(z+z_0)^2 + r^2}}\right]. \tag{24}$$

This formula has an interesting interpretation. It is easy to show, by using the formula of Chapter XXXIV, that the component of the vector potential (of a dipole in an infinite homogeneous medium) that is directed along the axis of the dipole is equal to

$$\frac{P\mu}{c}\,\frac{e^{ikR}}{R}, \tag{25}$$

where R is the distance from the dipole to the point of observation. Let us compare this expression with eq. (24). The quantity $\sqrt{[(z - z_0)^2 + r^2]}$ in eq. (24) also represents the distance from a dipole located at the point $r = 0$, $z = z_0$ to the point of observation $r = r$, $z = z$. (The coordinate φ, because of the symmetry of the field, can have an arbitrary value.) On the other hand, the quantity $\sqrt{[(z + z_0)^2 + r^2]}$ formally represents the distance from the point of observation to the point $r = 0$, $z = -z_0$, which is the mirror image of the point $r = 0$, $z = z_0$ in the lower half-space, relative to the boundary separating the media (the plane $z = 0$).

Thus, we reach the following conclusion. The field caused by a single vertically oriented dipole P at an arbitrary point above an ideal conducting earth is the same as the field that would be caused at that same point (if there were no earth) by two such vertically oriented dipoles oscillating with the same phase, with one situated at the same point as the original dipole P and the other situated at its mirror image with respect to the surface of the earth. In particular, it then follows that the field of a dipole situated on the surface of an ideal conducting earth is twice as strong as the field that would be created in the atmosphere if there were no earth. Formula (24) gives

$$A_z(r, z) = \frac{2P\mu}{c}\,\frac{e^{ikR}}{R} \qquad (R = \sqrt{r^2 + z^2}),$$

when $z = 0$; this is twice the value of expression (25).

Problems

1. Beginning with the equation (*) of problem 1 of the preceding section, show that, for a medium whose conductivity and dielectric constant de-

* See R.O.Kuz'min 14), p. 151.

pend on the coordinate z (that is, on the height), we have, instead of eq. (10), the equation

$$\frac{d}{dz}\left[\psi \frac{d\bar{A}_z(\gamma, z)}{dz}\right] - (1+\psi\gamma^2)\,\bar{A}_z(\gamma, z) = -4\pi\,\frac{\mu}{c}\;\psi\bar{j}_z^{(e)}(\gamma, z)\ ,$$

where the function $\psi = -1/k^2$.

2. Show that for $\psi = \alpha(z - z_0)^2$, where α and z_0 are constants, the homogeneous equation corresponding to the inhomogeneous equation of the preceding problem is reduced to the equation

$$\frac{d^2u}{d\xi^2} + \frac{2}{\xi}\,\frac{du}{d\xi} - \left(\gamma^2 + \frac{1}{\alpha\xi^2}\right) u = 0 \qquad (\xi = z - z_0)\ ,$$

whose solutions are the cylindrical functions

$$Z_\nu(\sqrt{-\gamma\xi}) \qquad \left(\nu = \sqrt{\frac{1}{\alpha} + \frac{1}{4}}\right) .$$

3. Prove the law of reflection of variable currents in an ideally conducting plane. According to this law, the electromagnetic field caused by a system of currents flowing over the surface coincides, at an arbitrary point on the surface, with the field that would be created at the same point (in the absence of the ideally conducting plane) by a system of currents formed by combining the initial system of currents with its mirror image on the other side of the plane in question. (This reverses the direction of the currents in the reflected system.)
Method: To prove this rule, it is sufficient to show that the reflected system of currents (after their direction is changed) will cause, at every point of the plane, a field that exactly offsets the field caused by the original system of currents.

4. Show that under the conditions of the problem solved in this section, an oscillating dipole located at the point N creates, at the point M, the same field that it would create at the point N if it were situated at the point M.

3. *A vertical emitter in a homogeneous medium over a sphere of finite conductivity*

Let us now consider the problem of the preceding section for a vertical emitter over a medium of finite conductivity σ_i. We shall accordingly speak of an upper and a lower medium. Where necessary, we shall indicate that a quantity applies to the upper medium by using the subscript e and to the lower medium by using the subscript i.

The statement of the problem obviously remains essentially the same as above; however, now the electromagnetic field will be different from zero in the half-space $z < 0$, so that Helmholtz' equation (3) for the upper medium must be extended to the lower medium, where it is homogeneous and has a different value for the parameter $k^2 = k_i^2$:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right)+\frac{\partial^2 A_z}{\partial z^2}+k_i^2 A_z=0 \quad \left(k_i^2=\frac{\omega^2\epsilon_i\mu_i+i\omega\mu_i\sigma_i}{c^2}\ ,\ z<0\right)\ . \tag{26}$$

The boundary condition must also be changed. Instead of the vanishing of the tangential component of the electric vector at the boundary separating the media, we now must require continuity of the tangential components of the field vectors (Chapter XXXIV, section 3). This requirement, on the basis of (4), will be satisfied if

$$\frac{1}{\mu_e}A_{ze}\Big|_{z=0}=\frac{1}{\mu_i}A_{zi}\Big|_{z=0}\ ,\qquad \frac{1}{k_e^2}\frac{\partial A_{ze}}{\partial z}\Big|_{z=0}=\frac{1}{k_i^2}\frac{\partial A_{zi}}{\partial z}\Big|_{z=0} \tag{27}$$

These expressions obviously retain the same form if we replace the components A_z by their Hankel transforms.

The relationship (12) remains valid for the vector potentials in the upper medium, since the boundary conditions were not used in its derivation. Similarly, eq. (13) remains valid since the conditions remain unchanged as z increases without bound.

The expression for the Hankel transform for the vector potential in the lower medium can be written down immediately on the basis of eq. (12). Noting that there are no external currents in the lower medium (the emitter, of course, is assumed to be located in the upper medium), so that $\bar{j}_{zi}^{(e)}(\gamma, z) = 0$, and noting that we must set $B_2^* = 0$, because of the vanishing of the field as z approaches $-\infty$, we obtain

$$\bar{A}_{zi}(\gamma, z)=\frac{2\pi}{c}\frac{\mu_i}{q_i}C\,e^{q_i z}\ , \tag{28}$$

where q_i is the root of $\gamma^2 - k_i^2$ that has a positive real part and where C is a constant.

We have the two constants C and B_2^* at our disposal for satisfying the two boundary conditions (27). From these boundary conditions, we obtain

$$\frac{C}{q_i}=\frac{1}{q_e}(B_1^*+B_2^*)\ ,\qquad \frac{C\mu_i}{k_i^2}=\frac{\mu_e}{k_e^2}(B_1^*-B_2^*)\ ,$$

so that

$$C=\frac{2\mu_e q_i k_i^2}{\mu_e q_e k_i^2+\mu_i q_i k_e^2}B_1^*\ ,\qquad B_2^*=\frac{\mu_e q_e k_i^2-\mu_i q_i k_e^2}{\mu_e q_e k_i^2+\mu_i q_i k_e^2}B_1^*\ .$$

We carry out further calculations only for the case of a vertical dipole located at the point $r = 0,\ z = z_0$. Then,

$$B_1^*=\frac{P}{2\pi}e^{-q_e z_0}\ ,$$

where P is the dipole moment. After some calculation, we obtain

$$\bar{A}_{ze}(\gamma, z) = \begin{cases} \dfrac{P\mu_e}{cq_e} \dfrac{(\mu_e q_e k_i^2 + \mu_i q_i k_e^2)\, e^{-q_e(z_0 - z)} + (\mu_e q_e k_i^2 - \mu_i q_i k_e^2)\, e^{-q_e(z_0+z)}}{\mu_e q_e k_i^2 - \mu_i q_i k_e^2} \\ \qquad \text{for } 0 < z < z_0 \,, \\ \dfrac{2P\mu_e}{c} \dfrac{k_i^2\, e^{-q_e(z_0+z)}}{\mu_e q_e k_i^2 + \mu_i q_i k_e^2} \quad \text{for } z > 0 \,, \end{cases} \tag{29}$$

$$\bar{A}_{zi}(\gamma, z) = \frac{2P}{c} \frac{\mu_i \mu_e k_i^2\, e^{-q_e z_0 + q_i z}}{\mu_e q_e k_i^2 + \mu_i q_i k_e^2} \quad \text{for} \quad z < 0 \,. \tag{30}$$

For $z_0 = 0$, we apply the transformation formula to the variable r and obtain Sommerfeld's solution for a dipole on a conducting earth:

$$A_z(r, z) = \frac{2P\mu_e}{c} k_i^2 \int_0^\infty \frac{e^{-z\sqrt{\gamma^2 - k_e^2}}\, J_0(\gamma r)\, \gamma\, d\gamma}{\mu_e k_i^2 \sqrt{\gamma^2 - k_e^2} + \mu_i k_e^2 \sqrt{\gamma^2 - k_i^2}} \quad (z > 0) \,, \tag{31}$$

$$A_z(r, z) = \frac{2P\mu_i\mu_e}{c} k_i^2 \int_0^\infty \frac{e^{z\sqrt{\gamma^2 - k_i^2}} J_0(\gamma r)\, \gamma\, d\gamma}{\mu_e k_i^2 \sqrt{\gamma^2 - k_e^2} + \mu_i k_e^2 \sqrt{\gamma^2 - k_i^2}} \quad (z < 0) \,. \tag{32}$$

These expressions are much more difficult to analyze than are the analogous relationships in the case of an ideally conducting earth.

4. *A magnetic antenna over a medium of finite conductivity*

Let us consider an emitter in the form of a circular cylindrical conducting coil along which a current is flowing. Such an emitter, with a vertical axis, is called a vertical magnetic antenna.

We shall assume that the base of the antenna in question is located on a plane separating a homogeneous dielectric (the atmosphere or the upper medium) and a conductor (the earth or the lower medium). We shall assume that the lines of current are concentric circles, lying in planes parallel to the surface dividing the media and having their centers at the axis of the coil.

Since, in view of the considerations explained in section 1, there are no vertical currents in the antenna being considered, we shall assume that the vertical component of the vector potential is equal to zero. Under this assumption, when we use Cartesian coordinates with x_1- and x_2-axes situated in the plane of the horizon, we obtain the following two equations for the magnetic field of the antenna:

$$\Delta A_1 + k^2 A_1 = -\frac{4\pi\mu}{c} j_1^{(e)} , \tag{33}$$

$$\Delta A_2 + k^2 A_2 = -\frac{4\pi\mu}{c} j_2^{(e)} . \tag{34}$$

If we transform these equations to cylindrical coordinates r, φ, and z, we can eliminate one more component of the vector potential and reduce the problem to a matter of solving a single scalar equation. Let us direct the axis of the system (r, φ, z) along the axis of the coil, and let us place the origin on the surface separating the media.

It is easy to see that the components of an arbitrary vector a, when expressed in Cartesian and cylindrical coordinate systems, are related as follows

$$a_r = a_1 \cos\varphi + a_2 \sin\varphi , \qquad a_\varphi = -a_1 \sin\varphi + a_2 \cos\varphi ,$$

so that, since $\cos\varphi = x_1/r$ and $\sin\varphi = x_2/r$, we obtain

$$r a_r = a_1 x_1 + a_2 x_2 , \qquad r a_\varphi = -a_1 x_2 + a_2 x_1 . \tag{35}$$

Let us use these formulae for making the transformation. If we multiply eq. (33) by x_1 and eq. (34) by x_2 and add the results, we obtain, in view of the first of formulae (35),

$$x_1 \Delta A_1 + x_2 \Delta A_2 + k^2 r A_r = -\frac{4\pi}{c} r j_r^{(e)} .$$

We also find that

$$x_1 \Delta A_1 = x_1 \frac{\partial^2 A_1}{\partial x_1^2} + \frac{\partial^2 x_1 A_1}{\partial^2 x_2} + \frac{\partial^2 x_1 A_1}{\partial^2 x_3} = \Delta x_1 A_1 - 2 \frac{\partial A_1}{\partial x_1} ,$$

$$x_2 \Delta A_2 = \Delta x_2 A_2 - 2 \frac{\partial A_2}{\partial x_2} , \qquad x_1 \Delta A_1 + x_2 \Delta A_2 = \Delta r A_r - 2 \left(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2}\right) .$$

By means of expression (48) of Chapter XVII for the divergence of a vector, we see that, in cylindrical coordinates,

$$\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} = \frac{1}{r}\left(\frac{\partial r A_r}{\partial r} + \frac{\partial A_\varphi}{\partial \varphi}\right) .$$

If we combine these expressions and use the expression for the Laplacian operator in cylindrical coordinates, we obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial r A_r}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 A_r}{\partial \varphi^2} + r \frac{\partial^2 A_r}{\partial z^2} + k^2 r A_r - \frac{2}{r}\left(\frac{\partial r A_r}{\partial r} + \frac{\partial A_\varphi}{\partial \varphi}\right) = -\frac{4\pi\mu}{c} j_r^{(e)} .$$

We expand the terms with derivatives and recall that, because of the axial symmetry of the field of the antenna that we are considering, the vector potential does not depend on the coordinate φ. We then obtain finally

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial A_r}{\partial r}\right) + \frac{\partial^2 A_r}{\partial z^2} + \left(k^2 - \frac{1}{r^2}\right) A_r = -\frac{4\pi\mu}{c} j_r^{(e)} . \tag{36}$$

Analogously, if we use the second of formulae (35), we obtain a second equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_\varphi}{\partial r}\right)+\frac{\partial^2 A_\varphi}{\partial z^2}+\left(k^2-\frac{1}{r^2}\right)A_\varphi=-\frac{4\pi\mu}{c}j_\varphi{}^{(e)}\,. \tag{37}$$

Since the current is annular, $j_r{}^{(e)}=0$ and eq. (36) will be satisfied if we set $A_r=0$. Therefore, we may seek a solution by using only eq. (37). If we find the component A_φ, the component of the field vectors can be found from the following equations:

$$\begin{gathered}E_\varphi=\frac{\mathrm{i}\omega}{c}A_\varphi\,,\qquad E_r=E_z=0\,,\\ H_r=-\frac{1}{\mu}\frac{\partial A_\varphi}{\partial z}\,,\qquad H_\varphi=0\,,\qquad H_z=\frac{1}{r}\frac{\partial rA_\varphi}{\partial r}\,.\end{gathered} \tag{38}$$

As the reader can easily verify, these equations follow from the formulae of problem 2 of section 1, and from the expressions for vector operations in cylindrical coordinates. We shall not write the equations for both media that we are considering separately, since eq. (37) is also valid for the lower medium, but without the right side and with the corresponding change in the value of the parameter k^2.

On the surface separating the media, the tangential components of the field vectors must be continuous (see section 3, Chapter XXXIV). On the basis of eqs. (38), this gives the following boundary condition:

$$A_{\varphi\mathrm{e}}|_{z=0}=A_{\varphi\mathrm{i}}|_{z=0}\,,\qquad \frac{1}{\mu_\mathrm{e}}\left.\frac{\partial A_{\varphi\mathrm{e}}}{\partial z}\right|_{z=0}=\frac{1}{\mu_\mathrm{i}}\left.\frac{\partial A_{\varphi\mathrm{i}}}{\partial z}\right|_{z=0}. \tag{39}$$

As before, the subscript e denotes quantities referring to the upper medium and the subscript i denotes quantities referring to the lower medium. The radiation condition must be satisfied at infinity. As in the preceding sections, we shall assume that the external medium has at least some conductivity. As we know, this automatically ensures that the radiation condition will be satisfied and that the field vectors decrease exponentially at infinity.

Let us now solve the problem. By means of an integral transformation, let us eliminate differentiation with respect to r. By the same method as that used in section 2, we find that the kernel of the operator must be the function $rJ_1(\gamma r)$. Therefore, we need to use the Hankel transform. We then reduce the boundary problem (37) - (39) to the form

$$\frac{\partial^2\bar{A}_\varphi}{\partial z^2}-(\gamma^2-k^2)\bar{A}_\varphi=-4\pi\frac{\mu}{c}\bar{j}_\varphi{}^{(e)}\,, \tag{40}$$

$$\bar{A}_{\varphi\mathrm{e}}|_{z=0}=\bar{A}_{\varphi\mathrm{i}}|_{z=0}\,,\qquad \frac{1}{\mu_\mathrm{e}}\left.\frac{\partial\bar{A}_{\varphi\mathrm{e}}}{\partial z}\right|_{z=0}=\frac{1}{\mu_\mathrm{i}}\left.\frac{\partial\bar{A}_{\varphi\mathrm{i}}}{\partial z}\right|_{z=0}, \tag{41}$$

where

$$\bar{A}_\varphi\equiv\int_0^\infty A_\varphi\,J_1(\gamma r)\,r\,\mathrm{d}r\,,\qquad \bar{j}_\varphi{}^{(e)}\equiv\int_0^\infty j_\varphi{}^{(e)}\,J_1(\gamma r)\,r\,\mathrm{d}r\,. \tag{42}$$

Noting that eq. (40) coincides with eq. (10), which we examined in section 2, we write its general solution in the same form:

$$\bar{A}_\varphi = \frac{2\pi}{c}\frac{\mu}{q}\left[e^{qz}\left(B_1 - \int_0^z \bar{j}_\varphi^{(e)} e^{-q\zeta}\, d\zeta\right) + e^{-qz}\left(B_2 + \int_0^z \bar{j}_\varphi^{(e)} e^{q\zeta}\, d\zeta\right)\right], \quad (43)$$

where B_1 and B_2 are arbitrary constants and where $q \equiv \sqrt{(\gamma^2 - k^2)}$ denotes the square root of $\gamma^2 - k^2$ with a positive real part. The solution (43) is generally speaking valid both for the upper and for the lower medium but, of course, with different values of the constants B_1 and B_2 and of the quantities which depend on the characteristics of the medium. To ensure the vanishing of the function $\bar{A}_\varphi$ as z either increases or decreases without bound, we must set

$$B_1 = B_{1e} = \int_0^\infty \bar{j}_\varphi^{(e)} e^{-q\zeta}\, d\zeta, \quad (44)$$

for the upper medium and

$$B_2 = B_{2i} = 0 \quad (45)$$

for the lower medium (since there are no external currents in it). By using the boundary conditions (41) for determining the constants B_{1i} and B_{2e}, we obtain the system of equations

$$\frac{\mu_e}{q_e}(B_{1e} + B_{2e}) = \frac{\mu_i}{q_i} B_{1i}, \qquad B_{1e} - B_{2e} = B_{1i},$$

so that

$$B_{2e} = \frac{\mu_i q_e - \mu_e q_i}{\mu_i q_e + \mu_e q_i}, \quad (46)$$

$$B_{1i} = \frac{2\mu_e q_i}{\mu_i q_e + \mu_e q_i}. \quad (47)$$

Let us now calculate B_{1e}. Let us consider the density of the current flowing through the coil of the antenna as being independent of r and z and equal to j_0 (uniform current distribution). Then,

$$\bar{j}_\varphi^{(e)} = \begin{cases} j_0 \int_{r_1}^{r_2} J_1(\gamma r)\, r\, d\gamma & \text{for} \quad 0 \leqslant z \leqslant z_1, \\ 0 & \text{for} \quad z < 0,\ z > z_1, \end{cases}$$

where z_1 is the coordinate of the highest level of the coil winding and r_1 and r_2 are the inner and outer radii of the coil.

Let us neglect the thickness of the coil by having r_2 approach r_1, but let us keep the total current flowing through the coil constant by assuming that the current density j_0 increases in such a way that the product $j_1 \equiv j_0(r_2 - r_1)$ remains constant. By using the mean-value theorem, we then obtain

$$\bar{j}_\varphi^{(e)} = \begin{cases} j_1 J_1(\gamma r_1) r_1 & \text{for} \quad 0 \leqslant z \leqslant z_1, \\ 0 & \text{for} \quad z < 0, \; z > z_1. \end{cases}$$

Here, we can consider the quantity j_1 as being the density of the circular surface current flowing along the cylinder $r = r_1$. Hence, because of eq. (45),

$$B_{1e} = j_1 \frac{1 - e^{-q_e z_1}}{q_e} J_1(\gamma r_1) \, r_1 \, .$$

We also note that when $z > z_1$, the integrals in eq. (43) have the values

$$\int_0^z \bar{j}_\varphi^{(e)} \, e^{-q\zeta} \, d\zeta = j_1 \frac{1 - e^{-q_e z_1}}{q_e} J_1(\gamma r_1) \, r_1 = B_{1e} \, ,$$

$$\int_0^z \bar{j}_\varphi^{(e)} \, e^{q\zeta} \, d\zeta = j_1 \frac{e^{q_e z_1} - 1}{q_e} J_1(\gamma r_1) \, r_1 \, .$$

If we let z_1 approach zero and at the same time increase j_1 in such a way that the product $I = j_1 z_1$ remains constant, we get the case of a single turn of radius r_1 through which a current I is flowing. We then obtain

$$\lim_{\substack{z_1 \to 0 \\ z > z_1}} \int_0^z \bar{j}_\varphi^{(e)} \, e^{-q\zeta} d\zeta = \lim_{\substack{z_1 \to 0 \\ z > z_1}} \int_0^z \bar{j}_\varphi^{(e)} \, e^{q\zeta} d\zeta = B_{1e} = I J_1(\gamma r_1) r_1 \quad (z > 0). \quad (48)$$

From relations (46) - (48) and (44) for a single circular turn, we obtain

$$\bar{A}_\varphi = \frac{4\pi \mu_e \mu_i r_1 I J_1(\gamma r_1)}{c(\mu_i q_e + \mu_e q_i)} \, e^{-q_e z} \qquad (z > 0) \, ,$$

$$\bar{A}_\varphi = \frac{4\pi \mu_e \mu_i r_1 I J_1(\gamma r_1)}{c(\mu_i q_e + \mu_e q_i)} \, e^{q_i z} \qquad (z < 0) \, .$$

Finally, let us consider the case of a magnetic dipole. Here, we let the radius r_1 of the turn approach zero and, at the same time, we increase I so that the magnetic moment of the turn $M = \pi r_1^2 I$ will remain unchanged. Recalling, on the basis of expansion (14) of Chapter XII, that the function $J_1(\gamma r_1) \to \frac{1}{2}\gamma r_1$ as $r_1 \to 0$, we obtain

$$\bar{A}_\varphi = \frac{2\mu_e \mu_i M \gamma}{c(\mu_i q_e + \mu_e q_i)} \, e^{-q_e z} \qquad (z > 0) \, ,$$

$$\bar{A}_\varphi = \frac{2\mu_e \mu_i M \gamma}{c(\mu_i q_e + \mu_e q_i)} \, e^{q_i z} \qquad (z < 0) \, .$$

By using the inverse Hankel transform (47) of Chapter XXXI, we find an expression for the component A_φ of the vector potential of the field of a magnetic dipole in the form

$$\bar{A}_\varphi = \frac{2\mu_e\mu_i}{c} M \int_0^\infty \frac{e^{-z\sqrt{\gamma^2-k_e^2}}\, J_1(\gamma r)\, \gamma^2\, d\gamma}{\mu_i\sqrt{\gamma^2-k_e^2} + \mu_e\sqrt{\gamma^2-k_i^2}} \quad (z > 0)\ , \tag{49}$$

$$\bar{A}_\varphi = \frac{2\mu_e\mu_i}{c} M \int_0^\infty \frac{e^{z\sqrt{\gamma^2-k_i^2}}\, J_1(\gamma r)\, \gamma^2\, d\gamma}{\mu_i\sqrt{\gamma^2-k_e^2} + \mu_e\sqrt{\gamma^2-k_i^2}} \quad (z < 0)\ . \tag{50}$$

These expressions formally solve the problem of the magnetic field of a dipole located on the boundary of a conducting medium. However, they can be put in a different form and, in certain cases, the field of a dipole can be expressed in terms of elementary functions.

Recalling that, on the basis of formula (22) of Chapter XII,

$$\gamma J_1(\gamma r) = -\frac{d}{dr} J_0(\gamma r)$$

and integrating eqs. (49) and (50) with respect to r, we obtain

$$A_\varphi = -\frac{2\mu_e\mu_i}{c} M \frac{\partial \Pi}{\partial r}\ , \tag{51}$$

where

$$\Pi = \int_0^\infty \frac{e^{-z\sqrt{\gamma^2-k_e^2}}\, J_0(\gamma r)\, \gamma\, d\gamma}{\mu_i\sqrt{\gamma^2-k_e^2} + \mu_e\sqrt{\gamma^2-k_i^2}} \quad (z > 0)\ , \tag{52}$$

$$\Pi = \int_0^\infty \frac{e^{z\sqrt{\gamma^2-k_i^2}}\, J_0(\gamma r)\, \gamma\, d\gamma}{\mu_i\sqrt{\gamma^2-k_e^2} + \mu_e\sqrt{\gamma^2-k_i^2}} \quad (z < 0)\ . \tag{53}$$

Since the magnetic permeability of the majority of actual media is very close to unity, we shall also assume that $\mu_e = \mu_i = 1$. Then, if we multiply the numerator and denominator of the integrand in eq. (52) by $\sqrt{(\gamma^2 - k_e^2)} - \sqrt{(\gamma^2 - k_i^2)}$, we obtain

$$\begin{aligned}
\Pi &= \frac{1}{k_i^2 - k_e^2} \int_0^\infty e^{-z\sqrt{\gamma^2-k_e^2}}\, J_0(\gamma r)\, \left(\sqrt{\gamma^2-k_e^2} - \sqrt{\gamma^2-k_i^2}\right) \gamma\, d\gamma \\
&= \frac{1}{k_i^2 - k_e^2} \left[\int_0^\infty e^{-z\sqrt{\gamma^2-k_e^2}}\, J_0(\gamma r)\, \sqrt{\gamma^2-k_e^2}\, \gamma\, d\gamma \right. \\
&\qquad - \int_0^\infty e^{-z\sqrt{\gamma^2-k_i^2}}\, J_0(\gamma r)\, \sqrt{\gamma^2-k_i^2}\, \gamma\, d\gamma \\
&\qquad \left. + \int_0^\infty \left(e^{-z\sqrt{\gamma^2-k_i^2}} - e^{-z\sqrt{\gamma^2-k_e^2}}\right) J_0(\gamma r)\, \sqrt{\gamma^2-k_i^2}\, \gamma\, d\gamma \right]
\end{aligned}$$

We now use a formula that we have encountered before:

$$\int_0^\infty \frac{e^{-x\sqrt{\gamma^2-k^2}}}{\sqrt{\gamma^2-k^2}} J(\gamma y)\, \gamma\, d\gamma = \frac{e^{ik\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} .$$

If we differentiate this formula twice with respect to x, we obtain

$$\int_0^\infty e^{-x\sqrt{\gamma^2-k^2}} J(\gamma y) \sqrt{\gamma^2-k^2}\, \gamma\, d\gamma = \frac{\partial^2}{\partial x^2}\left(\frac{e^{ik\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\right),$$

so that the expression written above can be put in the form

$$\Pi = \frac{1}{k_i^2-k_e^2}\frac{\partial^2}{\partial z^2}\left(\frac{e^{ik_e\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}} - \frac{e^{ik_i\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}}\right)$$

$$+ \frac{1}{k_i^2+k_e^2}\int_0^\infty \left(e^{-z\sqrt{\gamma^2-k_i^2}} - e^{-z\sqrt{\gamma^2-k_e^2}}\right) J_0(\gamma r)\sqrt{\gamma^2-k_i^2}\,\gamma\, d\gamma . \tag{54}$$

It is possible to show that the last integral approaches zero for positive values of r as z approaches zero *. We omit the proof, which is simple though somewhat long. Under these conditions, because of the identity

$$\frac{\partial}{\partial z}\sqrt{r^2+z^2} = \frac{z}{r}\frac{\partial}{\partial r}\sqrt{r^2+z^2},$$

we obtain

$$\left[\frac{\partial^2}{\partial z^2}\left(\frac{e^{ik\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}}\right)\right]_{z=0} = \frac{\partial}{\partial z}\left[\frac{z}{r}\frac{\partial}{\partial r}\left(\frac{e^{ik\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}}\right)\right]_{z=0} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\right).$$

Carrying out the appropriate transformations in eq. (54), we obtain the Van der Pol formula

$$\Pi|_{z=0} = \frac{1}{k_i^2-k_e^2}\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{e^{ik_e r}}{r} - \frac{e^{ik_i r}}{r}\right), \tag{55}$$

which makes it possible to determine the field on the surface separating the media (the surface of the earth).

Problems

1. Investigate the case in which the lower medium is an ideal conductor.
2. Use formula (55) to investigate the field of the electric and magnetic vectors on the surface separating two media.

* See, for example, G. A. Grinberg [27], p. 464.

5. *The field of an arbitrary system of emitters*

Let us suppose now that there is an arbitrary system of currents that is bounded in space above a plane surface separating two media, one of which is a dielectric and the other a conductor.

We assume the magnetic permeability of both media to be the same and equal to unity. Let us find the vector potential of the electromagnetic field of this system of currents.

We set up a rectangular Cartesian coordinate system at x_1, x_2, x_3 with the x_1, x_2-plane coinciding with the boundary between the two media and with the x_3-axis directed into the dielectric. In this coordinate system, the components of the vector potential of the system of currents satisfy the three scalar non-homogeneous Helmholtz' equations

$$\Delta A_\alpha + k^2 A_\alpha = -\frac{4\pi}{c} j_\alpha^{(e)} \qquad (\alpha = 1, 2, 3) \,, \tag{56}$$

and the field vectors are expressed in terms of the vector potential by

$$\begin{gathered} E_\alpha = \frac{i\omega}{c}\left[\frac{\partial}{\partial x_\alpha}\frac{1}{k^2}\left(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}\right) + A_\alpha\right] \,, \\ H_\alpha = \frac{\partial A_\gamma}{\partial x_\beta} - \frac{\partial A_\beta}{\partial x_\gamma} \,. \end{gathered} \tag{57}$$

According to section 3 of Chapter XXXIV, the tangential components of the electric and magnetic vectors must be continuous at the boundary between the media. On the basis of eqs. (57), this leads to the following boundary conditions for the components of the vector potential:

$$\begin{gathered} A_\alpha|_{x_3=+0} = A_\alpha|_{x_3=-0} \qquad (\alpha = 1, 2, 3) \,, \\ \left.\frac{\partial A_\alpha}{\partial x_3}\right|_{x_3=+0} = \left.\frac{\partial A_\alpha}{\partial x_3}\right|_{x_3=-0} \qquad (\alpha = 1, 2) \,, \\ \left.\frac{1}{k^2}\left(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}\right)\right|_{x_3=+0} = \left.\frac{1}{k^2}\left(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}\right)\right|_{x_3=-0} \,, \end{gathered} \tag{58}$$

where the subscript equations $x_3 = +0$ and $x_3 = -0$ mean that the limiting values of the corresponding quantities are taken as x_3 approaches zero from above and from below, respectively. The subscripts e and i, which we used in the preceding sections, are omitted here in order not to complicate the notation.

Let us perform integral transformations on the problem (57) - (58) to eliminate differentiation with respect to the variables x_1 and x_2. The differential expressions with respect to x_1 and x_2 that appear in the system (56) are of the form $\partial^2 A_\alpha/\partial x_j^2$. We have already encountered expressions of this form in section 1 of Chapter XXXIII, where it was shown that the Fourier transform should be used when the variable varies in the interval $(-\infty, \infty)$. Therefore, we need to apply the Fourier transform twice, with re-

spect to the coordinates x_1 and x_2. When we do this, we obtain the system of homogeneous differential equations:

$$\frac{d^2\overline{A}_\alpha}{dx_3^2} - q^2\overline{A}_\alpha = -\frac{4\pi}{c}\,\overline{j}_\alpha^{(e)} , \tag{59}$$

where

$$\overline{A}_\alpha = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_\alpha(x_1, x_2, x_3)\, e^{-i(\zeta_1 x_1 + \zeta_2 x_2)}\, dx_1\, dx_2 , \tag{60}$$

$$\overline{j}_\alpha^{(e)} = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} j_\alpha^{(e)}(x_1, x_2, x_3)\, e^{-i(\zeta_1 x_1 + \zeta_2 x_2)}\, dx_1\, dx_2 , \tag{61}$$

$$q^2 \equiv \zeta_1^2 + \zeta_2^2 - k^2 , \tag{62}$$

with the boundary conditions

$$\overline{A}_\alpha|_{x_3=+0} = \overline{A}_\alpha|_{x_3=-0} \qquad (\alpha = 1,2,3) , \tag{63}$$

$$\left.\frac{\partial\overline{A}_\alpha}{\partial x_3}\right|_{x_3=+0} = \left.\frac{\partial\overline{A}_\alpha}{\partial x_3}\right|_{x_3=-0} \qquad (\alpha = 1,2) , \tag{64}$$

$$\frac{1}{k^2}\left.\left(i\zeta_1\overline{A}_1 + i\zeta_2\overline{A}_2 - \frac{\partial\overline{A}_3}{\partial x_3}\right)\right|_{x_3=+0} = \frac{1}{k^2}\left.\left(i\zeta_1\overline{A}_1 + i\zeta_2\overline{A}_2 - \frac{\partial\overline{A}_3}{\partial x_3}\right)\right|_{x_3=-0} . \tag{65}$$

When we find the solution of the system (59) that satisfies the radiation conditions at infinity and the boundary conditions (63) - (65), we can then, by using the formula for the inverse Fourier transform, represent the components of the desired vector potential of the field in the form of integrals of known quantities

$$A_\alpha(x_1, x_2, x_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{A}_\alpha(\zeta_1, \zeta_2, x_3)\, e^{i(\zeta_1 x_1 + \zeta_2 x_2)}\, d\zeta_1\, d\zeta_2 , \tag{66}$$

which gives the solution to the problem.

The calculations leading to the determination of the quantities $\overline{A}_\alpha$ in the system (59) and the conditions (63) - (65), are considerably facilitated by the fact that each of the first two equations (59) can be solved independently of the remaining equations. The boundary conditions (63) - (65) can be broken down into groups:

$$\overline{A}_1|_{x_3=+0} = \overline{A}_1|_{x_3=-0} , \qquad \left.\frac{\partial\overline{A}_1}{\partial x_3}\right|_{x_3=+0} = \left.\frac{\partial\overline{A}_1}{\partial x_3}\right|_{x_3=-0} , \tag{67}$$

$$\overline{A}_2|_{x_3=+0} = \overline{A}_2|_{x_3=-0} , \qquad \left.\frac{\partial\overline{A}_2}{\partial x_3}\right|_{x_3=+0} = \left.\frac{\partial\overline{A}_2}{\partial x_3}\right|_{x_3=-0} , \tag{68}$$

$$\bar{A}_3|_{x_3=+0} = \bar{A}_3|_{x_3=-0} \,,$$
$$\frac{1}{k^2}\left(i\zeta_1\bar{A}_1 + i\zeta_2\bar{A}_2 - \frac{\partial\bar{A}_3}{\partial x_3}\right)\Bigg|_{x_3=+0} = \frac{1}{k^2}\left(i\zeta_1\bar{A}_1 + i\zeta_2\bar{A}_2 - \frac{\partial\bar{A}_3}{\partial x_3}\right)\Bigg|_{x_3=-0} . \tag{69}$$

Each of the first two groups of conditions is independent of the others, from which our assertion follows.

This fact is quite significant when the properties of the medium depend on one of the coordinates (variation with altitude). If we let x_3 be this coordinate, we can solve the problem by first finding the functions $\bar{A}_1$ and $\bar{A}_2$ and then substituting them into the third equation (see problems 2 and 3).

It also follows from conditions (67) - (69) that we may not set the component of the vector potential that is normal to the boundary between the media equal to zero when there is a system of currents parallel to this boundary, since then it would be impossible to satisfy the conditions (69). However, if the horizontal current is linear (for example, if it flows along the x_1-axis), we may set $A_2 = 0$. Then, as we shall see in the next section, the system of boundary conditions (67) - (69) can be satisfied.

Problems

1. Write a system of equations determining the vector potential of the field of an arbitrary system of emitters for the case in which the properties of the medium depend on one of the coordinates. Apply the Fourier transform to reduce this system to a system of ordinary differential equations.
   Method: Use the system of equations (*) of problem 1 of section 1.

2. Show that the system of equations of the preceding problem can be reduced to a system of two Helmholtz equations and one equation with variable coefficients. Show that these equations can be solved one after another. Consider the properties of the media as continuous (that is, assume that there is no dividing surface).

3. Set up the boundary conditions for the case of an arbitrary system of horizontal currents.

4. Show that it is possible to use the Fourier transform instead of the Hankel transform to solve the problem discussed in section 2.

6. *A horizontal emitter over a medium of finite conductivity*

By "horizontal emitter", we mean a system of currents that is bounded in space parallel to some direction in the plane separating two media. Let us direct the x_1-axis in this direction and, in accordance with what was said at the end of the preceding section, let us set the component A_2 of the vector potential equal to zero. (As above, the x_3-axis is assumed perpen-

dicular to the surface separating the media.) Then, to determine the field of the horizontal emitter, we obtain the two equations

$$\Delta A_1 + k^2 A_1 = -\frac{4\pi}{c} j_1^{(e)} , \tag{70}$$

$$\Delta A_3 + k^2 A_3 = 0 , \tag{71}$$

the solutions of which, on the basis of eqs. (58), must be found subject to the following boundary conditions on the separating surface:

$$A_1|_{x_3=+0} = A_1|_{x_3=-0} , \quad \left.\frac{\partial A_1}{\partial x_3}\right|_{x_3=+0} = \left.\frac{\partial A_1}{\partial x_3}\right|_{x_3=-0} , \tag{72}$$

$$\begin{gathered} A_3|_{x_3=+0} = A_3|_{x_3=-0} , \\ \frac{1}{k^2}\left(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_3}{\partial x_3}\right)\Bigg|_{x_3=+0} = \frac{1}{k^2}\left(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_3}{\partial x_3}\right)\Bigg|_{x_3=-0} . \end{gathered} \tag{73}$$

Furthermore, the radiation conditions must be satisfied at infinity. As in section 5, we assume that the magnetic permeability of both media is equal to unity.

If we apply the Fourier transform first to the variable x_1 and then to x_2, according to what was said in section 5, we will obtain a system of ordinary differential equations of the form (59):

$$\frac{d^2\bar{A}_1}{dx_3{}^2} - q^2\bar{A}_1 = -\frac{4\pi}{c}\bar{j}_1 , \tag{74}$$

$$\frac{d^2\bar{A}_3}{dx_3{}^2} - q^2\bar{A}_3 = 0 , \tag{75}$$

$$q^2 \equiv \zeta_1{}^2 + \zeta_2{}^2 - k^2 \tag{76}$$

and a system of boundary conditions of the form (63) - (65), which their solutions must satisfy:

$$\bar{A}_1|_{x_3=+0} = \bar{A}_1|_{x_3=-0} , \quad \left.\frac{\partial \bar{A}_1}{\partial x_3}\right|_{x_3=+0} = \left.\frac{\partial \bar{A}_1}{\partial x_3}\right|_{x_3=-0} , \tag{77}$$

$$\begin{gathered} \bar{A}_3|_{x_3=+0} = \bar{A}_3|_{x_3=-0} , \\ \frac{1}{k^2}\left(i\zeta_1\bar{A}_1 - \frac{\partial \bar{A}_3}{\partial x_3}\right)\Bigg|_{x_3=-0} = \frac{1}{k^2}\left(i\zeta_1\bar{A}_1 - \frac{\partial \bar{A}_3}{\partial x_3}\right)\Bigg|_{x_3=+0} . \end{gathered} \tag{78}$$

Also, the solutions of the system (74) - (75) must, by the radiation condition, approach zero as x_3 becomes infinitely large.

Let us first find the solutions to eq. (74) with boundary conditions (77). The general solution to eq. (74) is *

* See V.I.Smirnov [1]), Vol. 2, p. 28

$$\bar{A}_1 = \frac{2\pi}{cq}\left[e^{qx_3}\left(B_1 - \int_0^{x_3} \bar{j}_1\, e^{-q\zeta}\, d\zeta\right) + e^{-qx_3}\left(B_2 + \int_0^{x_3} \bar{j}_1\, e^{q\zeta}\, d\zeta\right)\right], \quad (79)$$

where B_1 and B_2 are arbitrary constants and q is the square root of q^2 with positive real part. We again denote by the subscripts e and i quantities referring to the upper and lower media.

From the given conditions, we have at infinity

$$B_{1e} = \int_0^{\infty} \bar{j}_1\, e^{-q\zeta}\, d\zeta\,, \qquad B_{2i} = 0\,. \quad (80)$$

To determine the constants B_{2e} and B_{1i}, we have the two equations

$$\frac{1}{q_e}(B_{1e} + B_{2e}) = \frac{1}{q_i} B_{1i}\,, \qquad B_{1e} - B_{2e} = B_{1i}\,,$$

so that

$$B_{2e} = \frac{q_e - q_i}{q_e + q_i} B_{1e}\,, \qquad B_{1i} = \frac{2q_i}{q_e + q_i} B_{1e}\,.$$

We define

$$\psi_1 \equiv \frac{1}{B_{1e}} \int_0^{x_3} \bar{j}_1\, e^{-q\zeta}\, d\zeta\,, \qquad \psi_2 \equiv \frac{1}{B_{1e}} \int_0^{x_3} \bar{j}_1\, e^{q\zeta}\, d\zeta\,. \quad (81)$$

With this notation, we have

$$\bar{A}_1 = \frac{2\pi B_{1e}}{cq_e}\left[e^{q_e x_3}(1-\psi_1) + e^{-q_e x_3}\left(\frac{q_e - q_i}{q_e + q_i} + \psi_2\right)\right] \quad (x_3 > 0)\,, \quad (82)$$

$$\bar{A}_1 = \frac{4\pi B_{1e}}{c(q_e + q_i)}\, e^{q_i x_3} \quad (x_3 < 0)\,. \quad (83)$$

Of course, it is assumed that there are no external currents in the lower medium. In particular, for $x_3 = 0$,

$$\bar{A}_1|_{x_3=+0} = \bar{A}_1|_{x_3=-0} = \frac{4\pi B_{1e}}{c(q_e + q_i)}\,. \quad (84)$$

Let us turn to eq. (75). Its general solution is

$$\bar{A}_3 = C_1\, e^{qx_3} + C_2\, e^{-qx_3}\,,$$

where C_1 and C_2 are arbitrary constants. From the given conditions, we have at infinity

$$C_{1e} = C_{2i} = 0\,,$$

for the upper and lower media. The first of conditions (78) gives

$$C_{2e} = C_{1i} \equiv C\,,$$

so that

$$\bar{A}_3 = C\, e^{-q_e x_3}\,, \quad x_3 > 0\,, \qquad \bar{A}_3 = C\, e^{q_i x_3}\,, \quad x_3 < 0\,. \tag{85}$$

To determine C, we use the second of conditions (78), which, in the light of eq. (84), yields

$$\frac{1}{k_e^2}\left(\frac{4\pi i \zeta_1 B_{1e}}{c(q_e + q_i)} + q_e C\right) = \frac{1}{k_i^2}\left(\frac{4\pi i \zeta_1 B_{1e}}{c(q_e + q_i)} - q_i C\right).$$

Since eq. (76) implies that

$$k_e^2 - k_i^2 = q_i^2 - q_e^2 = (q_i - q_e)(q_i + q_e)\,,$$

we have

$$C = \frac{4\pi i \zeta_1 B_{1e}}{c(q_e k_i^2 + q_i k_e^2)}\,. \tag{86}$$

The components A_1 and A_3 of the vector potential can now be represented in the form of integrals

$$A_\alpha = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{A}_\alpha\, e^{+i(\zeta_1 x_1 + \zeta_2 x_2)}\, d\zeta_1\, d\zeta_2 \qquad (\alpha = 1, 3)\,. \tag{87}$$

This completely solves the problem of determining the field of a horizontal emitter.

Let us examine the most interesting particular cases, namely, a linear antenna and a horizontally oriented dipole.

We shall consider a linear antenna as the limiting case of an emitter with square cross section (with current density j_0 uniformly distributed over the cross section) as the side of the square approaches zero. We denote the length of the side by $2a$. We shall assume that the emitter is situated at a distance $x_3 = h$ from the surface separating the two media, with its midpoint on the x_3-axis. We denote the length of the emitter by $2l$.

When we perform the Fourier transformation, we obtain

$$\bar{j}_1 = \begin{cases} j_0 \displaystyle\int_{-l}^{+l} e^{i\zeta_1 x_1}\, dx_1 \int_{-a}^{a} e^{i\zeta_2 x_2}\, dx_2 = \frac{4 j_0}{\zeta_1 \zeta_2} \sin \zeta_1 l \sin \zeta_2 a \\ \qquad\qquad \text{for} \quad h - a < x_3 < h + a\,, \\ 0 \quad \text{for} \quad x_3 < h - a\,,\; x_3 > h + a\,, \end{cases}$$

so that

$$\int_0^{x_3} \bar{j}_1\, e^{-q\zeta}\, d\zeta = \begin{cases} \dfrac{4 j_0}{\zeta_1 \zeta_2} \sin \zeta_1 l \sin \zeta_1 a\, \dfrac{e^{-q_e h}}{q_e} (e^{q_e a} - e^{-q_e a}) & \text{for} \quad x_3 > h + a\,, \\ 0 & \text{for} \quad x_3 \leqslant h - a\,. \end{cases}$$

Let us now have a approach zero and at the same time increase the current density j_0 in such a way that the product $I = 4\pi j_0 a^2$ (which is equal to the total current through a cross section of the emitter) will retain its former value. As a result, we obtain

$$\int_0^{x_3} \bar{j}_1 \, e^{-q\zeta} \, d\zeta = \begin{cases} \dfrac{2I \sin \zeta_1 l}{\zeta_1} \, e^{-q_e h} & \text{for} \quad x_3 > h \, , \\ 0 & \text{for} \quad x_3 < h \, . \end{cases}$$

Analogously, we obtain

$$\int_0^{x_3} \bar{j}_1 \, e^{q\zeta} \, d\zeta = \begin{cases} \dfrac{2I \sin \zeta_1 l}{\zeta_1} \, e^{q_e h} & \text{for} \quad x_3 > h \, , \\ 0 & \text{for} \quad x_3 < h \, . \end{cases}$$

By using these equations together with formulae (79), (80), (81), and (86), we obtain

$$B_{1e} = \frac{2I \sin \zeta_1 l}{\zeta_1} \, e^{-q_e h} \, , \qquad C = \frac{8\pi i I \sin \zeta_1 l}{c(q_e k_i^2 + q_i k_e^2)} \, ,$$

$$\psi_1 = \begin{cases} 1 & \text{for} \quad x_3 > h \, , \\ 0 & \text{for} \quad x_3 \leqslant h \, , \end{cases} \qquad \psi_2 = \begin{cases} e^{2qh} & \text{for} \quad x_3 > h \, , \\ 0 & \text{for} \quad x_3 \leqslant h \, , \end{cases}$$

so that, on the basis of formulae (82), (83), and (85), we have for a linear antenna

$$\bar{A}_1 = \frac{4\pi I \sin \zeta_1 l}{c\zeta_1 q_e(q_e + q_i)} \left[(q_e + q_i) \, e^{q_e h} + (q_e - q_i) \, e^{-q_e h}\right] e^{-q_e x_3} \qquad (x_3 > h) \, ,$$

$$\bar{A}_1 = \frac{4\pi I \sin \zeta_1 l}{c\zeta_1 q_e(q_e + q_i)} \, e^{-q_e h} \left[(q_e + q_i) \, e^{q_e x_3} + (q_e - q_i) \, e^{-q_e x_3}\right] \qquad (0 < x_3 \leqslant h) \, ,$$

$$\bar{A}_1 = \frac{8\pi I \sin \zeta_1 l}{c\zeta_1(q_e + q_i)} \, e^{-q_e h} \, e^{q_i x_3} \qquad (x_3 \leqslant 0) \, ,$$

$$\bar{A}_3 = \frac{8\pi i I \sin \zeta_1 l}{c(q_e k_i^2 + q_i k_e^2)} \, e^{-q_e h} \, e^{-q_e x_3} \qquad (x_3 > 0) \, ,$$

$$\bar{A}_3 = \frac{8\pi i I \sin \zeta_1 l}{c(q_e k_i^2 + q_i k_e^2)} \, e^{-q_e h} \, e^{q_i x_3} \qquad (x_3 < 0) \, .$$

To go from the case of a linear antenna to the case of a horizontally oriented dipole, we let l approach zero but increase I so that the product $P = 2Il$ remains constant. This gives

$$\bar{A}_1 = \frac{2\pi P}{c q_e(q_e + q_i)} \left[(q_e + q_i) \, e^{q_e h} + (q_e - q_i) \, e^{-q_e h}\right] e^{-q_e x_3} \qquad (x_3 > h) \, ,$$

$$\bar{A}_1 = \frac{2\pi P}{c q_e(q_e + q_i)} \, e^{-q_e h} \left[(q_e + q_i) \, e^{q_e x_3} + (q_e - q_i) \, e^{-q_e x_3}\right] \qquad (0 < x_3 \leqslant h) \, ,$$

$$\bar{A}_1 = \frac{4\pi P}{c(q_e + q_i)} \, e^{-q_e h} \, e^{q_i x_3} \qquad (x_3 \leqslant 0) \, ,$$

$$\bar{A}_3 = \frac{4\pi i \zeta_1 P}{c(q_e k_i^2 + q_i k_e^2)} e^{-q_e h} e^{-q_e x_3} \qquad (x_3 > 0) ,$$

$$\bar{A}_3 = \frac{4\pi i \zeta_1 P}{c(q_e k_i^2 + q_i k_e^2)} e^{-q_e h} e^{q_i x_3} \qquad (x_3 < 0) .$$

From these relations, it is easy to obtain the Herschelmann and Sommerfeld formulae for a dipole situated on the surface dividing two media. If we set $h = 0$ in these relations and apply the inverse Fourier transform, we obtain

$$A_1 = \frac{P}{\pi c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-(q_e x_3 + i x_1 \zeta_1 + i x_2 \zeta_2)}}{q_e + q_i} d\zeta_1 \, d\zeta_2 \qquad (x_3 > 0) ,$$

$$A_1 = \frac{P}{\pi c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{(q_i x_3 - i x_1 \zeta_1 - i x_2 \zeta_2)}}{q_e + q_i} d\zeta_1 \, d\zeta_2 \qquad (x_3 < 0) ,$$

$$A_3 = \frac{iP}{\pi c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\zeta_1 \, e^{-(q_e x_3 + i x_1 \zeta_1 + i x_2 \zeta_2)}}{q_e k_i^2 + q_i k_e^2} d\zeta_1 \, d\zeta_2 \qquad (x_3 > 0) ,$$

$$A_3 = \frac{iP}{\pi c} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\zeta_1 \, e^{(q_i x_3 - i x_1 \zeta_1 - i x_2 \zeta_2)}}{q_e k_i^2 + q_i k_e^2} d\zeta_1 \, d\zeta_2 \qquad (x_3 < 0) .$$

We introduce cylindrical coordinates (r, φ, z) with the z-axis coinciding with the x_3-axis and with the angle φ measured so that

$$x_1 = r \cos \varphi , \qquad x_2 = r \sin \varphi ,$$

and we set

$$\zeta_1 = \gamma \cos \psi , \qquad \zeta_2 = \gamma \sin \psi ,$$

so that, in particular,

$$q_i = \sqrt{\zeta_1^2 + \zeta_2^2 - k_i^2} = \sqrt{\gamma^2 - k_i^2} , \qquad q_e = \sqrt{\gamma^2 - k_e^2} .$$

The first of these integrals can be transformed to the form

$$A_1 = \frac{P}{\pi c} \int_0^{\infty} d\gamma \frac{\gamma \, e^{-z\sqrt{\gamma^2 - k_e^2}}}{\sqrt{\gamma^2 - k_e^2} + \sqrt{\gamma^2 - k_i^2}} \int_{-\pi}^{\pi} e^{-ir\gamma \cos(\psi - \varphi)} d\psi .$$

From a familiar formula in the theory of Bessel functions

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n\varphi - x \sin \varphi)} d\varphi ,$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ir\gamma\cos(\psi-\varphi)}\, d\psi = J_0(\gamma r) ,$$

so that we finally obtain

$$A_1 = \frac{2P}{c}\int_0^\infty \frac{e^{-z\sqrt{\gamma^2-k_e^2}}\, J_0(\gamma r)\,\gamma\, d\gamma}{N(\gamma)} \qquad (z > 0) ,$$

where

$$N(\gamma) = \sqrt{\gamma^2 - k_e^2} + \sqrt{\gamma^2 - k_i^2} .$$

In an analogous manner, we also obtain

$$A_1 = \frac{2P}{c}\int_0^\infty \frac{e^{z\sqrt{\gamma^2-k_i^2}}\, J_0(\gamma r)\,\gamma\, d\gamma}{N(\gamma)} \qquad (z < 0) ,$$

$$A_3 = \frac{2(k_i^2 - k_e^2)P}{c}\cos\varphi\int_0^\infty \frac{e^{-z\sqrt{\gamma^2-k_e^2}}\, J_1(\gamma r)\,\gamma^2\, d\gamma}{N_1(\gamma)\, N(\gamma)} \qquad (z > 0) ,$$

$$A_3 = \frac{2(k_i^2 - k_e^2)P}{c}\cos\varphi\int_0^\infty \frac{e^{z\sqrt{\gamma^2-k_i^2}}\, J_1(\gamma r)\,\gamma^2\, d\gamma}{N_1(\gamma)\, N(\gamma)} \qquad (z < 0) ,$$

where

$$N_1(\gamma) = k_i^2\sqrt{\gamma^2 - k_e^2} + k_e^2\sqrt{\gamma^2 - k_i^2} .$$

These are the Herschelmann and Sommerfeld formulae.

Problems

1. Show that on the boundary separating two media

$$A_1 = \frac{2P}{c(k_e^2 - k_i^2)}\,\frac{1}{r}\,\frac{d}{dr}\left(\frac{e^{ik_i r}}{r} - \frac{e^{ik_e r}}{r}\right) .$$

2. Investigate the case in which the lower medium is an ideal conductor.

Chapter XXXVI

DIRECTED ELECTROMAGNETIC WAVES

1. *Transverse electric, transverse magnetic, and transverse electromagnetic waves*

In this chapter, we shall examine a number of problems concerning steady-state processes of propagation of electromagnetic waves along systems that have the property of establishing conditions under which waves are propagated principally in a given direction. Such waves are called directed waves and the systems directing them are called waveguides.

The basic technique that we shall use for simplifying the examination of these problems is the representation of the electromagnetic field in the form of a superposition of waves of several types.

Suppose that the x_3-axis is directed along the direction of propagation of the waves. This can always be done, at least locally, at a given point. The electromagnetic field of a wave is determined by the six components E_1, E_2, E_3, H_1, H_2 and H_3 of the electric and magnetic vectors. Let us represent it in the form of a superposition of two fields determined, respectively, by the components

$$0,\ E_2,\ 0; \qquad H_1,\ 0,\ H_3 \tag{1}$$

and

$$E_1,\ 0,\ E_3; \qquad 0,\ H_2,\ 0. \tag{2}$$

The superposition of these fields clearly gives the original field. The electric vector of field (1) is perpendicular to the direction of propagation of the wave, whereas the magnetic vector has a non-zero component along the direction of propagation. Field (2) is characterized by the opposite arrangement of the electric and magnetic vectors. Specifically, the electric vector has a component along the direction of propagation and the magnetic does not. In connection with this, the waves characterized by field (1) are called transverse electric or TE waves and those characterized by field (2) are called transverse magnetic or TM waves. We shall also use these names for waves that have all transverse components different from zero; that is we shall consider the vanishing of E_3 and H_3 as the characteristic criterion for TE and TM waves, respectively.

Finally, we introduce yet a third type of wave, characterized by the absence of longitudinal components of both the electric and the magnetic vector. Such waves are characterized by the following table of components:

$$E_1,\ E_2,\ 0; \qquad H_1,\ H_2,\ 0 \tag{3}$$

and are called transverse electromagnetic or TEM waves.

These considerations do not yet indicate the possibility of independent existence of wave processes characterized by the fields (1)-(3). However, as we shall show below, waves of all three types can exist as separate and independent processes under suitable conditions.

Problem

Show that fields of the type (1)-(3) can be the fields of a travelling wave. Method: Start with Maxwell's equation.

2. *Waves between ideally conducting planes separated by a dielectric*

Let us study the propagation of a plane wave between two parallel ideally conducting planes at a distance s from each other. Let us place the coordinate origin on one of the planes. We shall assume the x_3-axis oriented in the direction of propagation of the wave, and the x_1-axis perpendicular to the planes and directed in such a way that the equations for the planes are $x_1 = 0$ and $x_1 = s$. All quantities will then be independent of x_2 and the equations of the field (12)-(13) of Chapter XXXIV will take the form

$$\begin{aligned} \frac{i\omega\mu}{c} H_1 &= -\frac{\partial E_2}{\partial x_3}, & \frac{\sigma - i\omega\epsilon}{c} E_1 &= = \frac{\partial H_2}{\partial x_3}, \\ \frac{i\omega\mu}{c} H_2 &= \frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}, & \frac{\sigma - i\omega\epsilon}{c} E_2 &= \frac{\partial H_1}{\partial x_3} - \frac{\partial H_3}{\partial x_1}, \\ \frac{i\omega\mu}{c} H_3 &= \frac{\partial E_2}{\partial x_1}, & \frac{\sigma - i\omega\epsilon}{c} E_3 &= \frac{\partial H_2}{\partial x_1}, \end{aligned} \tag{4}$$

where σ, ϵ and μ are, respectively, the conductivity, the dielectric permeability and the magnetic permeability of the dielectric between the two conducting planes.

Let us now try to represent the wave in question in the form of a superposition of TE and TM waves (section 1). When we examine the system of equations (4), we see that it can be divided into two systems

$$\frac{i\omega\mu}{c} H_1 = -\frac{\partial E_2}{\partial x_3}, \quad \frac{i\omega\mu}{c} H_3 = \frac{\partial E_2}{\partial x_1}, \tag{5a}$$

$$\frac{\sigma - i\omega\epsilon}{c} E_2 = \frac{\partial H_1}{\partial x_3} - \frac{\partial H_3}{\partial x_1} \tag{5b}$$

and

$$\frac{\sigma - i\omega\epsilon}{c} E_1 = -\frac{\partial H_2}{\partial x_3}, \quad \frac{\sigma - i\omega\epsilon}{c} E_3 = \frac{\partial H_2}{\partial x_1}, \tag{6a}$$

$$\frac{i\omega\mu}{c} H_2 = \frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}. \tag{6b}$$

Only the variables H_1, H_3 and E_2 appear in the first system, and only E_1, E_3 and H_2 appear in the second. Therefore, the solutions of these systems can be mutually dependent only when the boundary conditions for the two groups of variables (H_1, H_3 and E_2, on the one hand, and E_1, E_3 and H_2, on the other) are mutually dependent, which, in a number of interesting problems, we know is not the case.

Thus, if the boundary conditions can be broken into two groups such that only the variables H_1, H_3 and E_2 appear in one of them and only E_1, E_3 and H_2 appear in the other, and if system (4) has a solution, it can be constructed from the solutions of system (5) and system (6). But, on the basis of the relations (1) - (2), the solution of system (5) corresponds to TE waves and the solution of system (6) corresponds to TM waves. Consequently, under certain conditions, TE and TM waves can actually exist and be propagated independently of each other.

Finding the TE and TM waves then reduced to solving a single scalar Helmholtz equation. For it follows from eqs. (5a) and (6a) that the components H_1 and H_3, on the one hand, and E_1 and E_3, on the other, can be obtained by differentiating the components E_2 and H_2, respectively; hence it is sufficient to determine these last two. But, as we know (section 1 of Chapter XXXIV), each of the components of the field vectors satisfies Helmholtz' equation *. Therefore, we arrive at two Helmholtz equations, the first of which determines the TE waves and the second the TM waves:

$$\frac{\partial^2 E_2}{\partial x_1{}^2} + \frac{\partial^2 E_2}{\partial x_3{}^2} + k^2 E_2 = 0 , \tag{7a}$$

$$\frac{\partial^2 H_2}{\partial x_1{}^2} + \frac{\partial^2 H_2}{\partial x_3{}^2} + k^2 H_2 = 0 , \tag{7b}$$

where, from eq. (15) of Chapter XXXIV,

$$k^2 = \frac{\omega^2 \epsilon \mu + i\omega\mu\sigma}{c^2} . \tag{8}$$

Let us now examine the boundary conditions. As we know (Chapter XXV, section 1), the solution of Helmholtz' equation in an infinite region is determined when a linear combination of a desired function and its normal derivative is given on the boundary of the region, and when the conditions at infinity are given. If the radiation condition is satisfied at infinity, the solution is unique. In particular, in this case, the field at infinity approaches zero. However, we shall not be concerned with solutions of this type in the present chapter, since in the study of directed waves, the chief interest is in waves that are not damped in the direction of radiation and, therefore, do not decrease without limit at distant points. Therefore, for a condition at infinity, we make the requirement that the solution be bounded at an infinitely distant point. Here, the solutions to the Helmholtz equation will

* Of course, this can also be easily established by eliminating the variables from the systems (5) and (6).

clearly not be unique and our purpose is to show just which types of wave these solutions apply to.

Let us now consider the requirements that may be made on the solution at the boundary. According to section 3 of Chapter XXXIV, the tangential component of the electric vector and the normal component of the magnetic vector must vanish on the boundary of an ideal conductor; that is,

$$E_2 = E_3 = 0 \,, \tag{9}$$

$$H_1 = 0 \,. \tag{10}$$

Here, the second of these conditions is a consequence of the first. We do not need to write the boundary conditions for the components E_1, H_2 and H_3 since the jumps in E_1, H_2 and H_3 at the boundary are determined by the electric layers induced by the wave passing through it, and by the surface currents. The boundary values of E_1, H_2 and H_3 that are given by the solution make it possible to determine these layers and the currents.

Let us show that if we take

$$\text{for } x_1 = 0 \text{ and } x_1 = s \qquad E_2 = 0 \,, \tag{11a}$$

$$\text{for } x_1 = 0 \text{ and } x_1 = s \qquad \partial H_2/\partial x_1 = 0 \,, \tag{11b}$$

the conditions (9) will be satisfied. On the basis of the second of eqs. (6a) we have $E_3 = 0$ for $x_1 = 0$ and $x_1 = s$. On the basis of the first of eqs.(5a), $H_1 = 0$ since the variable E_2 (being equal to zero) does not vary along the boundary.

Let us solve eqs. (7) by separating the variables. We, therefore, write the desired solution in the form of the product of two functions $U(x_1)$ and $V(x_3)$. After substituting these functions into the equation and separating the variables, we obtain

$$\frac{d^2U}{dx_1{}^2} + k_n{}^2 U = 0 \,, \qquad \frac{d^2V}{dx_3{}^2} - \gamma_n{}^2 V = 0 \,,$$

where

$$\gamma_n{}^2 = k_n{}^2 - k^2$$

and $k_n{}^2$ is an arbitrary number. The general solutions of these equations are

$$U(x_1) = A_1 \sin k_n x_1 + A_2 \cos k_n x_1 \,, \tag{12}$$

$$U(x_3) = B_1\, e^{-\gamma_n x_3} + B_2\, e^{\gamma_n x_3} \tag{13}$$

When we impose the boundary condition (11a) on $U(x_1)$ we find that $A_2 = 0$ and thus obtain the following expression for the eigenvalues of the boundary-value problem determining TE waves:

$$k_n = \pi n/s \qquad (n = 0, 1, 2, \dots) \,. \tag{14}$$

Therefore, except for an arbitrary constant factor A_1, we have

$$U(x_1) = \sin\,(n\pi x_1/s) \qquad (n = 0, 1, 2, \dots) \,. \tag{15}$$

Here, the eigenvalue $k_n = 0$ corresponds to the trivial zero solution.

When we impose the boundary conditions (11b) on $U(x_1)$, we obtain $A_1 = 0$ and the same spectrum (14) of eigenvalues. From this it follows that, for TM waves, we have, up to an insignificant factor,

$$U(x_1) = \cos n\pi \frac{x_1}{s} . \tag{16}$$

To find $V(x_3)$, we substitute the value of k_2 given by (8) and $k_n{}^2 = \pi^2 n^2/s^2$ into the expression for $\gamma_n{}^2$. This gives

$$\gamma_n{}^2 = \frac{\pi^2}{s^2} n^2 - \frac{\epsilon\mu}{c^2} \omega^2 - i \frac{\omega\mu\sigma}{c} . \tag{17}$$

If the conductivity σ of the dielectric between the conducting planes is non-zero, $\gamma_n{}^2$ will be a complex number. Therefore, its root γ_n will have a non-zero real part. Let us substitute γ_n into eq. (13). Because of the condition at infinity, we need to keep only that term that remains bounded when x_3 increases. We therefore conclude that when $\sigma \neq 0$ the solution for E_2 or H_2 retains an exponentially decreasing factor. In this case, the wave process that we are considering is exponentially damped in the direction of propagation.

Let us now assume that we have a perfect dielectric, that is, $\sigma = 0$. Then,

$$\gamma_n{}^2 = \frac{\pi^2}{s^2} n^2 - \frac{\epsilon\mu}{c^2} \omega^2 .$$

For purposes of further analysis, it is convenient to transform this equation by expressing the angular frequency ω in terms of the wavelength λ.

We have

$$\omega = 2\pi c_1/\lambda ,$$

where c_1 is the velocity of propagation of the electromagnetic field in the dielectric between conducting planes. On the other hand,

$$c_1 = c/\sqrt{\epsilon\mu} ,$$

so that

$$\omega = 2\pi c\lambda/\sqrt{\epsilon\mu} ,$$

and

$$\gamma_n{}^2 = \frac{\pi^2 n^2}{s^2\lambda^2} \left(\lambda^2 - \frac{4s^2}{n^2}\right) . \tag{18}$$

There are several different cases, depending on the value of $\gamma_n{}^2$.

If

$$\lambda > 2s/n ,$$

γ_n will be a real number, and considerations analogous to those made in the case of non-zero conductivity show that the wave process is damped exponentially in the direction of propagation (for given values of λ and n). In

particular, for $\lambda > 2s$, only damped TE and TM waves are possible between the conducting planes.

If

$$\lambda = 2s/n \ ,$$

that is, if the distance between the two planes is equal to a half-integral number of wavelengths, $\gamma_n = 0$ and, on the basis of (14)-(16), the given values of λ and n correspond, up to a constant factor, to

$$E_2 = \sin \pi n \frac{x_1}{s}$$

for TE waves and

$$H_2 = \cos \pi n \frac{x_1}{s}$$

for TM waves. These solutions do not depend on the coordinate x_2. This means that at points of an arbitrary plane x_3 = constant ($x_1 \leqslant s$), electromagnetic oscillations take place in both waves with a single phase and with the same amplitude. Obviously, these oscillations represent standing waves. In the case of TM waves, we obtain a constant magnetic field for $n = 0$.

Finally, if

$$\lambda < 2s/n \ ,$$

we have

$$\gamma_n = \mathrm{i}\beta_n \ ,$$

where β_n is a non-zero real number. On the basis of eqs. (14) - (16), we have, for these values of λ and n, the solutions

$$E_2 = \sin \pi n \frac{x_1}{s} (B_1 \, \mathrm{e}^{-\mathrm{i}\beta_n x_3} + B_2 \, \mathrm{e}^{\mathrm{i}\beta_n x_3})$$

for TE waves and

$$H_2 = \cos \pi n \frac{x_1}{s} (B_1 \, \mathrm{e}^{-\mathrm{i}\beta_n x_3} + B_2 \, \mathrm{e}^{\mathrm{i}\beta_n x_3})$$

for TM waves, which correspond to waves with amplitudes that vary as

$$\mathrm{Re} \, \sin \pi n \frac{x_1}{s} \, \mathrm{e}^{-\mathrm{i}\beta_n x_3} \ , \qquad \mathrm{Re} \, \sin \pi n \frac{x_1}{s} \, \mathrm{e}^{\mathrm{i}\beta_n x_3} \ ,$$

$$\mathrm{Re} \, \cos \pi n \frac{x_1}{s} \, \mathrm{e}^{-\mathrm{i}\beta_n x_3} \ , \qquad \mathrm{Re} \, \cos \pi n \frac{x_1}{s} \, \mathrm{e}^{\mathrm{i}\beta_n x_3} \ .$$

For greater visual clarity, we write these expressions in terms of the time factor $\mathrm{e}^{-\mathrm{i}\omega t}$. This gives

$$\mathrm{Re} \, {\sin \atop \cos} \, \pi n \frac{x_1}{s} \, \mathrm{e}^{-\mathrm{i}(\beta_n x_3 + \omega t)} \ , \qquad \mathrm{Re} \, {\sin \atop \cos} \, \pi n \frac{x_1}{s} \, \mathrm{e}^{\mathrm{i}(\beta_n x_3 - \omega t)} \ ,$$

from which it is clear that these waves are travelling waves. In the first

and third waves, the phase is propagated along the x_3-axis in the negative direction, and in the second and fourth it is propagated in the positive direction. The phase velocity is the same in all cases:

$$v_\varphi = \omega/\beta_n \ . \tag{19}$$

The field of TE waves necessarily depends on the coordinate x_1. On the other hand, in the case of TM waves, we have, for $n = 0$, waves

$$H_2 = e^{\pm i\beta_n x_3} \ ,$$

which do not depend on the coordinate x_1. That is, they are plane waves. Because of the second of eqs. (6a), the longitudinal component of the electric vector E_3 is equal to zero in this case. We therefore conclude that this type of wave represents TEM waves.

Problems

1. Consider waves that are being propagated between two parallel ideally conducting planes. Investigate the behaviour of the H_1 and H_3 components in a TE wave, and the E_1 and E_3 components in a TM wave.
2. Investigate the distribution of currents and charges in conducting planes when waves pass through them.
   Method: Use the boundary conditions for E_1, H_2, and H_3.
3. Show that a standing wave corresponds to an infinitely large velocity of phase propagation of a travelling wave.
   Method: Use formula (19).
4. What new feature will be introduced into the solution of the problem of the propagation of waves between two planes by the requirement that the solution satisfy not only the boundary conditions (11), but also the radiation condition (Chapter XXV, section 5)?
5. Investigate the propagation of waves in a waveguide of the form of a of rectangular cross section with ideally conducting walls.

3. *Further examination of directed waves*

Each of the components of the vectors of an electromagnetic field, as we know, satisfies Helmholtz' equation. If we seek a solution of Helmholtz' equation in the form of a product $U(x_1, x_2)V(x_3)$, we obtain the equation

$$\frac{1}{U}\left(\frac{\partial^2 U}{\partial x_1{}^2} + \frac{\partial^2 U}{\partial x_2{}^2}\right) + \frac{1}{V}\frac{\partial^2 V}{\partial x_3{}^2} + k^2 = 0 \ ,$$

which, as is easy to see, can be broken into two equations,

$$\frac{\partial^2 U}{\partial x_1{}^2} + \frac{\partial^2 U}{\partial x_2{}^2} + (k^2 - \gamma^2)U = 0 \tag{20}$$

and

$$\frac{\partial^2 V}{\partial x_3{}^2} + \gamma^2 V = 0 , \tag{21}$$

where γ^2 is an arbitrary complex or real number. If γ^2 is a complex or a negative real number, the solutions $V(x_3)$ of eq. (21) that are bounded at infinitely distant points represent functions that decrease exponentially in the direction of x_3 (see, for example, section 2). The corresponding solutions $U(x_1,x_2)V(x_3)$ of the Helmholtz' equations therefore represent waves that are exponentially damped in the direction of x_3. However, if γ^2 is a positive real number, the general solution of eq. (21) will be

$$V(x_3) = B_1 \, e^{-i\gamma x_3} + B_2 \, e^{i\gamma x_3} \tag{22}$$

and the solution of Helmholtz' equation represents the superposition of two waves that are not damped in the x_3 direction. As we can easily see by multiplying eq. (22) by $e^{-i\omega t}$, the first of these is propagated along the negative x_3-axis for positive values of γ and the second along the positive axis. This class of waves is of especial interest in the theory of directed waves, since it includes waves that can be propagated along waveguides without being damped. In this connection, we shall concentrate our attention primarily on the problem of finding out just what waves of this class can be propagated along waveguides. Here, assuming that the parameter γ can be of either sign, we shall calculate the dependence on x_3 of each individual wave, by introducing into the solution the factor

$$e^{i\gamma x_3} , \tag{23}$$

where γ is an arbitrary real number called the constant of propagation. Obviously, it will be sufficient to consider the cases in which γ is positive, since when γ is negative only the direction of the travelling waves will change and the general picture will remain basically the same. We need only remember that if a forward wave is possible, so is a backward wave.

Thus, we have reduced the problem of directed waves to a study of fields which depend on the coordinate x_3 as shown by the expression (23). We note that the components of the field vectors will then satisfy not only the three-dimensional, but also the two-dimensional Helmholtz equation (20), and that the derivatives with respect to the coordinate x_3 satisfy the equations

$$\frac{\partial E_\alpha}{\partial x_3} = i\gamma E_\alpha , \qquad \frac{\partial H_\alpha}{\partial x_3} = i\gamma H_\alpha \qquad (\alpha = 1,2,3) . \tag{24}$$

We shall henceforth make the following simplifications.

In the preceding section, it was shown (for the case that we were examining then) that the amplitude of travelling waves decreases exponentially when propagated through an imperfect dielectric (one with conductivity different from zero). From a physical point of view, this phenomenon is a simple consequence of the fact that, when passing through an imperfect dielectric, waves cause currents and this leads to a dispersal of the energy

of the wave as a result of Joule heating. Since a study of this process is not part of our problem, we shall assume that the waves are propagated in a perfect dielectric with conductivity $\sigma = 0$.

If we substitute eqs. (24) into the field equations (12)-(13) of Chapter XXXIV, set σ equal to zero, and solve the resulting equations for the transverse components, we obtain

$$\begin{aligned}
(k^2-\gamma^2)E_1 &= \mathrm{i}\gamma\frac{\partial E_3}{\partial x_1} + \frac{\mathrm{i}\omega\mu}{c}\frac{\partial H_3}{\partial x_2}, \\
(k^2-\gamma^2)E_2 &= \mathrm{i}\gamma\frac{\partial E_3}{\partial x_2} - \frac{\mathrm{i}\omega\mu}{c}\frac{\partial H_3}{\partial x_1}, \\
(k^2-\gamma^2)H_1 &= \mathrm{i}\gamma\frac{\partial H_3}{\partial x_1} - \frac{\mathrm{i}\omega\epsilon}{c}\frac{\partial E_3}{\partial x_2}, \\
(k^2-\gamma^2)H_2 &= \mathrm{i}\gamma\frac{\partial H_3}{\partial x_2} + \frac{\mathrm{i}\omega\epsilon}{c}\frac{\partial E_3}{\partial x_1},
\end{aligned} \tag{25}$$

$$k^2 \equiv \omega^2\epsilon\mu/c^2 , \tag{26}$$

from which it is clear that all the transverse components of the vectors of the field can, for $k^2-\gamma^2 \neq 0$, be found by simple differentiation of the longitudinal components. Let us find these longitudinal components as solutions of the two-dimensional Helmholtz equations of the form (20):

$$\frac{\partial^2 E_3}{\partial x_1^2} + \frac{\partial^2 E_3}{\partial x_2^2} + (k^2-\gamma^2)E_3 = 0 , \tag{27}$$

$$\frac{\partial^2 H_3}{\partial x_1^2} + \frac{\partial^2 H_3}{\partial x_2^2} + (k^2-\gamma^2)H_3 = 0 . \tag{28}$$

The case of TEM waves for which $E_3 = H_3 = 0$ occupies a special position. It follows from (25) that this type of wave is possible only when $k^2 = \gamma^2$, since when $k^2 \neq \gamma^2$ all components of TEM waves must be equal to zero. If we substitute the value of k^2 given by eq. (26), we see that for TEM waves

$$\gamma = \omega\sqrt{\epsilon\mu}/c . \tag{29}$$

The system (25) is inapplicable for determining the transverse components of a TEM wave. If we substitute γ for $\omega\sqrt{\epsilon\mu}/c$ in the field equations (12) - (13) of Chapter XXXIV, we find, after some simple manipulation, that the system (12) - (13) of that chapter is reduced, in the case of TEM waves, to the four relationships

$$\sqrt{\epsilon}\,E_1 = \sqrt{\mu}\,H_2 , \qquad \sqrt{\epsilon}\,E_2 = -\sqrt{\mu}\,H_1 , \tag{30}$$

$$\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} = 0 , \qquad \frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} = 0 . \tag{31}$$

If we substitute the values of H_1 and H_2 given by eqs. (30) into eqs. (31), we obtain

$$\frac{\partial E_1}{\partial x_2} - \frac{\partial E_2}{\partial x_1} = 0 \, .$$

If we differentiate this equation with respect to x_2, then differentiate the first of eqs. (31) with respect to x_1, and add the results, we obtain

$$\frac{\partial^2 E_1}{\partial x_1{}^2} + \frac{\partial^2 E_1}{\partial x_2{}^2} = 0 \, .$$

In an analogous manner, we obtain equations of the same form for the remaining component, so that, in general,

$$\Delta_{12}E_1 = 0 \, , \quad \Delta_{12}E_2 = 0 \, , \quad \Delta_{12}H_1 = 0 \, , \quad \Delta_{12}H_2 = 0 \, , \tag{32}$$

where

$$\Delta_{12} = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2}$$

is the two-dimensional Laplacian operator. (We note that eqs. (32) are also simple consequences of eq. (20) for $\gamma^2 = k^2$.)

Thus, in the general case, the situation is somewhat analogous to that examined in section 2. To solve the problem of the propagation of TM or TE waves, we need to find the solution of the scalar Helmholtz equation which determines the longitudinal component of the electric and magnetic vectors; the transverse components of the field vectors can be found by differentiation. The problem of the propagation of TEM waves reduces to Laplace's equation; that is, the harmonic functions that we studied in detail earlier are solutions of it.

If a waveguide is an ideal conductor, as was shown in section 3 of Chapter XXXIV, the tangential component E_T of the electric vector must vanish on its boundaries. Therefore, the solutions of eqs. (27) - (28) or (32) must satisfy the boundary condition (on the boundary of the waveguide)

$$E_T = 0 \, . \tag{33}$$

If we choose admissible values of γ^2 that give a solution to the problem, this condition will be satisfied.

Let us draw some conclusions from these results.

We begin with transverse electromagnetic waves. If we substitute the expression given by eq. (29) for γ into expression (23) and multiply by $e^{-i\omega t}$, we obtain

$$e^{ik(x_3 - c_1 t)} \, , \tag{34}$$

where $c_1 = c/\sqrt{\epsilon\mu}$ is the velocity of propagation of electromagnetic waves in a dielectric with permeabilities ϵ and μ (see section 1 of Chapter XXXIV). Thus, a TEM wave always represents a travelling wave whose velocity of propagation c_1 is independent of the frequency ω. Therefore, in particular,

any combination of TEM waves of different frequencies which forms a wave of composite profile, is propagated in such a way that this profile is maintained. As we know from the theory of Fourier integrals, we can, by a superposition of harmonic waves that extend infinitely far out into space, obtain a composite wave (wave packet) whose amplitude is non-zero only in a finite region of space. From what has been said, such a wave formed by superposition of TEM waves will be propagated with no distortion in shape, all the while remaining in a bounded portion of space. This fact indicates that TEM waves are propagated without dispersion.

It is easy to see that eqs. (30) imply perpendicularity of the electric and magnetic vectors in a TEM wave, and that the absolute values of the mutually perpendicular components of the electric and magnetic vectors are proportional. We give the reader an elementary proof of this assertion. The coefficient of proportionality

$$\eta = \sqrt{\mu/\epsilon}$$

depends only on the properties of the dielectric in which the wave is being propagated and is called the characteristic resistance of the dielectric.

We note finally that, on the basis of eqs. (32), the components of the field vectors of a TEM wave are harmonic functions of the arguments x_1 and x_2 in an arbitrary plane x_3 = constant. This indicates, in the first place, that the picture of the field of a TEM wave at an arbitrary fixed instant of time coincides with the picture of static electric and magnetic fields that arise under analogous boundary conditions. In the second place, it follows that these components are derivatives with respect to x_1 and x_2 of the corresponding potentials, which are also harmonic functions.

It is obvious from this last fact that TEM waves cannot be propagated inside a waveguide having conducting boundaries which enclose the field of the TEM wave and whose intersection with the plane x_3 = constant forms a closed simply-connected curve. This is true because the charges on the surface of the conductor form, as we know (Chapter XXII, section 2), a homogeneous layer whose electrostatic potential is the same at all points. But a harmonic function that is constant over some closed curve has the same constant value within the curve as well. Therefore, the components of the electric vector (and the magnetic vector as well, as a result of the relations (30)), being derivatives of the potential, are in this case identically equal to zero, thus there is no field in the waveguide. However, if the field of the wave is not enclosed in a conductor, this phenomenon does not occur. Therefore, a wire *outside* which a wave is being propagated (or a system of wires) can serve as a waveguide for TEM waves, but the inner surface of an empty cylinder, for example, cannot.

Let us now consider transverse magnetic (TM) waves. In waves of this type, the component $H_3 = 0$. The problem of the propagation of TM waves, by virtue of our previous remarks, is reduced to solving a single scalar Helmholtz equation.

First of all, let us consider the problem of the boundary conditions, which is analogous to the problem considered in section 2. The solution of Helmholtz' equation is completely determined by one boundary condition,

which on the basis of (33), is the condition that, on the boundary of the waveguide,

$$E_3 = 0 \, . \tag{35}$$

But the solution must satisfy yet another boundary condition, namely that the tangential component of the electric vector in the transverse plane must be equal to zero. Let us show that this condition is automatically satisfied on the basis of eqs. (25).

For $H_3 = 0$, the system of eqs. (25) can be written in the following form

$$\frac{k^2 - \gamma^2}{i\gamma} E_1 = \frac{(k^2 - \gamma^2)c}{i\omega\epsilon} H_2 = \frac{\partial E_3}{\partial x_1} \, , \tag{36}$$

$$\frac{k^2 - \gamma^2}{i\gamma} E_2 = - \frac{(k^2 - \gamma^2)c}{i\omega\epsilon} H_1 = \frac{\partial E_3}{\partial x_2} \, , \tag{37}$$

from which we find that

$$E_1 = \frac{c\gamma}{\omega\epsilon} H_2 \, , \qquad E_2 = - \frac{c\gamma}{\omega\epsilon} H_1 \, ;$$

that is, the transverse components of the electric and magnetic vectors are mutually perpendicular and the absolute values of the mutually perpendicular components are proportional. In contrast with the TEM waves, the proportionality coefficients also depend on those properties of the waveguide which exert influence on the value of the constant of propagation γ. Since the magnetic vector of a TM wave is transverse, it follows from what has been said that the magnetic vector of a TM wave is perpendicular to the electric vector.

Suppose that ψ is the angle between the tangent T to the surface of the conductor at some point on the cross section of the waveguide and the x_1-axis. The derivative of the component E_3 in the direction of T, which is equal to

$$\frac{\partial E_3}{\partial T} = \frac{\partial E_3}{\partial x_1} \cos \psi + \frac{\partial E_3}{\partial x_2} \sin \psi \, ,$$

vanishes on the surface of the conductor because of condition (35). Since, on the basis of eqs. (36), the tangential component of the electric vector is equal to

$$E_T = E_1 \cos \psi + E_2 \sin \psi = \frac{i\gamma}{k^2 - \gamma^2} \frac{\partial E_3}{\partial T} \, ,$$

we conclude that, when the boundary condition (35) is satisfied for points on the boundary, the general boundary condition $E_T = 0$ will also be satisfied. Thus, on the basis of the field equations, the boundary condition (35) implies that the general boundary condition for the electric vector is satisfied.

For future analysis, let us use the integral relationship shown in problem 7 of section 6 of Chapter XXV (obtained in studying Helmholtz' equation). We set $u = E_3$ in that equation. If, in accordance with eq. (27), we replace k^2 by $k^2 - \gamma^2$, we obtain

$$\iiint_V \left[\left(\frac{\partial E_3}{\partial x_1}\right)^2 + \left(\frac{\partial E_3}{\partial x_2}\right)^2 + \left(\frac{\partial E_3}{\partial x_3}\right)^2 - (k^2-\gamma^2)E_3{}^2\right] \mathrm{d}V = \iint_{\mathcal{F}V} E_3 \frac{\partial E_3}{\partial n}\,\mathrm{d}S\,.$$

Let us assume that the waveguide in question is closed, that is, that the field is bounded by conducting surfaces in the x_1x_2-plane. We take as V the volume bounded by the conducting walls and two arbitrary cross sections of the waveguide. Then, the integral on the right side of the above equation vanishes since, on the walls of the waveguide, $E_3 = 0$, and on the cross sections

$$\frac{\partial E_3}{\partial n} = \frac{\partial E_3}{\partial x_3} = 0\,,$$

since the component E_3 does not depend on x_3. For the same reason, one term drops out of the left side of this equation so that we finally have

$$\iiint_V \left[\left(\frac{\partial E_3}{\partial x_1}\right)^2 + \left(\frac{\partial E_3}{\partial x_2}\right)^2 - (k^2-\gamma^2)E_3{}^2\right] \mathrm{d}V = 0\,.$$

If E_3 is not identically equal to zero, this equation can hold only if

$$k^2 - \gamma^2 > 0\,.$$

Recalling eq. (26), we write this condition in the form

$$\gamma^2 = \frac{\omega^2}{c_1{}^2} - \delta^2\,, \tag{38}$$

where $c_1{}^2 = c^2/\epsilon\mu$ is the square of the velocity of propagation of the electromagnetic field in the dielectric filling the waveguide and δ is a real number. As we know from the general theory of Chapter XXV, under these boundary conditions non-trivial solutions of the Helmholtz eq. (27) do not generally exist for all values of the difference $\delta^2 = k^2 - \gamma^2$. The values of δ^2 at which non-trivial solutions exist (the eigenvalues of the problem) form an infinite sequence. Suppose that $\delta_1{}^2$ is the smallest of the numbers in this sequence. Then, we find from (38) that γ^2 is negative if

$$\omega^2 < \omega_0{}^2 \equiv c_1{}^2\delta_1{}^2\,; \tag{39}$$

that is, γ is an imaginary number and we are dealing with a process that is damped in the x_3 direction. Thus, in the general case, there exists a critical frequency ω_0 (usually called the cut-off frequency) characterizing the waveguide, with the property that TM waves with frequency less than ω_0 cannot be propagated in the waveguide without being damped.

If we make the substitution

$$\gamma = \frac{\omega}{c_1}\sqrt{1 - \frac{\omega_0{}^2}{\omega^2}} \tag{40}$$

in eq. (23) and then multiply by $\mathrm{e}^{-\mathrm{i}\omega t}$, we find that the dependence of the amplitude of the wave on the coordinate x_3 and on time is given, for TM waves in a closed waveguide, by the factor

$$\exp\left[i \frac{2\pi}{\lambda} (x_3 - c^* t)\right],$$

where

$$c^* = \frac{c_1}{\sqrt{1 - (\omega_0{}^2/\omega^2)}} \tag{41}$$

in the phase velocity and

$$\lambda = \frac{2\pi c_1}{\sqrt{\omega^2 - \omega_0{}^2}} \tag{42}$$

is the length of a TM wave with frequency of vibration ω.

It follows from eq. (41) that the phase velocity of TM waves is greater than the velocity of propagation of the electromagnetic field in the dielectric filling the waveguide. The phase velocity also depends on the frequency of oscillation. This last fact indicates that dispersion occurs when TM waves are propagated. The wave packet formed by TM waves, which is originally localized in the bounded portion of the waveguide, "spreads out" more and more with the passage of time, increasing in length. The group velocity of TM waves (Chapter XXIV, section 2) is equal to

$$c_2^* \equiv \frac{\partial \omega}{\partial \gamma} = c_1 \sqrt{1 - \frac{\omega_0{}^2}{\omega^2}}\,.$$

The center of the wave packet is propagated with this velocity.

The general case of the propagation of TE waves in waveguides can be studied in an analogous manner. The difference consists only in the fact that the boundary conditions for the Helmholtz equation (28) must be written in the form

$$\partial H_3/\partial n = 0\,, \tag{43}$$

on the boundary of the waveguide; that is, it is the homogeneous Neumann problem and not the Dirichlet problem, that must be solved. The same expressions for the cut-off frequency and the phase and group velocities will be obtained as for TM waves. Verification of this statement is left to the reader.

Problems

1. The ratio of the absolute values of the transverse components of the electric and magnetic vectors of a wave is called the wave resistance. Show that the wave resistances Z_{TM} and Z_{TE} for TM and TE waves are given, respectively, by the expressions

$$Z_{TM} = c\gamma/\omega\epsilon\,, \qquad Z_{TE} = \omega\mu/c\gamma\,,$$

which, in the case of hollow waveguides, can be represented as follows

$$Z_{TM} = \sqrt{\frac{\mu}{\epsilon}\left(1 - \frac{\omega_0^2}{\omega^2}\right)}\,, \qquad Z_{TE} = \frac{1}{\sqrt{\frac{\epsilon}{\mu}\left(1 - \frac{\omega_0^2}{\omega^2}\right)}}\,.$$

2. Show that, in cylindrical coordinates (r, φ, z) with the z-axis directed along the x_3-axis, eqs. (25) take the form

$$(k^2 - \gamma^2)E_r = \mathrm{i}\gamma \frac{\partial E_z}{\partial r} + \frac{\mathrm{i}\omega\mu}{c}\frac{1}{r}\frac{\partial H_z}{\partial \varphi}\,,$$

$$(k^2 - \gamma^2)E_\varphi = \mathrm{i}\gamma \frac{1}{r}\frac{\partial E_z}{\partial \varphi} - \frac{\mathrm{i}\omega\mu}{c}\frac{\partial H_z}{\partial r}\,,$$

$$(k^2 - \gamma^2)H_r = \mathrm{i}\gamma \frac{\partial H_z}{\partial r} - \frac{\mathrm{i}\omega\epsilon}{c}\frac{1}{r}\frac{\partial E_z}{\partial \varphi}\,,$$

$$(k^2 - \gamma^2)H_\varphi = \mathrm{i}\gamma \frac{1}{r}\frac{\partial H_z}{\partial \varphi} + \frac{\mathrm{i}\omega\epsilon}{c}\frac{\partial E_z}{\partial r}\,.$$

3. Show that in cylindrical coordinates oriented as in the preceding problem the system of equations (30) - (31) takes the following form for **TEM** waves:

$$\sqrt{\epsilon}\, E_r = \sqrt{\mu}\, H_\varphi\,, \qquad \sqrt{\epsilon}\, E_\varphi = \sqrt{\mu}\, H_r\,,$$

$$\frac{\partial rE_\varphi}{\partial r} - \frac{\partial E_r}{\partial \varphi} = 0\,, \qquad \frac{\partial rH_\varphi}{\partial r} - \frac{\partial H_r}{\partial \varphi} = 0\,.$$

Use these equations to show that the products rE_r, rR_φ, rH_r, and rH_φ satisfy the two-dimensional Laplace equation

$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 u}{\partial \varphi^2} = 0\,.$$

4. *TM wave in a waveguide of circular cross section*

Let us examine the problem of the propagation of TM waves in a hollow circular conducting cylinder of infinite length.

According to section 3, this problem is reduced to the homogeneous Dirichlet problem for Helmholtz' equation (27), which in cylindrical coordinates (r, φ, z) is of the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \varphi^2} + \delta^2 E_z = 0\,, \tag{44}$$

$$\delta^2 \equiv k^2 - \gamma^2\,. \tag{45}$$

Here, the z-axis is assumed to be directed along the axis of the cylinder (corresponding to the x_3-axis of the coordinate system used in section 3).

We can separate the variables in (44) by making the substitution $E_2 = u(\varphi)v(r)$. We then have the equations

$$\frac{d^2u}{d\varphi^2} + n^2u = 0 , \quad \frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} + \left(\delta^2 - \frac{n^2}{r^2}\right) v = 0 \tag{46}$$

Since the function u is obviously of period 2π with respect to the coordinate φ, the admissible values of n are

$$n = 0, 1, 2, \ldots .$$

The second equation is Bessel's equation. The boundary condition for E_3 gives the following boundary conditions for v:

$$v(r_0) = 0 , \tag{47}$$

where r_0 is the inner radius of the cylinder. The solutions of the second of eqs. (46) that satisfy the condition (47) and that are bounded for $r = 0$ are the n-th order Bessel functions $J_n(\delta_{nm}r)$, where the δ_{nm} are the roots of the equation

$$J_n(\delta_{nm}r_0) = 0 . \tag{48}$$

Thus, the particular solutions of eq. (44) that satisfy the boundary condition $E_z = 0$ are of the form

$$E_{z,nm} = J_n(\delta_{nm}r)\,(A_{nm}\cos n\varphi + B_{nm}\sin n\varphi) , \tag{49}$$

where A_{nm} and B_{nm} are arbitrary constants. Each of the solutions of this form corresponds to a TM wave with some position or other of the nodal line. If we use the equations listed in problem 2 of section 3, we can, by differentiating the expressions (49), find all the transverse components of the field.

For $n = 0$ and $m = 1$, the smallest root of eq. (48) is $\delta_1 = \delta_{01} = 2.405/r_0$. According to (39), this value determines the cutoff frequency $\omega_0 = 2.405c_1/r_0$. In air, the wavelength corresponding to the frequency ω_0 is equal to $\lambda_0 = 2.606 r_0\sqrt{\mu\epsilon}$. Longer waves cannot be propagated in air in such a waveguide without being damped. As is clear from expression (49), the field at $n = 0$ does not depend on the angular coordinate φ. If n is not equal to zero, the field does depend on φ, and there are $2n$ radial nodal lines $E_z = 0$. If $m > 1$, we have $m - 1$ nodal lines in the form of concentric circles with centers on the axis of the waveguide. To each of the TM waves characterized by definite values of n and m there corresponds a definite cutoff frequency

$$\omega_{nm} = c_1\frac{x_{nm}}{r_0} , \tag{50}$$

where x_{nm} is the m-th root of the equation $J_n(x) = 0$. Undamped waves with given n and m and with frequency less than ω_{nm} cannot be propagated in a waveguide without being damped. It should be noted that the excitation of high-frequency oscillations in a hollow with given n and m is in practice quite difficult. However, they appear to some extent as overtones, for example, in the case of the vibrations of a membrane.

Problems

1. Use the graph of the functions $J_m(x)$* to find the geometric positions of the points at which the components of the field vectors attain a maximum.
2. Find expressions determining the constant of propagation and the wavelength for given n and m in a waveguide of circular cross section.

5. *TE waves in a waveguide of circular cross section*

TE waves can be studied in the same way as TM waves (section 4), except that here we have the homogeneous Neumann problem rather than the Dirichlet problem for Helmholtz' equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial H_z}{\partial\varphi^2}+\delta^2 H_z=0 \qquad (\delta^2 \equiv k^2-\gamma^2)\ , \tag{51}$$

and on the boundary of the waveguide,

$$\partial H_z/\partial n = 0\ . \tag{52}$$

We shall examine only the differences in the propagation of TM and TE waves.

Particular solutions of eq. (51) satisfying the boundary condition (52) are of the form (49):

$$H_{z,nm} = J_n(\delta_{nm}r)\,(A_{nm}\cos n\varphi + B_{nm}\sin n\varphi)\ .$$

However, the eigenvalues δ_{nm} are now expressed by the equation

$$\delta_{mn} = q_{nm}/r_0\ , \tag{53}$$

where q_{nm} is the m-th root of the equation

$$\mathrm{d}J_n(x)/\mathrm{d}x = 0 \qquad (n = 0,1,2,\dots)\ . \tag{54}$$

Since $\mathrm{d}J_0/\mathrm{d}x = -J_1(x)$, the smallest of the numbers q_{0m} is equal to $x_{11} = 3.832$. This gives us for the cutoff frequency with $n = 0$ and $m = 1$, $\omega_{01} = 3.832c_1/r_0$. The wavelength in air corresponding to a frequency ω_{01} is equal to $\lambda_{01} = 1.639r_0\sqrt{\mu\epsilon}$. Thus, for TE waves, the cutoff frequency with $n = 0$ and $m = 1$ is 1.59 times as great as for TM waves. However, q_{01} is not the smallest of the numbers q_{nm}. The smallest is the root $q_{11} = 1.84$. This yields the smallest value for the cutoff frequency $\omega_0 = 1.84c_1/r_0$, corresponding to a wavelength in air of $\lambda_0 = 3.41r_0\sqrt{\mu\epsilon}$. This means that, for a given frequency, a waveguide can be used for transmitting TE waves that is approximately 23% smaller than the waveguide needed for the transmission of TM waves.

* See, for example, Jahnke and Emde [41], fig. 98a.

Problems

1. Solve problems 1 and 2 of section 4 for TE waves.
   Method: Use the relationship

$$dJ_n(x)/dx = \tfrac{1}{2}[J_{n-1}(x) - J_{n+1}(x)] \, .$$

2. What kind of undamped wave processes can be excited in a hollow ideally conducting cylinder of finite length.

6. *Waves in a coaxial cable*

A coaxial cable (or a coaxial line) is a directing system in which waves are propagated through a dielectric filling the space between two circular conducting cylinders with a common axis (fig. 69). The simplicity of construction and the effective screening of the field by the outer cylinder has led to widespread technical use of coaxial cables. Since the boundary of the dielectric in a coaxial cable is not simply-connected, TEM waves, whose frequencies are not restricted either by the cutoff conditions or by dispersion, can be propagated along it. We shall begin with an examination of these waves.

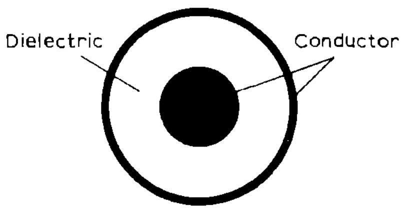

Fig. 69.

We introduce a cylindrical coordinate system (r, φ, z) with the z-axis directed along the common axis of the conducting cylinders. In this coordinate system, the tangential components of the electric vector and the normal component of the magnetic vector at the boundary of the conductors will be, respectively, equal to E_φ and H_r, so that the boundary conditions are of the form

$$E_\varphi = H_r = 0$$

at the boundary of the waveguide. But we know from problem 3 of section 3 that the product rE_φ and rH_r are harmonic functions in the annular region bounded by the conductors. Since they vanish on the boundary of the region, they are, by the theorem of section 4 of Chapter XVIII, identically equal to zero within the region. Thus, in a TEM wave,

$$E_\varphi = E_r = 0 \, .$$

From the equations given in problem 3 of section 3, we obtain

$$\frac{\partial E_r}{\partial \varphi} = 0 , \quad \frac{\partial r H_r}{\partial r} = 0 , \quad E_r = \sqrt{\frac{\mu}{\epsilon}} H_\varphi .$$

These relationships show that the field of a TEM wave in a coaxial cable does not depend on φ. The second and third give

$$H_\varphi = A/r , \quad E_r = \sqrt{\frac{\mu}{\epsilon}} \frac{A}{r} ,$$

where A is a constant. Thus, on the basis of (34) the field of a TEM wave in a coaxial cable is determined, up to an arbitrary constant factor, by the expression

$$\mathcal{H}_\varphi = \frac{e^{ik(z-c_1 t)}}{r} , \quad \mathcal{E}_r = \frac{e^{ik(z-c_1 t)}}{r} . \tag{55}$$

Let us turn now to TM and TE waves. Their longitudinal components satisfy Helmholtz' equations (44) and (51) and the boundary conditions

$$E_z \Big|_{\substack{r=r_i \\ r=r_a}} = 0 , \quad \frac{\partial H_z}{\partial r} \Big|_{\substack{r=r_i \\ r=r_a}} = 0 , \tag{56}$$

where r_i and r_a are the radii of the inner and outer cylindrical surfaces. As in sections 4 and 5, particular solutions of these equations can be represented in the form of the product

$$Z_n(\delta_{nm} r) \, (A_{nm} \cos n\varphi + B_{nm} \sin n\varphi) , \tag{57}$$

where Z_n is a cylindrical function of order n, n is an arbitrary positive integer, and δ_{nm} is the eigenvalue of the corresponding problem. In contrast with the case of waveguides of circular cross sections, there is now no basis for discarding those solutions of the second of eqs. (46) that become infinite at $r = 0$, since the point $r = 0$ does not belong to the field. Therefore, we must set

$$Z_n(\delta_{nm} r) = a_n J_n(\delta_{nm} r) + b_n Y_n(\delta_{nm} r) ,$$

which, because of the boundary conditions (56), yields the following equations in a_n and b_n in the case of TM waves:

$$a_n J_n(\delta_{nm} r_i) + b_n Y_n(\delta_{nm} r_i) = 0 , \quad a_n J_n(\delta_{nm} r_a) + b_n Y_n(\delta_{nm} r_a) = 0 . \tag{58}$$

Non-zero solutions of this system exist only when its determinant vanishes, that is, when

$$J_n(\delta_{nm} r_i) \, Y_n(\delta_{nm} r_a) - J_n(\delta_{nm} r_a) \, Y_n(\delta_{nm} r_i) = 0 . \tag{59}$$

For given n, the eigenvalues $\delta_{n1}, \delta_{n2}, \ldots$, giving the solution of the problem, are determined by this equation.

In the case of TE waves, it is not the functions J_n and Y_n, but their derivatives with respect to r that appear in the system of equations for determining the coefficients a_n and b_n.

Further investigation of the propagation of waves in a coaxial cable is left to the reader. The tables given by Jahnke and Emde can be used for determining the roots of eq. (59) [41].

Problem

Study the case in which r_a approaches infinity (the case of a cylindrical wire).

7. *Waves in a dielectric rod*

In the preceding sections, we examined waveguides constructed by means of conducting surfaces. It is noteworthy that a waveguide can be formed by a rod made of a dielectric.

Let us consider such a rod. Suppose that it has the form of a circular cylinder of radius r_0 and that it is located in a dielectric medium. In what follows, we shall denote quantities referring to the medium and the field in it by the subscript a and retain, for the field inside the rod, the same notations as we used in the preceding sections for the field within waveguides having conducting walls. Let us examine the problem in terms of cylindrical coordinates r, φ and z, where the z-axis is placed along the axis of the rod.

The general procedure of solution remains the same as that explained in section 3 and applied in section 4. To solve the problem, it is thus sufficient to know only the longitudinal components of the electric vector; the remaining components can then be found by differentiation (see problem 2 of section 3). However, in the present case, this component will satisfy two Helmholtz' equations of the form (44); one of them will have to do with the rod and the other with the medium. Specifically, for the field in the rod, we have the equations

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial E_z}{\partial \varphi}+\delta^2 E_z=0\,, \tag{60}$$

$$\delta^2=\frac{\omega^2\epsilon\mu}{c^2}-\gamma^2\,, \tag{61}$$

and for the field in the medium, we have the equations

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_{za}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial E_{za}}{\partial \varphi}+\delta_a{}^2 E_{za}=0\,, \tag{62}$$

$$\delta_a{}^2=\frac{\omega^2\epsilon_a{}^2\mu_a{}^2}{c^2}-\gamma_a{}^2\,. \tag{63}$$

Here, in view of formula (26), we have written the expression (45) for the parameters δ and δ_a.

Let us show that it is possible for undamped TM waves, with amplitude

that rapidly decreases with increasing distance from the electric rod, to be propagated along the rod.

When we examine TM waves on the boundary of the rod, the conjugacy conditions (42) of Chapter XXXIV (appearing in the equation for the tangential components of the field inside and outside the rod) must be satisfied:

$$E_z = E_{z\mathrm{a}}\,,\qquad E_\varphi = E_{\varphi\mathrm{a}}\,,\qquad H_\varphi = H_{\varphi\mathrm{a}}\qquad \text{when } r = r_0\,.$$

Here, there is no condition on the components H_z and $H_{z\mathrm{a}}$, since in TM waves they are equal to zero. It follows from the equations given in problem 2 of section 3 that these conditions are equivalent to the following ones:

$$E_z|_{r=r_0} = E_{z\mathrm{a}}|_{r=r_0}\,, \tag{64}$$

$$\frac{\gamma}{\delta^2}\frac{\partial E_z}{\partial\varphi}\bigg|_{r=r_0} = \frac{\gamma_\mathrm{a}}{\delta_\mathrm{a}^2}\frac{\partial E_{z\mathrm{a}}}{\partial\varphi}\bigg|_{r=r_0}\,, \tag{65}$$

$$\frac{\epsilon}{\delta^2}\frac{\partial E_z}{\partial r}\bigg|_{r=r_0} = \frac{\epsilon_\mathrm{a}}{\delta_\mathrm{a}^2}\frac{\partial E_{z\mathrm{a}}}{\partial r}\bigg|_{r=r_0}\,, \tag{66}$$

which contain only the components E_z and $E_{z\mathrm{a}}$.

In addition to eqs. (60) and (62), the longitudinal component of the electric vector must also satisfy equations of the form (21):

$$\frac{\partial^2 E_z}{\partial z^2} + \gamma^2 E_z = 0\,,\qquad \frac{\partial^2 E_{z\mathrm{a}}}{\partial z^2} + \gamma_\mathrm{a}^2 E_{z\mathrm{a}} = 0\,, \tag{67}$$

which determine its dependence on the coordinate z. As we saw in section 3, we need to choose from among the solutions of these equations those for which the dependence on the coordinate z is given by factors of the form $\mathrm{e}^{\mathrm{i}\gamma z}$ and $\mathrm{e}^{\mathrm{i}\gamma_\mathrm{a} z}$. Since the tangential components are then equal, it immediately follows that

$$\gamma_\mathrm{a} = \gamma\,. \tag{68}$$

As we know (Chapter XXV, section 4) particular solutions of the system of eqs. (60), (62) and (67) that meet this requirement on the dependence on the coordinate z and that are of period 2π with respect to the angular coordinate φ can be written in the form

$$E_z = A_n Z_n(\delta r)\,\mathrm{e}^{\mathrm{i}\gamma z}\cos(n\varphi + \psi_n)\,, \tag{69}$$

$$E_{z\mathrm{a}} = B_n Z_n(\delta_\mathrm{a} r)\,\mathrm{e}^{\mathrm{i}\gamma_\mathrm{a} z}\cos(n\varphi + \psi_{n\mathrm{a}})\,, \tag{70}$$

where A_n, B_n, ψ_n, and $\psi_{n\mathrm{a}}$ are constants and $Z_n(\delta r)$ and $Z_n(\delta_\mathrm{a} r)$ are cylindrical functions.

For the component of the electric vector in the medium surrounding the rod, we shall seek a solution that decreases rapidly with increasing r. It follows from the asymptotic formulae of Chapter XII that the solutions of Bessel's equation that decrease rapidly with increasing values of the argument are the Hankel functions of the first kind $H_n^{(1)}(\zeta)$ with imaginary values of the argument. Therefore, let us set

$$Z_n(\delta_a r) = H_n^{(1)}(i\beta r) ,$$

where β is a real number and

$$i\beta \equiv \delta_a . \tag{71}$$

It is more convenient to change from Hankel's functions with imaginary argument to Macdonald's functions (Chapter XII, section 7):

$$K_n(\zeta) = \tfrac{1}{2}\pi\, i^{n+1}\, H_n^{(1)}(i\zeta) ,$$

which have real values for all real values of ζ. Then, eq. (70) can be written in the form

$$E_{za} = B_n K_n(\beta r)\, e^{i\gamma z} \cos(n\varphi + \psi_{na}) . \tag{72}$$

Since the field in the rod must be bounded, we need to set

$$Z_n(\delta r) = J_n(\delta r) ,$$

in expression (69), which yields

$$E_z = A_n J_n(\delta r)\, e^{i\gamma z} \cos(n\varphi + \psi_n) . \tag{73}$$

When we substitute these expressions into the boundary condition (64), we obtain

$$A_n J_n(\delta r_0) \cos(n\varphi + \psi_n) = B_n K_n(\beta r_0) \cos(n\varphi + \psi_{na}) .$$

Since the functions $J_n(\zeta)$ and $K_n(\zeta)$ have no common roots, this equation can be satisfied for all values of φ only when

$$\cos(n\varphi + \psi_n) = \cos(n\varphi + \psi_{na}) ,$$

so that

$$\psi = \psi_{na} . \tag{74}$$

Let us turn to the boundary condition (65). For $n = 0$, that is, for TM waves with amplitude depending only on the coordinate r, it is satisfied identically. However, if n is not zero, it follows from eqs. (68) and (71) and from the expressions for $E_z = E_{za}$ that the boundary condition (65) is equivalent to

$$\frac{1}{\delta^2} E_z = -\frac{1}{\beta} E_{za} .$$

It then follows on the basis of the boundary condition (64) that

$$\beta^2 = -\delta^2 \quad \text{when} \quad n \neq 0 , \tag{75}$$

which, in view of the relations (61) and (63), can be written in the form

$$\epsilon_a \mu_a = \epsilon\mu \quad \text{when} \quad n \neq 0 . \tag{76}$$

Consequently, in the general case, if $\epsilon_a\mu_a \neq \epsilon\mu$, TM waves of the type in which we are interested, with n not equal to zero, cannot exist in a dielectric rod.

Finally, let us turn to the boundary condition (66). From the recursion rmulae of Chapter XII, we find that

$$\frac{\partial}{\partial r} J_n(\delta r) = \frac{n}{r} J_n(\delta r) - \delta J_{n+1}(\delta r) , \qquad \frac{\partial}{\partial r} K_n(\beta r) = \frac{n}{r} K_n(\beta r) - \beta K_{n+1}(\beta r) .$$

Therefore, when we substitute the expressions (72) and (73) into the boundary condition (66), we obtain

$$\frac{\epsilon}{\delta^2}\left[n - \frac{\delta r_0 J_{n+1}(\delta r_0)}{J_n(\delta r_0)}\right] = -\frac{\epsilon_a}{\beta^2}\left[n - \frac{\beta r_0 K_{n+1}(\beta r_0)}{K_n(\beta r_0)}\right] . \tag{77}$$

To this equation, we add the following

$$\delta^2 = \frac{\omega^2 \epsilon \mu}{c^2} - \gamma^2 , \tag{78}$$

$$\beta^2 = \gamma^2 - \frac{\omega^2 \epsilon_a \mu_a}{c^2} . \tag{79}$$

The three eqs. (77)-(79) relate the four quantities β, δ, γ and ω. One of them, for example ω, can therefore be given independently of the others. The values of β, δ and γ are then determined from eqs. (77)-(79). As we have shown, the parameter β must be real for the field to decrease rapidly with increasing distance from the rod. The constant of propagation γ must also be real, since otherwise the waves would be damped in the direction of propagation. Thus, the question arises as to whether eqs. (77) - (79) have real solutions for γ and β (in some region of variation of ω) that are compatible with the boundary conditions.

Let us first assume that condition (76) is satisfied. Then either the parameters δ and β are both equal to zero or the parameter δ is imaginary for real values of β. The reader will not find it difficult to show that if $\beta = \delta = 0$ waves of the type in which we are interested are not possible. Therefore, there remains only the second possibility, of which we shall speak below.

Let us turn to the general case. If γ is real, it follows from formula (78) that the parameter δ has either a real or a purely imaginary value.

Let us assume that δ is imaginary. Then, $\delta = \mathrm{i}\kappa$, where κ is a real number. From the formula given in Chapter XII,

$$I_n(\zeta) = \mathrm{i}^{-n} J_n(\zeta) .$$

Let us transform eq. (73) to the form

$$E_z = A_n^* I_n(\kappa r)\, e^{\mathrm{i}\gamma z} \cos(n\varphi + \psi_n) , \tag{80}$$

where A_n^* is constant. Let us substitute this expression and also the expression (72) for E_{za} into the boundary condition (64). We then find that

$$A_n^* I_n(\kappa r_0) = B_n K_n(\beta r_0) .$$

Since the functions $I_n(\kappa r_0)$ and $K_n(\beta r_0)$ are non-negative, the constants A_n^* and B_n have the same sign. Let us now substitute the expressions (80) and (72) into the boundary condition (66). This gives us the equation

$$\frac{\epsilon}{\kappa} A_n^* I_n'(\kappa r_0) = \frac{\epsilon_a}{\beta} B_n K_n'(\beta r_0) .$$

This equation is insoluble because its left and right sides are different from zero when A_n^* and B_n are non-zero and have different signs for all values of κ, β and n. Specifically, the functions $I_n(\zeta)$ increase monotonically with increasing ζ and the functions $K_n(\zeta)$ decrease monotonically. Therefore, the derivatives $I_n'(\kappa r)$ and $K_n'(\kappa r_0)$ are of opposite sign. Our assertion follows.

Thus, when the parameter δ is imaginary, we cannot satisfy the boundary conditions and hence, solutions of the type that we are interested in do not exist. Therefore, they also cannot exist when (76) holds, so that δ cannot be real when γ and β are real. In accordance with the idea of condition (76), it then follows that TM waves of the kind that we are interested in are in general impossible when n is non-zero.

Let us now assume that $n = 0$ and that the parameter δ is real. Let us show that eqs. (77) - (79) have real solutions for β and γ in some region of variation of ω if

$$\epsilon\mu > \epsilon_a\mu_a \,. \tag{81}$$

The necessity of the last condition is clear if we add eqs. (78) and (79) and note that $\delta^2 + \beta^2 > 0$. We now show the sufficiency.

Let us assign some non-zero value to the quantity β. The right side of eq. (77) will then have a fixed value. Let us now vary δ. Since the functions $J_0(\zeta)$ and $J_1(\zeta)$ have no common roots, the ratio

$$v(\delta r_0) = J_1(\delta r_0)/J_0(\delta r_0)$$

will, with increasing values of δ, take on values infinitely great and infinitely small infinitely many times. Since the ratio $v(\delta r_0)$ is continuous when $J_0(\delta r_0) \neq 0$, there are as infinite number of these values that satisfy eq. (77). Thus, to every fixed value of β there correspond an infinite number of values $\delta = \delta_{0m}(\beta)$, for $m = 0, 1, 2, \ldots$, that increase without bound with increasing m and that satisfy eq. (77). Since the right side of eq. (77) is continuous, the quantities $\delta_{0m}(\beta)$ are continuous functions of β.

If we set $\delta = \delta_{0m}(\beta)$ in eq. (78) and add this equation to eq. (79), we obtain

$$\frac{\omega^2}{c^2}(\epsilon\mu - \epsilon_a\mu_a) = \delta_{0m}{}^2(\beta) + \beta^2 \,, \tag{82}$$

which determines the parameters β and $\delta_{0m}(\beta)$ as continuous real functions $\beta_{0m}(\omega)$ and $\delta_{0m}(\omega)$ of ω in some region of variation of ω when the condition (81) is satisfied. Finally, from eq. (79), we obtain

$$\gamma^2 = \beta_{0m}{}^2(\omega) + \frac{\omega^2}{c^2}\epsilon_a\mu_a > 0 \,. \tag{83}$$

Thus, for real values of δ, the system of eqs. (77)-(79) has real roots in some region of variation of ω. This completes the proof of the existence of TM waves of the type referred to at the beginning of the section.

Let us determine the region of frequencies for which TM waves are propagated in a dielectric rod without damping.

Since, from what was said, to any real value of β there corresponds a

set of values $\delta_{om}{}^2(\beta) > 0$, we conclude from eq. (82) that to any sufficiently high value of the frequency ω there corresponds a real value of the parameter β. It then follows from the relationship (83) that γ^2 is positive. Therefore, the region of frequency for which TM waves are propagated in a rod without damping is unbounded above.

On the other hand, it follows from eq. (82) that the frequency of the waves with given m is bounded below by the critical value

$$\omega_{om} = \frac{M(m)}{\frac{1}{c}\sqrt{\epsilon\mu - \epsilon_a\mu_a}},$$

where $M(m)$ is the smallest value of the expression $\sqrt{[\delta_{om}{}^2(\beta) + \beta^2]}$.

Let us determine the lowest critical frequency ω_0. For $n = 0$, eq. (77) is of the form

$$\frac{\delta J_0(\delta r_0)}{\epsilon J_1(\delta r_0)} = -\frac{\beta K_0(\beta r_0)}{\epsilon_a K_1(\delta r_0)}\,. \tag{84}$$

It follows from expression (72) that the critical conditions under which the field outside the rod will cease to decrease exponentially with increasing r are attained when $\beta = 0$. For $\beta = 0$, eq. (84) takes the form

$$\delta J_0(\delta r_0) = 0\,.$$

On the basis of formulae (82) and (83), the root $\delta = 0$ corresponds to those values $\omega = \gamma = 0$ at which there is no travelling wave. The corresponding root is equal to

$$\delta_{01}\alpha_{01}/r_0\,,$$

where $\alpha_{01} = 2.405$ is the smallest root of the equation $J_0(\zeta) = 0$. By means of formulae (82) and (79), we now find the desired smallest critical frequency:

$$\omega_0 = \omega_{01} = \frac{c\alpha_{01}}{r_0\sqrt{\epsilon\mu - \epsilon_a\mu_a}}$$

and the corresponding value of the constant

$$\gamma = \frac{\alpha_{01}}{r_0\sqrt{(\epsilon\mu/\epsilon_a\mu_a) - 1}}\,.$$

Problems

1. Show that TE waves, with a positive value of n, that are rapidly damped at increasing distances from a dielectric rod cannot be propagated along the rod.

2. By using Hertz' vectors (Chapter XXXIV, section 3), show that waves representing a linear combination of TE and TM waves can be propagated along a dielectric rod that has the form of a circular cylinder. Method: The z components of the Hertz vectors should be put in the form

$$\left.\begin{aligned}\Pi_z &= A_n J_n(\delta r)\, e^{i\gamma z} \sin(n\varphi + \psi_n) ,\\ \Pi_z^* &= A_n^* J_n(\delta r)\, e^{i\gamma z} \cos(n\varphi + \psi_n) ,\end{aligned}\right\} \quad r < r_0 ,$$

$$\left.\begin{aligned}\Pi_z &= B_n K_n(\delta_a r)\, e^{i\gamma z} \sin(n\varphi + \psi_n) ,\\ \Pi_z^* &= B_n^* K_n(\delta_a r)\, e^{i\gamma z} \cos(n\varphi + \psi_n) ,\end{aligned}\right\} \quad r > r_0 .$$

Chapter XXXVII

ELECTROMAGNETIC HORNS AND RESONATORS

1. *Sectorial horns and resonators*

We use cylindrical coordinates r, φ, and z and consider the infinite region $0 \leqslant z \leqslant a$, $0 \leqslant \varphi \leqslant \alpha$, where a and α are positive constants. We shall call this region a sectorial horn and its boundary the walls of the horn. We shall consider the walls of the horn to be ideal conductors and we shall assume that the horn is filled with an ideal dielectric with $\epsilon = \mu = 1$, $\sigma = 0$.

Let us study the types of travelling electromagnetic waves that are possible in a sectorial horn. We use the representation of an electromagnetic field that was developed in section 4 of Chapter XXXIV (in terms of the scalar functions u and v).

For example, let us consider those solutions of Maxwell's equations for a horn that are waves of the electric type, that is, waves for which the component H_z of the magnetic vector is equal to zero. These waves are expressed in terms of the function u by formulae (62) - (63) of Chapter XXXIV, which in the present case are of the form

$$E_r = \frac{\partial^2 u}{\partial r\, \partial z}, \quad E_\varphi = \frac{1}{r}\frac{\partial^2 u}{\partial \varphi\, \partial z}, \quad E_z = \frac{\partial^2 u}{\partial z^2} + k^2 u\,; \tag{1}$$

$$H_r = -\frac{ik}{r}\frac{\partial u}{\partial \varphi}, \quad H_\varphi = ik\frac{\partial u}{\partial r}, \quad H_z = 0\,, \tag{2}$$

where

$$k^2 = \omega^2/c^2\,. \tag{3}$$

The function u must satisfy eq. (60) of Chapter XXXIV, which in the present problem represents Helmholtz' equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0\,. \tag{4}$$

On the walls of the horn, the tangential components of the electric vector must vanish (Chapter XXXIV, section 3); that is,

$$E_r = E_\varphi = 0 \quad \text{when} \quad z = 0 \text{ and } z = a\,, \tag{5}$$

$$E_r = E_z = 0 \quad \text{when} \quad \varphi = 0 \text{ and } \varphi = \alpha\,. \tag{6}$$

To satisfy these conditions, we must set

$$\partial u/\partial z = 0 \quad \text{when} \quad z = 0 \text{ and } z = a\,, \tag{7}$$

$$u = 0 \quad \text{when} \quad \varphi = 0 \text{ and } \varphi = \alpha\,. \tag{8}$$

Let us seek solutions of the problem (4) - (8) in the form

$$u = u_1(r)\, u_2(\varphi)\, u_3(z)\ .$$

The general expression for the solutions of Helmholtz' equation of this form is given by formula (50) of Chapter XXV:

$$u = AZ_n(\mu r) \cos (n\varphi + \psi_n) \cos (\nu z + \psi_\nu)\ , \tag{9}$$

where

$$\mu^2 + \nu^2 = k^2\ ,$$

A is an arbitrary constant, and $Z_n(\mu r)$ is a solution to Bessel's equation. When we substitute this expression into conditions (8) and (9), we find that

$$\psi_n = \tfrac{1}{2}\pi\ , \qquad n = \pi m/\alpha\ , \tag{10}$$

$$\psi_\nu = 0\ , \qquad \nu = \pi l/a \qquad (m, l = 0, 1, 2, \ldots)\ , \qquad \mu^2 = k^2 - \pi^2 l^2/a^2\ .$$

For the functions $Z_n(\mu r)$, we take the Hankel functions of the first kind $H_n^{(1)}(\mu r)$. As we know (Chapter XXV, section 5), with such a choice of functions $Z_n(\mu r)$ (and only with such a choice), our solution represents a system of travelling waves diverging to infinity. If $r = 0$, our solution will have an infinite discontinuity. The interpretation is as follows: The interval $r = 0$, $0 \leqslant z \leqslant a$ is a linear source of waves. We could avoid introducing discontinuous solutions by eliminating the sector $r < \epsilon$ from the region in question, and by giving the current on its boundary $r = \epsilon$, which would play the role of a source of the field. However, this would not change anything in the general expression for the solution.

Thus, the general expression for the function u for a travelling wave of the electric type in a sectorial horn is

$$u_{ml} = A_{ml} H^{(1)}_{\pi m/\alpha}(\mu r) \sin \pi m \frac{\varphi}{\alpha} \cos \pi l \frac{z}{a}\ , \tag{11}$$

$$\mu = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{a^2} l^2}\ , \tag{12}$$

where, instead of k^2, we have substituted its value as given by eq. (3). We note that, in contrast with the problems that we examined earlier, the solution of this problem contains Bessel functions of, generally speaking, non-integral order.

Let us consider some peculiarities of travelling waves of the electric type in a sectorial horn. We use the asymptotic expression (64) of Chapter XII for Hankel's functions of the first kind. For sufficiently large values of r, we find

$$H^{(1)}_{\pi m/\alpha}(\mu r) \approx \sqrt{\frac{2\pi}{\mu r}} \exp\left[-\tfrac{1}{2}\pi i(\pi m/\alpha + \tfrac{1}{2})\right] \exp\left[i\mu r\right]\ . \tag{13}$$

Let us suppose that

$$\omega > \pi c l/a\ . \tag{14}$$

Then, the number μ is real, and it follows from expression (13) that travelling waves in a sectorial horn are propagated with a phase velocity c when r is large and that their amplitude decreases with increasing r as $r^{-\frac{1}{2}}$.

However, if the direction of inequality (14) is reversed, travelling waves are generally impossible. For then, $\mu = i\kappa$, where κ is a real number. Consequently,

$$H^{(1)}_{\pi m/\alpha}(\mu r) = H^{(1)}_{\pi m/\alpha}(i\kappa r) \approx \sqrt{\frac{2\pi}{i\kappa r}} \exp\left[-\kappa r\right] \exp\left[-\tfrac{1}{2}\pi i(\pi m/\alpha + \tfrac{1}{2})\right] .$$

When we multiply this expression by $e^{-i\omega t}$, we see that it corresponds to a system of standing waves with amplitude decreasing according to an exponential law. Thus, a critical frequency

$$\omega_{0l} = \pi c l/a \qquad (l = 0, 1, 2, \dots) \tag{15}$$

exists such that, for all smaller frequencies, travelling waves with given l are impossible. Here, the value $l = 0$ corresponds to a wave that does not depend on the coordinate z. This wave may be a travelling wave for all frequencies. When $l > 1$, waves along the coordinate z are periodic.

We introduce the wavelength

$$\lambda_{0l} = 2\pi c/\omega_{0l} ,$$

corresponding to electromagnetic oscillations with angular frequency ω_{0l}. From eq. (15), we obtain

$$l = 2a/\lambda_{0l} ;$$

that is, for a travelling wave with $l \neq 0$ to be possible, its length must not exceed $2a/l$.

Let us turn now to the expressions (1) - (2) for the field vectors. It is easy to see that, in the general case, for large values of r, the components E_r, E_z, and H_φ of the field vectors decrease with increasing r as $r^{-\frac{1}{2}}$ and that the components E_φ and H_r decrease as $r^{-\frac{3}{2}}$; thus, in comparison with the first three components, we may neglect them. If, in addition, $l = 0$, then $E_r = 0$ and only the components E_z and H_φ are significant; that is, the wave is an almost purely transverse wave.

Let is now suppose that by means of an ideal conducting partition whose surface is situated at $r = b$, we isolate the finite portion of the horn $0 \leqslant r \leqslant b$, $0 \leqslant z \leqslant a$, $0 \leqslant \varphi \leqslant \alpha$. We shall call the hollow that is then formed (with ideal conducting walls) a sectorial resonator.

As we know (sections 2 and 3 of Chapter XXV), free oscillations are possible in the bounded region. Let us determine just what oscillations of the electric type are possible in a sectorial resonator. To do this, we need to find the eigenfunctions of the problem for Helmholtz' equation (4) in the region occupied by the resonator, under the boundary conditions corresponding to oscillations of the electric type. Each of the eigenfunctions determines one of the possible oscillations (an oscillation that cannot be represented in the form of a superposition of oscillations corresponding to the other eigenfunctions). Conversely, since the system of eigenfunctions is complete, any possible oscillation of the electric type can be represented

in the form of a combination of the oscillations determined by the eigenfunctions.

In addition to the boundary conditions (5) and (6) for a sectorial resonator, there is the boundary condition

$$E_z = E_\varphi = 0 \quad \text{when} \quad r = b \ .$$

To satisfy this condition, we must set

$$u = 0 \quad \text{when} \quad r = b \ . \tag{16}$$

We also require that the field in the resonator be bounded, so that, in expression (9), we need to set

$$Z_n(\mu r) = J_n(\mu r) \ .$$

For the boundary condition (16) to be satisfied, the numbers μ must satisfy the equation

$$J_n(\mu b) = 0 \ .$$

We denote by μ_{ms} (for $s = 1, 2, 3, \ldots$), the roots of this equation numbered in increasing order of magnitude, with the value of n determined by the relationship (10).

The choice of the remaining quantities in expression (9) will be determined by the same conditions as in the case of a sectorial horn. Therefore, the eigenfunctions of the problem in question can be represented in the form

$$u_{mls} = A_{mls} J_{\pi m/\alpha}(\mu_{ms} r) \sin \pi m \frac{\varphi}{\alpha} \cos \pi l \frac{z}{a}$$

$$(m, s = 1, 2, 3, \ldots ; \ l = 0, 1, 2, \ldots) \ .$$

Oscillations of the electric type corresponding to these eigenfunctions can be characterized by the three numbers m, l, and s.

We find the possible frequencies ω_{mls} of the oscillations from eq. (12) (solving it for the frequency ω). This gives

$$\omega_{mls} = c \sqrt{\pi^2 \frac{l^2}{a^2} + \mu_{ms}{}^2} \ .$$

It is clear from this expression that the lowest of the frequencies of free oscillations is equal to

$$\omega_{101} = c\mu_{11} \ .$$

Problems

Show that the components of the field vectors for waves of the magnetic type in a sectorial horn can be calculated from the formulae

$$E_r = \frac{ik}{r} \frac{\partial v}{\partial \varphi}, \qquad E_\varphi = -ik \frac{\partial v}{\partial r}, \qquad E_z = 0 \ ,$$

$$H_r = \frac{\partial^2 v}{\partial r \, \partial z}, \qquad H_\varphi = \frac{1}{r} \frac{\partial^2 v}{\partial \varphi \, \partial z}, \qquad H_z = \frac{\partial^2 v}{\partial z^2} + k^2 v \ ,$$

where v is the solution of eq. (4) satisfying the boundary conditions

$$v = 0 \quad \text{when} \quad z = 0 \text{ and } z = a \ ,$$
$$dv/d\varphi = 0 \quad \text{when} \quad \varphi = 0 \text{ and } \varphi = a \ .$$

2. *Spherical resonators*

A spherical hollow in a conducting material is called a spherical resonator. Let us find the type of electromagnetic oscillations that are possible in such a hollow. Here, we shall assume that the walls of the hollow are ideal conductors and that the medium inside the hollow is an ideal dielectric with $\epsilon = \mu = 1$.

As in the preceding section, from a mathematical point of view, this problem is reduced to the problem of finding the eigenfunctions for the spherical region.

We introduce spherical coordinates r, θ, φ with origin at the center of the hollow. We saw in section 4 of Chapter XXXIV that electromagnetic fields of the electric and magnetic type in spherical coordinates can be expressed in terms of the Debye potentials $\bar{u}$ and $\bar{v}$ satisfying Helmholtz' equation. Since the Debye potential $\bar{u} = u/r$, where u is the function introduced in section 4 of Chapter XXXIV, we find from the relationships (62)-(63) of Chapter XXXIV that the components of the field vectors for the electric-type field can be expressed in terms of the Debye potential $\bar{u}$ by means of the formulae

$$E_r = \frac{\partial}{\partial r} r\left(\frac{\partial \bar{u}}{\partial r} + \frac{\bar{u}}{r}\right) + k^2 r\bar{u} \ , \quad E_\theta = \frac{\partial}{\partial \theta}\left(\frac{\partial \bar{u}}{\partial r} + \frac{\bar{u}}{r}\right) , \quad E_\varphi = \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{\partial \bar{u}}{\partial r} + \frac{\bar{u}}{r}\right);$$

$$H_r = 0 \ , \quad H_\theta = -\frac{ik}{c}\frac{\partial \bar{u}}{\partial \varphi} \ , \quad H_\varphi = \frac{ik}{c}\frac{\partial \bar{u}}{\partial \theta} \ .$$

Analogously, by means of the formulae (64) - (65) of Chapter XXXIV, we see that the components of the field vectors for the magnetic-type field can be expressed in terms of the Debye potential $\bar{v}$ by means of the formulae

$$E_r = 0 \ , \quad E_\theta = \frac{ik}{c}\frac{\partial \bar{v}}{\partial \varphi} \ , \quad E_\varphi = -\frac{ik}{c}\frac{\partial \bar{v}}{\partial \theta};$$

$$H_r = \frac{\partial}{\partial r} r\left(\frac{\partial \bar{v}}{\partial r} + \frac{\bar{v}}{r}\right) + k^2 r\bar{v} \ , \quad H_\theta = \frac{\partial}{\partial \theta}\left(\frac{\partial \bar{v}}{\partial r} + \frac{\bar{v}}{r}\right) , \quad H_\varphi = \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{\partial \bar{v}}{\partial r} + \frac{\bar{v}}{r}\right) .$$

At the boundary $r = a$ of the spherical resonator, the conditions

$$E_\theta = E_\varphi = 0 \quad \text{when} \quad r = a \ ,$$

must be satisfied. This leads us to the following boundary conditions for the Debye potentials

$$\frac{\partial \bar{u}}{\partial r} + \frac{\bar{u}}{r} = 0 \quad \text{when} \quad r = a \ , \tag{17}$$

$$\bar{v} = 0 \quad \text{when} \quad r = a \ . \tag{18}$$

In section 4 of Chapter XXV, we examined the solutions of Helmholtz' equation of the form $u_1(r)u_2(\theta)u_3(\varphi)$ and found their general expression for $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant \theta \leqslant \pi$:

$$\frac{1}{\sqrt{r}} Z_{n+\frac{1}{2}}(kr)\, P_{nm}(\cos\theta)\cos(m\varphi+\psi_m) , \tag{19}$$

where m and n are integers. The solution of the type that we are considering is regular at $r = 0$ if

$$Z_{n+\frac{1}{2}}(kr) = J_{n+\frac{1}{2}}(kr) .$$

By using the notation

$$j_n(kr) \equiv \sqrt{\tfrac{1}{2}\pi}\,\frac{1}{\sqrt{r}}\,J_{n+\frac{1}{2}}(kr) ,$$

we transform expression (19) to the form

$$j_n(kr)\, P_{nm}(\cos\theta)\cos(m\varphi+\psi_m) .$$

If we impose the boundary condition (17) on this expression, we obtain the eigenfunctions of the problem in question for Helmholtz' equation with the following boundary condition:

$$\bar{u}_{mnl} = j_n(k_l r)\, P_{nm}(\cos\theta)\cos(m\varphi+\psi_m) , \tag{20}$$

where the k_l (for $l = 1,2,3,\ldots$) are the roots of the equation

$$kaj_n'(ka) + j_n(ka) = 0 .$$

In an analogous manner, we find the eigenfunctions for the boundary condition (18):

$$\bar{v}_{mnl} = j_n(k_l r)\, P_{nm}(\cos\theta)\cos(m\varphi+\psi_m) , \tag{21}$$

where the k_l (for $l = 1,2,3,\ldots$) are the roots of the equation

$$j_n(ka) = 0 .$$

Formulae (20) and (21) determine all the possible types of free electromagnetic oscillations in a spherical resonator.

Problems

1. Show that to each characteristic frequency $\omega_l = k_l c$ of electromagnetic oscillations in a spherical resonator there correspond $2n+1$ possible types of oscillations.

2. Show that the spectrum of frequencies of natural oscillations in a spherical resonator is bounded below, and that, with increase in the radius of the resonator, the lowest of the possible frequencies of natural oscillations decreases.

Chapter XXXVIII*

MOTION OF A VISCOUS FLUID

1. *Equations of motion of a viscous fluid*

In section 2 of Chapter VI, we gave the equations of motion of an ideal fluid (Euler's equations), namely

$$\frac{\partial v_i}{\partial t} + \sum_{\alpha=1}^{3} v_\alpha \frac{\partial v_i}{\partial x_\alpha} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} \quad (i = 1,2,3) \tag{1}$$

and pointed out that in order to satisfy the law of conservation of mass, the motion of the fluid must satisfy not only system (1) but also the continuity equation

$$\frac{\partial \rho}{\partial t} + \sum_{\alpha=1}^{3} \frac{\partial \rho v_\alpha}{\partial x_\alpha} = 0 . \tag{2}$$

We recall that ρ is the density of the fluid, that the v_i (for $i = 1,2,3$) are the components of the vector $\boldsymbol{v}$ (representing the velocity of the fluid), and that p is the pressure.

With the intention of generalizing Euler's equations to the case of a viscous fluid, let us determine how the momentum $\rho\boldsymbol{v}$ of a unit volume of an ideal fluid changes. We have

$$\frac{\partial}{\partial t}\rho v_i = \rho\frac{\partial v_i}{\partial t} + v_i\frac{\partial \rho}{\partial t} .$$

Keeping formulae (1) and (2) in mind, we have, after some manipulations,

$$\frac{\partial}{\partial t}\rho v_i = -\frac{\partial p}{\partial x_i} - \sum_{\alpha=1}^{3}\frac{\partial}{\partial x_\alpha}\rho v_i v_\alpha .$$

We introduce the symbol

$$\delta_{ij} \equiv \begin{cases} 1 , & i = j , \\ 0 , & i \neq j , \end{cases}$$

and write the last equation in the form

$$\frac{\partial}{\partial t}\rho v_i = -\sum_{\alpha=1}^{3}\frac{\partial \Pi_{i\alpha}}{\partial x_\alpha} \quad (i = 1,2,3) , \tag{3}$$

where

$$\Pi_{ij} \equiv p\delta_{ij} + \rho v_i v_j . \tag{4}$$

The system (3) represents the transformed system of Euler's equations.

To clarify the physical meaning of the quantities Π_{ij}, let us integrate the equations (3) over an arbitrary volume V, and let us apply the Ostrogradskii-Gauss formula to the right side of these equations. This gives us

$$\frac{\partial}{\partial t}\iiint_V \rho v_i \, dV = -\iint_{\mathcal{F}V}\left(\sum_{\alpha=1}^{3} \Pi_{i\alpha} n_\alpha\right) dS \qquad (i = 1, 2, 3) ,$$

where $\mathcal{F}V$ is the surface bounding the volume V and the n_α (for $\alpha = 1, 2, 3$) are the direction cosines of the outward normal to $\mathcal{F}V$. On the left sides of these equations are the rates of change in the i-th components of the momentum of the fluid contained in the volume V. Consequently, the quantities

$$-\left(\sum_{\alpha=1}^{3} \Pi_{i\alpha} n_\alpha\right) dS$$

represent the momentum flows of the corresponding components through the surface $\mathcal{F}V$ per element dS of the surface per unit of time. If the normal to the surface $\mathcal{F}V$ is directed along the j-axis, then

$$-\left(\sum_{\alpha=1}^{3} \Pi_{i\alpha} n_\alpha\right) dS = -\Pi_{ij} \, dS .$$

This implies that $(-\Pi_{ij})$ is the magnitude of the i-th component of the momentum transmitted in the direction of the j-axis (through a unit area normal to that axis per unit of time). The vector Π_i with components Π_{i1}, Π_{i2}, and Π_{i3} will be referred to as the flow of the i-th component of the momentum.

Expression (4) shows that the flow of momentum in an ideal fluid results from the action of the forces of pressure and mechanical transfer of the fluid. Between adjacent parts of a viscous fluid there are (in addition to forces of pressure which are normal to the boundary between the parts) forces lying in the plane tangential to the boundary. These latter forces tend to decrease the relative velocity of adjacent portions of the fluid. This last phenomenon is known as *viscosity*. The presence of viscosity obviously causes terms characterizing the viscous transmission of momentum to appear in the expression for the component of the vector Π_i. We shall denote the set of these terms by $(-\sigma'_{ij})$. Similarly, we shall determine the vector Π_i of the flow of the i-th component of the momentum in a viscous fluid by means of the equations

$$\Pi_{ij} \equiv p\delta_{ij} - \sigma'_{ij} + \rho v_i v_j . \tag{5}$$

The dependence of the quantities σ_{ij} on the velocity of the fluid can be established from the following considerations. When there is no displacement of any part of the fluid with respect to the other parts, the term σ_{ij} must vanish. Consequently, the quantity σ'_{ij} depends not on the velocity of the fluid itself, but only on its derivatives $\partial v_i/\partial x_j$ with respect to the coordinates. As a first approximation, we shall consider this dependence to be linear.

When the fluid as a whole rotates uniformly, there are no relative mo-

tions of the parts in the fluid. Consequently, the quantities σ'_{ij} do not depend directly on the derivatives $\partial v_i/\partial x_j$ but only on those combinations of the derivatives for which the equations

$$v_\alpha = \omega_\beta x_\gamma - \omega_\gamma x_\beta \quad \text{for} \quad \alpha,\beta,\gamma = \begin{cases} 1,2,3, \\ 3,2,1, \\ 2,1,3, \end{cases}$$

(where ω_1, ω_2, and ω_3 are the components of the vector representing the angular velocity of the fluid) imply that $\sigma'_{ij} = 0$. Linear combinations of the derivatives satisfying this requirement are

$$\sum_{\alpha=1}^{3} \frac{\partial v_\alpha}{\partial x_\alpha} \quad \text{and} \quad \frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} \quad (\alpha \neq \beta) .$$

The most general kind of vector that can be formed from these expressions has components

$$\sigma'_{ij} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \eta_1 \delta_{ij} \sum_{\alpha=1}^{3} \frac{\partial v_\alpha}{\partial x_\alpha},$$

where the quantities η and η_1 do not depend on the velocity of the fluid. This expression is usually transformed into the form

$$\sigma'_{ij} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \tfrac{2}{3}\delta_{ij} \sum_{\alpha=1}^{3} \frac{\partial v_\alpha}{\partial x_\alpha}\right) + \zeta \delta_{ij} \sum_{\alpha=1}^{3} \frac{\partial v_\alpha}{\partial x_\alpha}.$$

Here, the sum of the terms with $i = j$ does not depend on η. The quantities η and ζ are called the coefficients of viscosity. For actual fluids, both these components are positive.

Note that system (3) relates the derivatives of the i-th component of the momentum of a unit of volume of a fluid to the flow of these components. Consequently, its form does not depend on the specific nature of the forces acting on the fluid. Therefore, when we seek the equations of motion of a viscous fluid, we substitute into system (3) the expressions (5), with the values just obtained for the terms σ'_{ij}. We then obtain the most general system of equations for the motion of a viscous fluid:

$$\rho \left(\frac{\partial v_i}{\partial t} + \sum_{\alpha=1}^{3} v_\alpha \frac{\partial v_\alpha}{\partial x_i}\right) = -\frac{\partial p}{\partial x_i} + \sum_{\alpha=1}^{3} \frac{\partial}{\partial x_\alpha}\left[\eta \left(\frac{\partial v_i}{\partial x_\alpha} + \frac{\partial v_\alpha}{\partial x_i}\right) - \tfrac{2}{3}\delta_{i\alpha} \sum_{\beta=1}^{3} \frac{\partial v_\beta}{\partial x_\beta}\right]$$
$$+ \frac{\partial}{\partial x_j} \zeta \sum_{\alpha=1}^{3} \frac{\partial v_\alpha}{\partial x_\alpha} \quad (i = 1,2,3) . \tag{6}$$

If the coefficients η and ζ are constant throughout the fluid, this system takes the form

$$\rho \left(\frac{\partial v_i}{\partial t} + \sum_{\alpha=1}^{3} v_\alpha \frac{\partial v_\alpha}{\partial x_i}\right) = -\frac{\partial p}{\partial x_i} + \eta \Delta v_i + (\zeta + \tfrac{1}{3}\eta) \frac{\partial}{\partial x_i} \sum_{\alpha=1}^{3} \frac{\partial v_\alpha}{\partial x_\alpha} \tag{7}$$

$$(i = 1,2,3) ,$$

where Δ is the Laplacian operator. Finally, if the fluid is incompressible, we have

$$\sum_{\alpha=1}^{3} \frac{\partial v_\alpha}{\partial x_\alpha} = 0$$

and we arrive at the system of Navier-Stokes' equations:

$$\frac{\partial v_i}{\partial t} + \sum_{\alpha=1}^{3} v_\alpha \frac{\partial v_\alpha}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\Delta v_i \quad (i = 1, 2, 3) , \tag{8}$$

where $\nu \equiv \eta/\rho$ is the coefficient of kinematic viscosity.

To the system of equations of motion of a viscous fluid we must add the continuity equation (2) when solving specific problems. The form of this equation is the same for an ideal and for a viscous fluid.

The boundary conditions for a system of equations of motion of a viscous fluid can be established from physical considerations.

As a result of the forces of molecular adhesion, the layer of the fluid directly adjacent to a solid wall moves (or remains at rest) with that wall. Therefore, on the boundary separating the solid and liquid media, we need to take

$$v_i = v_{ic} \quad (i = 1, 2, 3) , \tag{9}$$

where the v_{ic} are the components of the displacement of the boundary (solid wall).

From the same considerations, the velocities of adjacent fluids on the boundary separating the fluids must be equal. However, since the boundary between the fluids is deformable, we must add the condition that the forces with which the fluids act on each other at the boundary are equal in magnitude and opposite in direction. For a mathematical formulation of this last condition, we note that, on the basis of Newton's second law, the components of the vector representing the force acting on an element of surface dS are equal to the flow of the corresponding components of the momentum through this element of surface. From this, we obtain the following expression for the components P_i of the force acting on a unit of area:

$$P_i = \sum_{\alpha=1}^{3} \Pi_{i\alpha} n_\alpha = \rho v_i \sum_{\alpha=1}^{3} v_\alpha n_\alpha - \sum_{\alpha=1}^{3} \sigma_{i\alpha} n_\alpha \quad (i = 1,2,3) ,$$

where the n_α (for $\alpha = 1, 2, 3$) are the direction cosines of the normal to the surface and

$$\sigma_{ij} \equiv -p\delta_{ij} + \sigma'_{ij} .$$

We shall denote by a superscript a those quantities referring to one of the fluids and by a superscript b those referring to the other. Then, the condition for equilibrium of the forces takes the form $P_i^{(a)} = P_i^{(b)}$. Noting that $n_j^{(a)} = -n_j^{(b)}$ and $v_j^{(a)} = v_j^{(b)}$, we write the condition for equilibrium in the form

$$\sum_{\alpha=1}^{3} \sigma_{i\alpha}^{(a)} n_\alpha = \sum_{\alpha=1}^{3} \sigma_{i\alpha}^{(b)} n_\alpha \quad (i = 1,2,3)\ , \tag{10}$$

where, for the quantities n_α, we may substitute either $n_\alpha^{(a)}$ or $n_\alpha^{(b)}$.

We note that the force applied to a unit of area oriented in an arbitrary manner in the viscous fluid can be broken down into two components, namely a normal component

$$P_n = \sum_{\alpha=1}^{3} P_\alpha n_\alpha = \rho \sum_{\alpha=1}^{3} v_\alpha{}^2 n_\alpha{}^2 - \sum_{\beta=1}^{3}\sum_{\alpha=1}^{3} \sigma_{\beta\alpha} n_\alpha n_\beta$$

and a tangential component, the expression for which we omit. The terms in the expressions for these components do not depend on the components of the velocity v_i; they are called the normal and tangential (cleavage) stresses. Thus, boundary condition (10) implies that the stresses are the same in both fluids on the boundary separating them.

Finally, let us turn to the boundary conditions of the free surface of a viscous fluid. The stress on the free surface is obviously equal to zero, so that

$$\sum_{\alpha=1}^{3} \sigma_{i\alpha} n_\alpha = 0 \quad (i = 1,2,3)\ ,$$

or

$$p n_i - \sum_{\alpha=1}^{3} \sigma'_{i\alpha} n_\alpha = 0\ , \tag{11}$$

where the n_α (for $\alpha = 1,2,3$) are the direction cosines of the normal to the free surface.

Problems

1. Show that, on a solid immovable wall, the solutions of the system of Euler's equations (1) cannot, in the general case, satisfy a boundary condition of the form $v_i = 0$ (for $i = 1,2,3$), which is valid in the case of a viscous fluid.

2. Show that the components of the vector representing the force acting on a unit of area of a rigid immovable plane surface placed in a viscous fluid in a position perpendicular to the x_1-axis are equal to

$$P_1 = p - 2\eta \frac{\partial v_1}{\partial x_1}\ , \quad P_2 = -\eta\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right)\ , \quad P_3 = -\eta\left(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}\right)\ .$$

3. Show that if the coefficients η and ζ are constant throughout an entire viscous fluid, the tangential stresses in the fluid do not depend on the pressure, but only on the internal motion of the fluid.

4. Show that in cylindrical coordinates the system of Navier-Stokes' equations and the equations of continuity are of the form

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\varphi}{r} \frac{\partial v_r}{\partial \varphi} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\varphi{}^2}{r}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \varphi^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi} - \frac{v_r}{r^2}\right);$$

$$\frac{\partial v_\varphi}{\partial t} + v_r \frac{\partial v_\varphi}{\partial r} + \frac{v_\varphi}{r} \frac{\partial v_\varphi}{\partial \varphi} + v_z \frac{\partial v_\varphi}{\partial z} - \frac{v_r v_\varphi}{r}$$

$$= -\frac{1}{\rho r}\frac{\partial p}{\partial \varphi} + \nu\left(\frac{\partial^2 v_\varphi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\varphi}{\partial \varphi^2} + \frac{\partial^2 v_\varphi}{\partial z^2} + \frac{1}{r}\frac{\partial v_\varphi}{\partial r} + \frac{2}{r^2}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r^2}\right);$$

$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\varphi}{r} \frac{\partial v_z}{\partial \varphi} + v_z \frac{\partial v_z}{\partial z}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \varphi^2} + \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right);$$

$$\frac{\partial v_z}{\partial r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r} = 0 .$$

2. *Motion of a viscous fluid in the space over a rotating disk of infinite radius*

In all cases in which the non-linear terms in the equations of motion of a viscous fluid are not equal to zero due to the given conditions of a problem, the exact solution of these equations presents great difficulties and depends to a greater extent on intuition and guess than on any kind of widely applicable method. Examples of exact solutions of the equations of motion of a viscous fluid are given in this and the following sections.

Let us consider the Karman problem. An infinite plane disk is rotating around its axis with a constant angular velocity ω. Find the steady-state motion of a viscous fluid adjacent to the disk and filling the half-space above it.

We use a cylindrical coordinate system (r,φ,z) with the z-axis directed along the axis of the disk and with the plane $z=0$ coinciding with the surface of the disk.

In each of the planes z = constant, we shall distinguish between two motions: the angular motion caused by viscous forces in the fluid dragged along by the disk, and a radial motion, directed along the radii away from the z-axis, caused by inertial forces. Furthermore, there must be a vertical motion filling the vacuum resulting from the flow of the fluid away from the z-axis in its radial motion. The velocity of the vertical motion, which is directed toward the disk, must increase with increasing distance from the disk (corresponding to the increasing net amount of the fluid that is

flowing away from the z-axis in that part of space between the disk and the plane under consideration).

We shall require that the axial component of the velocity v_z at infinity remain finite. This is possible only in the case in which the velocity v_r of the radial flow of the fluid away from the z-axis decreases without bound with increase in z. This, in turn, presupposes unbounded decrease in the inertial forces, which is possible only if the velocity v_φ of the angular motion approaches zero as z approaches infinity. When $z = 0$, the velocity of the fluid coincides with the velocity of the surface of the disk.

Thus, we have the following boundary conditions:

$$\begin{aligned} &v_r|_{z=0} = 0, \quad v_\varphi|_{z=0} = \omega r, \quad v_z|_{z=0} = 0, \\ &v_r|_{z=\infty} = 0, \quad v_\varphi|_{z=\infty} = 0, \quad |v_z||_{z=\infty} < \infty. \end{aligned} \tag{12}$$

Let us seek a solution, assuming that the velocities of the radial and angular motions are proportional to the distance from the axis of rotation of the disk and that the vertical velocity and the pressure are constant in each of the planes parallel to the plane of the disk. The compatibility of these assumptions with the given conditions will follow from the consistency of the results that we shall obtain. If a contradiction were to arise, we would need to advance some other hypothesis concerning the properties of the desired solution. That is, we would have to introduce additional physical considerations and make a new attempt to obtain results that would not be contradictory.

On the basis of these assumptions, let us seek a solution in the form

$$v_r = r\omega F(\zeta), \quad v_\varphi = r\omega G(\zeta), \quad v_z = \sqrt{\nu\omega}\, H(\zeta), \quad p = -\rho\nu\omega P(\zeta).$$

where

$$\zeta = z\sqrt{\omega/\nu}.$$

The coefficients of the functions F, G, H, and P are chosen so that these functions are dimensionless. Also, instead of z, we have the dimensionless argument ζ.

Let us substitute the quantities (5) into the Navier-Stokes equations and the equations of continuity, both written in cylindrical coordinates (see problem 4 of section 1), and into the relations (12) for determining the functions F, G, H, and P. This leads us to the system of ordinary differential equations:

$$\begin{aligned} &F^2 - G^2 + F'H = F'', \quad 2FG + G'H = G'', \\ &HH' = P' + H'', \quad 2F + H' = 0, \end{aligned} \tag{13}$$

and the boundary conditions

$$\begin{aligned} &F|_{\zeta=0} = 0, \quad G|_{\zeta=0} = 1, \quad H|_{\zeta=0} = 0, \\ &F|_{\zeta=\infty} = 0, \quad G|_{\zeta=\infty} = 0, \quad |H||_{\zeta=\infty} < \infty. \end{aligned} \tag{14}$$

Thus, we have reduced a problem involving a system of partial differential equations to a problem involving a system of ordinary differential

equations. We drop the matter at this point, noting only that the solution of this problem belongs to the class of functions with continuous first and second derivatives. The functions F, G, and H have been calculated by means of numerical integration. The reader can find their graphs in the book by Landau and Lifshitz [25].

If we can assume, on the basis of physical considerations, that solutions of steady-state problems involving the hydrodynamics of a viscous fluid are unique in the class of bounded functions with continuous first and second derivatives, the solution of the system (13) satisfying the boundary conditions (14) is the desired solution.

3. *Motion of a viscous fluid in a plane diffuser*

Consider a dihedral angle formed by plane walls. Suppose that a viscous incompressible fluid is flowing between the walls and along the line of their intersection. Let us find the steady-state motion of the fluid.

We take the line of intersection of the walls as the z-axis of cylindrical coordinates. The coordinate φ is measured from the plane bisecting the dihedral angle.

From symmetry considerations, we shall consider the motion plane (purely radial). Therefore, we set

$$v_\varphi = v_z = 0\ , \qquad v_r = v(r,\varphi)\ .$$

Here, the Navier-Stokes equations and the equation of continuity (see problem 4 of section 1) are of the form

$$v\,\frac{\partial v}{\partial r} = -\frac{1}{\rho}\,\frac{\partial v}{\partial r} + \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\,\frac{\partial^2 v}{\partial \varphi^2} + \frac{1}{r}\,\frac{\partial v}{\partial r} - \frac{v}{r^2}\right)\ , \tag{15}$$

$$-\frac{1}{\rho r}\,\frac{\partial p}{\partial \varphi} + \frac{2\nu}{r^2}\,\frac{\partial v}{\partial \varphi} = 0\ , \tag{16}$$

$$\frac{\partial (rv)}{\partial r} = 0\ . \tag{17}$$

On the walls, the velocity of the fluid is equal to zero; that is

$$v\,|_{\varphi=\pm\frac{1}{2}\alpha} = 0\ , \tag{18}$$

where α is the angle between the walls.

It is clear from eq. (17) that the product rv depends only on φ.

If we introduce the dimensionless function

$$u(\varphi) = \frac{1}{6\nu}\,rv\ ,$$

we obtain from eq. (16)

$$\frac{1}{\rho}\,\frac{\partial p}{\partial \varphi} = \frac{12\nu^2}{r^2}\,\frac{du}{d\varphi}\ ,$$

so that

$$\frac{p}{\rho} = \frac{12\nu^2}{r} u(\varphi) + f(r) .$$

If we substitute this expression into eq. (15), we obtain

$$\frac{d^2u}{d\varphi^2} + 4u + 6u^2 = \frac{1}{6\nu^2} r^3 f'(r) .$$

The left side of this equation depends only on φ and the right side only on r. Consequently, they are equal to a single constant, which we denote by $2\mu_1$. In view of this, we obtain

$$\frac{d^2u}{d\varphi^2} + 4u + 6u^2 = 2\mu_1 , \qquad f'(r) = 12\mu_1\nu^2 \frac{1}{r^3} . \tag{19}$$

From the last equation, it follows that

$$f(r) = -\frac{6\mu_1\nu^2}{r^2} + \text{constant} ,$$

as a result of which the pressure is determined by the expression

$$p(r, \varphi) = \frac{6\nu^2\rho}{r^2} (2u - \mu_1) + p_0 ,$$

where p_0 is an arbitrary constant.

If we multiply the first of eqs. (19) by du and integrate, we obtain

$$\frac{1}{2}\left(\frac{du}{d\varphi}\right)^2 + 2u^2 + 2u^3 - 2\mu_1 u - 2\mu_2 = 0 , \tag{20}$$

where μ_2 is a constant. If we separate the variables, we obtain

$$\frac{1}{2} \frac{du}{\sqrt{-u^3 - u^2 + \mu_1 u + \mu_2}} = + d\varphi , \tag{21}$$

so that

$$2\varphi = \int_{u_1}^{u} \frac{d\xi}{\pm\sqrt{-\xi^3 - \xi^2 + \mu_1\xi + \mu_2}} , \tag{22}$$

where u_1 is an arbitrary constant.

We obtain the following three equations for determining the constants μ_1, μ_2, and u_1:

$$\int_{-\frac{1}{2}\alpha}^{\frac{1}{2}\alpha} \rho v r \, d\varphi = 6\nu\rho \int_{-\frac{1}{2}\alpha}^{\frac{1}{2}\alpha} u \, d\varphi = Q , \tag{23}$$

$$u(\varphi)\big|_{\varphi=\frac{1}{2}\alpha} = 0 , \qquad u(\varphi)\big|_{\varphi=-\frac{1}{2}\alpha} = 0 . \tag{24}$$

Eq. (23) expresses the requirement that the same amount ($Q > 0$) of fluid pass through each cross section r = constant. Eqs. (24) are a consequence of the boundary condition (18).

The dimensionless quantity

$$R \equiv Q/\nu\rho \,, \tag{25}$$

which characterizes the ratio of the outflow of the liquid to its kinematic viscosity, is called the Reynolds number for the flow.

Thus, we have reduced the original problem to a problem in which the unknown function u is determined by certain integral relationships. Thereby, we are able to study a number of the properties of the solutions of the original problem.

Let us find the conditions under which the motion of the fluid is symmetric about the plane $\varphi = 0$ and the velocity v of the fluids in the surfaces r = constant changes monotonically from zero (when $\varphi = +\frac{1}{2}\alpha$) to $v_0 = ru_0$ (when $\varphi = 0$).

At the maximum, $du/d\varphi = 0$. Therefore, it follows from eq. (20) that u_0 is a root of the equation

$$-\xi^3 - \xi^2 + \mu_1\xi + \mu_2 = 0 \,,$$

and, therefore, the radical in eq. (22) can be written in the form

$$-\xi^3 - \xi^2 + \mu_1\xi + \mu_2 = (u_0 - \xi)\,[\xi^2 + (1+u_0)\xi + q_0] \,,$$

where

$$q_0 \equiv u_0^{\,2} + u_0 - \mu_1 \,.$$

Noting that $u = u_0$ when $\varphi = 0$, we write eq. (22) in the form

$$2\varphi = \int_0^{u_0} \frac{d\xi}{\pm\sqrt{(u_0 - \xi)\,[\xi^2 + (1+u_0)\xi + q_0]}} \,. \tag{26}$$

Here, instead of the constants μ_1 and μ_2, we have two new unknown constants u_0 and q_0. By using the boundary condition (24), we obtain the following equation for these constants

$$\int_0^{u_0} \frac{d\xi}{\sqrt{(u_0 - \xi)\,[\xi^2 + (1+u_0)\xi + q_0]}} = \alpha \,. \tag{27}$$

If we substitute expression (21) for $d\varphi$ into the boundary condition (23) and remember that the function $u(\varphi)$ is symmetric about the plane $\varphi = 0$, we obtain a second equation:

$$6\int_0^{u_0} \frac{\xi\, d\xi}{\sqrt{(u_0 - \xi)\,[\xi^2 + (1+u_0)\xi + q_0]}} = R \,. \tag{28}$$

It is easy to see from eq. (27) that α is a monotonically increasing function with respect to both u_0 and q_0. Consequently, this equation defines u_0 as a monotonic decreasing function of q_0 for a fixed value of α. Analysis of eq. (28) shows that R, being a function of u_0 and q_0, decreases with increasing q_0 and increases with increasing u_0.

Let us fix the value of α. The lowest possible value of q_0 is zero since, if q_0 were negative, the integral on the left side of formula (27) would be

complex. For a given α, the largest possible value of u_0 corresponds to the value q_0. This value of u_0 determines the largest value of R, $R = R_{cr}$, that is compatible with the given value of α. With increasing α, the quantity R_{cr} obviously decreases. When R exceeds R_{cr}, motion of the type that we are examining is impossible in a diffuser, since it is incompatible with eqs. (27) and (28). A more detailed analysis (see problem 1) shows that as α approaches π, the value of R_{cr} approaches zero and that as α approaches zero, the value of R_{cr} becomes infinite.

Thus, with increasing R, where $R = R_{cr}$, symmetrically divergent motion in the diffuser, $(v \geqslant 0)$ for all φ, becomes impossible and is changed to another kind of motion. Formal investigation of eqs. (22) - (24) shows that the function $u(\varphi)$ has several maxima and minima when R exceeds R_{cr}. Also, with increasing R, the number of alternating maxima and minima increases and $u(\varphi)$ is negative at the minima. Consequently, there must be regions of flow out of the diffuser and regions of flow into the diffuser. However, in practice, such motions are not realized when R greatly exceeds R_{cr}, since they are unstable. Specifically, when R exceeds R_{cr}, small external disturbances and also small deviations in the boundary conditions cause a sharp change in the motion, and this brings about a non-steady-state turbulent situation. The number $R = R_{cr}$ is called the critical Reynold's number.

Problems

1. Derive the formulae

$$\alpha = 2\sqrt{1-2k^2}\int_0^{\frac{1}{2}\pi} \frac{d\zeta}{\sqrt{1-k^2\sin^2\zeta}},$$

$$R_{cr} = -6\alpha\,\frac{1-k^2}{1-2k^2} + \frac{12}{\sqrt{1-2k^2}}\int_0^{\frac{1}{2}\pi} \sqrt{1-k^2\sin^2\zeta}\,d\zeta\,,$$

which relate (parametrically) the critical Reynold's number R_{cr} and the angle α between the walls of the diffuser. Use these formulae to show (1) that k and R_{cr} approach zero as α approaches π and (2) that as α approaches zero, the parameter k approaches $2^{-\frac{1}{2}}$ and R_{cr} approaches infinity.

Method: Start with formulae (27) and (28) with $q_0 = 0$ and make the substitutions

$$\zeta = u_0\cos^2\zeta\,, \qquad k^2 = \frac{u_0}{1+2u_0}\,.$$

2. Show that when there is nozzle-type flow, that is, when a drainage rather than an outflow of the viscous liquid $(Q < 0)$ takes place along the line of intersection of the walls, there will be no return motion of the liquid. Specifically, show that when r is positive and α is less than π, the mo-

tion will be symmetric about the plane $\varphi = 0$ and directed along the line of drainage, for all values of the Reynold's number

$$R_{cr} \equiv |Q|/\rho\nu .$$

Method: With nozzle-type flow, the function $u(\varphi)$ is a solution of the equation

$$2\varphi = \pm \int_{-u_0}^{u} \frac{d\xi}{\sqrt{(\xi + u_0)\,[-\xi^2 - (1 - u_0)\xi + q_0]}} ,$$

where the quantities u_0 and q_0 are determined from the conditions

$$\alpha = \int_{-u_0}^{0} \frac{d\xi}{\sqrt{(\xi + u_0)\,[-\xi^2 - (1 - u_0)\xi + q_0]}} ,$$

$$R = 6 \int_{-u_0}^{0} \frac{\xi\, d\xi}{\sqrt{(\xi + u_0)\,[-\xi^2 - (1 - u_0)\xi + q_0]}} .$$

Chapter XXXIX*

GENERALIZED FUNCTIONS †

1. *Introduction*

The concept of a continuous medium or a continuous field, as used in the mathematical formulation of physical problems, ignores, in effect, the discrete structure of matter. Therefore, from a physical standpoint, the quantities that appear in the equations of mathematical physics do not have any direct physical meaning. A physical meaning can be ascribed to them only to the extent that they make possible a description of the state of the medium or field, not at a point, but in some region which, though it may be quite small, is still large enough so that the discrete structure of the substance can be ignored.

A classic example of this is the concept of temperature, which is meaningful only in connection with a portion of a typical medium that contains a sufficiently large number of molecules. A physical meaning can be given to the mathematical concept of temperature at a geometric point only to the extent that the averaging of temperatures, regarded as point functions, over arbitrary (not too small) regions, leads to correct values of mean temperature in these regions.

Thus, from a physical point of view, a description of the physical phenomena in terms of functions of the region rather than of the point, would be more adequate. This indicates a discrepancy between the content of the problems of mathematical physics and their classical analytical formulation, which requires the satisfaction of certain conditions at every point of the region in question.

In particular, this discrepancy shows up in the fact that many problems of mathematical physics do not have solutions in the exact analytic sense, although exact solutions are known to exist in an implicit physical sense. We have already encountered an example of this kind in the simple problem of the oscillations of a string, when, in order to avoid a contradiction, we needed to introduce the concept of generalized solutions (Chapter III, section 6).

In the last two decades, extensive progress has been made in analysis. These new developments (although they have not given any results that extend the classical methods of solution of the problems of mathematical physics) have made it possible to perfect the analytical formulation of problems and to study more thoroughly the problem of the existence of solu-

† In the presentation of the material in this chapter, we are basically following the monograph of I. M. Gel'fand and G. E. Shilov [40].

tions. Without giving rigorous proofs, we shall present certain facts concerning these new developments.

2. *Generalized functions*

In this chapter, we shall study one-, two- and three-dimensional regions. We shall denote the points of these regions by the letters x, ξ and so on. We shall denote the coordinates of the corresponding points by the same letters, but with subscripts. For clarity, we shall use the notation for three dimensions, but everything that is said can, with obvious modifications, be applied for any number of dimensions.

Suppose that $f(x)$ is a function of the point x describing some physical phenomenon. In accordance with the above, a physical meaning must be given not to the function $f(x)$ itself but to some function of the region that is defined in terms of $f(x)$. This can be the average value of $f(x)$ in the region (as, for example, will be the case if $f(x)$ is the temperature) or the integral of $f(x)$ over the region (if, for example, $f(x)$ is the density), and so forth. In both these examples, the transition from a function of the point x to a function of the region is made by means of integration. A more detailed analysis shows that such a situation is typical in physics. Specifically, a meaning can be ascribed to point functions only to the extent that they determine the value of some integral having a direct physical meaning.

This fact leads to the view that mathematical analysis can be more appropriate if we construct it as a system of concepts relating suitably chosen integrals and if we introduce point functions only at certain intermediate steps in the construction. It is very significant that this idea can be applied even more widely, encompassing those cases in which a function of the region which corresponds naturally to a physical phenomena cannot be represented by means of an integral of a point function.

As an auxiliary means of building up this theory, we use real functions $\varphi(x)$ of the point x which are infinitely many times differentiable with respect to the coordinates of the point x and which are finite (that is, different from zero only in finite regions). We shall call these functions *fundamental* and we shall refer to the set of these functions as the fundamental function space or, more simply, as the fundamental space *. The sum of two or

* By a "space" (or, more precisely, a topological space), we mean a set in which the topological structure is defined. The family Ω of subsets of the set E defines the topological structure in E, if this family represents a set such that the intersection of a finite number of these subsets and the union of an arbitrary number of these subsets are again members of the family. (The union of two or more subsets in the set of elements of the original set that belong to at least one of the subsets in the union. The intersection of two or more subsets is the set of elements of the original set that belong to all the subsets in the intersection.) The reader will have no difficulty in showing that the set of fundamental functions is a space. Various spaces have been defined, depending on the choice of elements of the space and the relationships between them (for example, linear, functional, and so on). The set of fundamental functions is a function space since its elements are funotions. The relations between the functions are defined by the usual operations of analysis.

more fundamental functions and the product of a fundamental function and a real number are fundamental functions. A space whose elements have these properties is called a *linear space*. Thus, the fundamental space is a linear space. A sequence $\varphi_1(x), \varphi_2(x), \ldots \varphi_\nu(x), \ldots$ of fundamental functions is said to *converge in the fundamental space* if all the functions $\varphi_\nu(x)$ vanish in a *single* finite region and if the difference $\varphi_{\nu+k} - \varphi_\nu$ (for $k > \nu$) and its first derivatives approach zero uniformly (for all x) as ν increases.

We shall assume that the function $f(x)$, which describes some physical phenomenon, is locally integrable *. We put in correspondence with this function the set of integrals of the form

$$\int\int\int f(x)\ \varphi(x)\ \mathrm{d}V\ , \tag{1}$$

where $\varphi(x)$ is a fundamental function. The integration can be considered taken over all space **, though in practice it is taken over the finite region in which the corresponding fundamental function $\varphi(x)$ is not identically equal to zero.

The integral (1) for the given function $\varphi(x)$ represents a number. If we choose a different fundamental function, we shall generally obtain a different number. A rule which assigns some number to each function of a given function space is called a *functional*. Thus, the integral (1) represents a functional defined in some space for the given locally integrable function $f(x)$. For brevity, we use the notation

$$(f, \varphi) \equiv \int\int\int f(x)\ \varphi(x)\ \mathrm{d}V\ . \tag{2}$$

It can be shown that the set of values that the functional (f, φ) assumes in the fundamental space (that is, for any possible choice of the functions $\varphi(x)$) uniquely determines the function $f(x)$ at all points at which it is continuous. In particular, two distinct continuous functions $f_1(x)$ and $f_2(x)$ correspond to two functionals (f_1, x) and (f_2, x) which assume different values at least for some of the fundamental functions. It can also be shown that the average values of a locally integrable function $f(x)$ in an arbitrary finite region are uniquely determined by the functional (f, x). (The reader is advised to carry out the proof of the last assertion.) Thus, the defining of a functional determines a point function with the desired degree of accuracy and, in that sense, is equivalent to defining the point function itself.

The functional (f, φ) determined by expression (2) is linear (that is,

$$(f, \alpha_1\varphi_1 + \alpha_2\varphi_2) = \alpha_1(f, \varphi_1) + \alpha_2(f, \varphi_2)\ , \tag{3}$$

where α_1 and α_2 are arbitrary constants) and continuous (that is, for any positive ϵ a positive number ν_0 exists such that

* If the integral of the absolute value of a function over some finite region exists, this function is said to be locally integrable.

** Henceforth, when we use the word "space" without the adjective "fundamental", what is meant is the "geometrical space" in which the physical phenomenon takes place.

$$|(f,\varphi_\nu)-(f,\varphi_{\nu+k})|<\epsilon \quad \text{if} \quad \nu>\nu_0,\ k>0, \tag{4}$$

where $\varphi_1, \varphi_2, \ldots, \varphi_\nu, \ldots$ is a sequence of fundamental functions that converges in the fundamental space. The properties of linearity and continuity are at the basis of the definition of a special class of functionals, known as generalized functions.

If a rule of correspondence f assigns to each fundamental function $\varphi(x)$ some number (f,φ) satisfying the relations (3) and (4), a linear continuous functional or a generalized function is said to be defined in the fundamental space.

A generalized function is said to be *regular* if its values can be determined by means of a locally integrable point function from the rule defined by expression (2). However, as we shall see below, not all generalized functions can be obtained from this rule. So-called *singular* generalized functions exist, and these, in general, cannot be put in correspondence with point functions. Nonetheless, these functions are necessary for describing certain physical phenomena.

To denote generalized functions, we use the symbols f, $f(x)$, (f,φ), or $\iiint f(x)\varphi(x)\mathrm{d}V$, where f, generally speaking, is the symbol for the rule that assigns to each fundamental function $\varphi(x)$ some number. If a generalized function is singular, the symbols (f,φ) and $\iiint f(x)\varphi(x)\mathrm{d}V$ refer only to the number (and not to an integral, which would have no meaning). The use of the notation $f(x)$, which formally coincides with the notation for a point function, is related to the fact that (as noted above) defining a regular generalized function corresponding to some locally integrable point function is equivalent to defining the point function. (In this sense, it is often stated that a regular generalized function *coincides* with some point function.) Therefore, an extension of the meaning of the symbol $f(x)$ is natural; it should be clear from the text whether one is speaking of an ordinary function or a generalized function. This notation is also convenient in that it shows the argument x on which the fundamental functions depend.

Our immediate purpose is to establish rules for operations on generalized functions. The most important principle that we will use here is the requirement that, in the particular case in which the generalized functions are regular, the rules governing the operations on them will be a consequence of the rules of classical analysis and of definition (2) of a regular generalized function. This ensures that the theory of generalized functions will be consistent with classical analysis.

Problems

1. Suppose that $\varphi(x)$ is a fundamental function. Show which, if any, of the following sequences

$$\text{(a) } \frac{1}{\nu}\varphi(x)\ , \quad \text{(b) } \frac{1}{\nu}\varphi(\nu x)\ , \quad \text{(c) } \frac{1}{\nu}\varphi(x/\nu) \quad (\nu=1,2,3,\ldots)\ ,$$

converge in the fundamental space.

Answer: The sequences (a) and (b) converge to zero in the fundamental space.

2. Show that the generalized function, called the "constant C"

$$(C, \varphi) \equiv C \int\int\int \varphi(x)\, dV ,$$

where C is an arbitrary real number, is regular.

3. *Properties of fundamental and generalized functions. The most important operations on generalized functions*

We present without proof a number of properties of fundamental and generalized functions and we define the most important operations on the latter.

1. The derivatives of fundamental functions are also fundamental functions. They vanish outside the same region as the function from which they were derived by differentiation. The product of a fundamental function and an arbitrary infinitely many times differentiable function (in particular, a constant or another fundamental function) is a fundamental function and the sum or difference of two fundamental functions is a fundamental function.

2. Suppose that $\varphi_1(x), \varphi_2(x), \ldots, \varphi_\nu(x), \ldots$ is a sequence of fundamental functions that converges in the fundamental space (see section 2). Then, the sequence of linear differential expressions

$$\mathcal{M}\varphi_\nu(x) \equiv \sum_{\alpha,\beta=1}^{3} a_{\alpha\beta} \frac{\partial^2 \varphi_\nu}{\partial x_\alpha\, \partial x_\beta} + \sum_{\alpha=1}^{3} b_\alpha \frac{\partial \varphi_\nu}{\partial x_\alpha} + c\varphi_\nu ,$$

where the a_{ij}, b_i, and c are constant coefficients, also concerges in the fundamental space.

3. For any bounded continuous function $g(x)$ and any positive ϵ, there exists a fundamental function $\varphi(x)$ such that

$$|g(x) - \varphi(x)| < \epsilon \qquad \text{for all } x .$$

For any given region and any positive number η, there exist fundamental functions that (1) are equal to unity for points in the given region at a distance greater than or equal to η from its boundary, (2) are equal to zero at all points outside the given region at a distance greater than or equal to η from its boundary, and (3) have values between zero and unity at all remaining points.

Examples of the construction of such fundamental functions are given in problems 4 and 5.

It follows from the existence of fundamental functions with these properties that, for a suitable choice of fundamental function $\varphi(x)$, the value of a regular generalized function corresponding to a given function $f(x)$ of a point x can be made arbitrarily close to the mean value of $f(x)$ in an arbitrary finite region.

4. Generalized functions can be defined on regions of a (geometric) space.

A generalized function f is said to be equal to zero in a region Ω if the number (f, φ) is equal to zero for any fundamental function that is equal to zero outside the region Ω. In particular, a generalized function f is equal to zero in the neighbourhood Ω_x of a point x if $(f, \varphi) = 0$ for all fundamental functions that vanish outside Ω_x.

If a generalized function f is non-zero in some neighbourhood of a point, the point is called an essential point. The set of essential points is called the carrier of the function f. A generalized function is said to be finite if its carrier is bounded.

It follows from these definitions that a regular generalized function f is equal to zero in any region in which the corresponding function $f(x)$ of the point x is equal to zero. The carrier of the generalized function is the set of points such that in any neighbourhood of these points a set of points (of positive measure) exists at which $f(x) \neq 0$.

5. The sum of two generalized functions and the product of a generalized function and a number can be defined in a natural manner. The results of these operations are also generalized functions. We define

$$(f+g, \varphi) = (f, \varphi) + (g, \varphi) , \qquad (af, \varphi) = a(f, \varphi) = (f, a\varphi) ,$$

where f and g are generalized functions and a is a number. Obviously, these operations obey the usual rules of addition and multiplication.

Two generalized functions f and g are said to be *equal* if

$$(f, \varphi) = (g, \varphi)$$

for all fundamental functions $\varphi(x)$.

6. The *product* f_1 of a generalized function f and an infinitely many times differentiable function $\alpha(x)$ is defined so that, in the regular case, it will be the generalized function generated by the product $f(x)\alpha(x)$, that is, so that

$$(f_1, \varphi) \equiv \int\int\int f(x)\ \alpha(x)\ \varphi(x)\ \mathrm{d}V . \tag{5}$$

If f is a singular function, the right side of this equation is not defined (in the classical sense) and the product must be defined by means of a rule according to which a number (f, φ) is assigned to each fundamental function $\varphi(x)$. Noting that $\alpha(x)\varphi(x)$ is also a fundamental function, we define the product $f\alpha \equiv f_1$ by the formula

$$(f_1, \varphi) \equiv (f, \alpha\varphi) . \tag{6}$$

The right side of eq. (6) can be calculated from the rule of formation of the functional (f, φ). In the regular case, formulae (5) and (6) are identical.

The product of two generalized functions or the product of a generalized function and an arbitrary point function is not defined. We note that the product of two generalized functions cannot be defined by the formula

$$(fg, \varphi) = (f, \varphi)\ (g, \varphi) ,$$

since its right member is not a generalized function.

7. As we stated, the notation $f(x)$, which formally coincides with the

notation for a function of the point x, is used for a generalized function defined according to the rule f. This notation is especially useful when making a change of variables.

Let us first examine a classical function $f(x)$ of a point x. Suppose that we have carried out a transformation u of variables, thus introducing new variables:

$$y_i = u_i(x) \qquad (i = 1, 2, 3) \ , \tag{7}$$

where the $u_i(x)$ are functions of the coordinates x_1, x_2, x_3 of the point x (the old variables x_1, x_2, x_3). We know that the transformation (7) is one-to-one if the determinant of the transformation

$$|U| = \begin{vmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{vmatrix} , \tag{8}$$

whose elements are the derivatives

$$\partial u_i/\partial x_j \equiv u_{ij} \ ,$$

does not vanish at any point of the region in question. We shall assume that this is the case. Thus, there exists an inverse transformation from the new variables to the old:

$$x_i = u_i^{-1}(y) \qquad (i = 1, 2, 3) \ ,$$

where the $u_i^{-1}(y)$ are functions of the coordinates y_1, y_2, y_3. We shall write the direct and inverse transformations in the form

$$y = Ux \ , \qquad x = U^{-1}y \ .$$

We denote by $f_1(x) = Uf(x)$ the function of the new variables y_i into which the transformation (7) maps the function $f(x)$. In other words,

$$Uf(x) = f(U^{-1}y) \ . \tag{9}$$

As a simple example, let us take the one-dimensional case. Suppose that $f(x) = \sin x$ and that $y = Ux = \cos x$. Then, $x = U^{-1}y = \text{arc cos } y$ and $f_1(y) = Uf(x) = \sin(\text{arc cos } y)$.

Let us turn to the general transformation. Suppose that

$$(f, \varphi) = \int\int\int f(x)\ \varphi(x)\ \mathrm{d}V \tag{10}$$

is the generalized function corresponding to a locally integrable function $f(x)$. When the transformation U is performed, the function $f(x)$ is mapped into the function $Uf(x) = f(U^{-1}y)$ and the generalized function (10) is mapped into the generalized function

$$(f(U^{-1}y), \varphi(y)) \equiv \int\int\int f(U^{-1}y)\ \varphi(y)\ \mathrm{d}V_y \ .$$

To make this expression meaningful even when the generalized function (f, φ) is singular, we make a change of variables under the integral sign *,

* See V.I.Smirnov [1]), Vol. 2, p. 98.

returning to the variable $x = U^{-1}y$. We then obtain

$$dV_y = |U|\ dV_x\ ,\qquad f(U^{-1}y) = f(x)\ ,\qquad \varphi(y) = \varphi(Ux)\ ,$$

where $|U|$ is the functional determinant (8). Then,

$$\int\!\!\int\!\!\int f(U^{-1}y)\ \varphi(y)\ dV_y = \int\!\!\int\!\!\int f(x)\ |U|\ \varphi(Ux)\ dV_x\ .$$

If the determinant $|U|$ is infinitely many times differentiable and if the transformation U is such that a finite region is always mapped into a finite region, the product $|U|\varphi(Ux)$ is a fundamental function. This relationship can then be written in the form

$$(f(U^{-1}x), \varphi(x)) = (f(x), |U|\varphi(Ux))\ . \tag{11}$$

Here, we write x on both sides of the equation because the symbol used for denoting the variable of integration plays no role.

Eq. (11) can be taken as the definition of the transformation of a generalized function that occurs upon change of variable, since its right side can be calculated if we know the rule of formation of the functional (f,φ). As is easily shown, it is linear and continuous.

As an example, let us examine the transformation

$$y_i = x_i + h_i\ ,$$

called a *translation* of the vector $h(h_1, h_2, h_3)$. We have

$$y = Ux = x + h\ ,\qquad |U| = 1\ ,\qquad x = U^{-1}y = y - h\ .$$

When we apply formula (11), we obtain

$$(f(x-h), \varphi(x)) = (f(x), \varphi(x+h))$$

or

$$\int\!\!\int\!\!\int f(x-h)\ \varphi(x)\ dV = \int\!\!\int\!\!\int f(x)\ \varphi(x+h)\ dV\ .$$

8. (a) A sequence of generalized functions f_ν (for $\nu = 1, 2, 3, \ldots$) is said to converge to the generalized function f if

$$\lim_{\nu\to\infty} (f_\nu, \varphi) = (f, \varphi)$$

for an arbitrary fundamental function $\varphi(x)$.

(b) A series of generalized functions

$$u_1 + u_2 + u_3 + \ldots + u_\nu + \ldots$$

is said to converge to a generalized function f if the sequence of partial sums

$$f_\nu = u_1 + u_2 + \ldots + u_\nu \qquad (\nu = 1, 2, 3, \ldots)$$

converges to a generalized function f.

Let us note that the convergence of the sequence of functions $u_1, u_2, \ldots, u_\nu, \ldots$ to a locally integrable function $f(x)$ in the mean implies convergence of the generalized function (u_ν, φ) to a generalized function (f, φ). To show this, suppose that

$$f(x) - u_\nu(x) \equiv \zeta_\nu(x) \ .$$

It follows from the identity

$$\left[\int\int\int \zeta_\nu(x)\ \varphi(x)\ \mathrm{d}V\right]^2$$

$$= \int\int\int \int\int\int \{\zeta_\nu{}^2(x)\ \varphi^2(\xi) - \tfrac{1}{2}[\zeta_\nu(x)\ \varphi(\xi) - \zeta_\nu(\xi)\ \varphi(x)]^2\}\ \mathrm{d}V_x\ \mathrm{d}V_\xi$$

that

$$(f - u_\nu, \varphi)^2 \leq \left[\int\int\int (f - u_\nu)^2\ \mathrm{d}V\right]\cdot\left[\int\int\int \varphi^2\ \mathrm{d}V\right] .$$

The integral

$$\int\int\int \varphi^2\ \mathrm{d}V$$

is bounded and the integral

$$\int\int\int (f - u_\nu)^2\ \mathrm{d}V$$

by hypothesis approaches zero as ν increases. Consequently, the left side of this inequality, being non-negative, also approaches zero as ν increases. The assertion made above then follows.

9. A generalized function can be a function of some parameter. If a generalized function (f_λ, φ) depends on a parameter λ and if the limit

$$\lim_{\lambda - \lambda_0 \to 0} \frac{(f_\lambda, \varphi) - (f_{\lambda_0}, \varphi)}{\lambda - \lambda_0}$$

exists at a point $\lambda = \lambda_0$, this limit is called the derivative of the generalized function f_λ with respect to the parameter λ and is denoted by $\partial f_\lambda / \partial \lambda$.

Problems

1. Show that the fundamental function $\varphi_i(x)$ is the derivative $\partial\varphi/\partial x_i$ with respect to the coordinate x_i of another fundamental function $\varphi(x)$ if and only if

$$\int_{-\infty}^{\infty} \varphi_i(x)\ \mathrm{d}x_i = 0 \ .$$

Method: Necessity is shown by direct calculation of the integral

$$\int_{-\infty}^{\infty} \frac{\partial \varphi}{\partial x_i}\ \mathrm{d}x_i \ ,$$

sufficiency follows from the fact that

$$\int_{-\infty}^{x_i} \varphi_i(x)\, dx_i$$

is a fundamental function.

2. Use the preceding problem to show that every fundamental function $\varphi(x)$ can be represented in the form

$$\varphi(x) = \varphi_0(x) \int_{-\infty}^{\infty} \varphi(x)\, dx_i + \varphi_i(x) \ , \tag{*}$$

where $\varphi_i(x)$ is a fundamental function that can be represented as the derivative with respect to x_i of another fundamental function and $\varphi_0(x)$ is a fundamental function satisfying the condition

$$\int_{-\infty}^{\infty} \varphi_0(x)\, dx_i = 1$$

Method: Show that $\varphi_i(x)$ defined by the equation (*) satisfies the condition of the preceding problem.

3. Show that the function

$$\varphi(|\xi|,\eta) \equiv \begin{cases} \exp[-\eta^2/(\eta^2 - |\xi|^2)] & \text{for} \quad |\xi| < \eta \ , \\ 0 & \text{for} \quad |\xi| \geqslant \eta \ , \end{cases}$$

where η is a constant and $|\xi|$ is the distance between the point ξ and the coordinate origin, is a fundamental function that vanishes outside the sphere of radius η with center at the coordinate origin.

4. Suppose that $g(x)$ is a continuous function of the point x and that $\varphi(|\xi|,\eta)$ is the function defined in the preceding problem. Show that the function

$$\varphi_\eta(x) = \frac{\displaystyle\iiint_{|x-\xi|\leqslant\eta} g(\xi)\, \varphi(|x-\xi|,\eta)\, dV_\xi}{\displaystyle\iiint_{|\xi|\leqslant\eta} \varphi(|\xi|,\eta)\, dV}$$

is infinitely many times differentiable and that, as η approaches zero, it approaches $g(x)$ uniformly in every finite region; that is, for any positive number ϵ and for any fixed finite region, a sufficiently small number η exists such that for all x belonging to this region, the inequality

$$|g(x) - \varphi_\eta(x)| < \epsilon$$

holds. If $g(x)$ is a finite continuous function, show that $\varphi_\eta(x)$ is a fundamental function.

Method: Choose η so that the inequality $|g(x) - g(\xi)| < \epsilon$ holds for $|x-\xi| < \eta$ in the region in question and use the mean-value theorem.

5. Suppose that the function $g(x)$ is equal to 1 insice a region V and that it is equal to zero outside it. Show that the function $\varphi_\eta(x)$, defined in problem 4, is equal to 1 at all points of the region V that are at a distance greater than or equal to η from its boundary, is equal to zero at points outside the region V at a distance greater than or equal to η from its boundary, and has a value between zero and 1 at all other points.

6. Show that under the similarity transformation $Ux = \alpha x$, where α is a real number, generalized functions are transformed according to

$$(f(\alpha x), \varphi(x)) = \alpha^3 (f(x), \varphi(\alpha x)) .$$

7. A generalized function $f(x)$ satisfying the equality

$$f(\alpha x) = \alpha^\lambda f(x) \tag{*}$$

is said to be a homogeneous generalized function of degree λ. Write the equation (*) in the form of a relationship between functionals.

8. Show that if the locally integrable functions $f_\nu(x)$ $(\nu = 1, 2, 3, \ldots)$ converge uniformly to a locally integrable function $f(x)$, the generalized functions (f_ν, φ) converge to the generalized functions (f, φ)

4. *Differentiation of generalized functions. The concept of generalized solutions of differential equations*

Let us suppose that a locally integrable linear differential expression $\mathcal{M}u$ can be formed from a function $u(x)$ of the coordinates of the point x. In this case, the regular generalized function

$$(\mathcal{M}u, \varphi) \equiv \int\int\int (\mathcal{M}u)\ \varphi\ \mathrm{d}V \tag{12}$$

is naturally called a differential expression $\mathcal{M}u$ of the generalized function u.

To extend this definition to singular generalized functions, we transform the expression (12) by means of Green's formula (6) of Chapter XVII by keeping the region of integration sufficiently large that the function $\varphi(x)$ vanishes on its boundary. Then, we obtain

$$\int\int\int (\mathcal{M}u)\ \varphi\ \mathrm{d}V = \int\int\int u(\mathcal{N}\varphi)\ \mathrm{d}V ,$$

or

$$(\mathcal{M}u, \varphi) = (u, \mathcal{N}\varphi) . \tag{13}$$

The expression on the right side of this equation is defined in the general case only when $\mathcal{N}\varphi$ is a fundamental function, that is, one that vanishes outside some finite region and is infinitely many times differentiable. The first condition is satisfied since φ is a fundamental function. The second condition, however, holds only when the coefficients of the expression $\mathcal{N}\varphi$ are also infinitely many times differentiable. Under this assumption, we

shall call the functional $(u, \mathcal{N}\varphi)$ the differential expression $\mathcal{M}u$ of the generalized function u. Obviously, the functional on the right side of eq. (13) is linear. On the basis of the properties of fundamental functions that we listed in paragraph 2 of section 3, it is also continuous. This is true because if the sequence φ_ν (for $\nu = 1, 2, 3, \ldots$) converges in the fundamental space, the sequence $\mathcal{N}\varphi_\nu$ will also converge in it. Since the functional (u, φ) is continuous by hypothesis, the inequalities (4) will also be satisfied in the particular case of a sequence of fundamental functions $\mathcal{N}\varphi_\nu$; this is, the functional $(u, \mathcal{N}\varphi)$ will also be continuous. Thus, the differential expression $\mathcal{M}u$ of a generalized function is also a generalized function.

In the particular case in which

$$\mathcal{M}u = \partial u/\partial x_i ,$$

we obtain

$$\left(\frac{\partial u}{\partial x_i}, \varphi\right) = \left(u, -\frac{\partial \varphi}{\partial x_i}\right) . \tag{14}$$

Using this formula, we can construct the derivatives of a generalized function of arbitrary order; these derivatives will also be generalized functions. Consequently, a generalized function is infinitely many times differentiable. It is easy to show that the derivative of a sum of generalized functions is equal to the sum of their derivatives.

Suppose that f_ν (for $\nu = 1, 2, 3, \ldots$) is a sequence of generalized functions that converges to the generalized function f, that is, that

$$\lim_{\nu \to \infty} (f_\nu, \varphi) = (f, \varphi) . \tag{15}$$

Then, the sequence of derivatives $\partial f_\nu/\partial x_i$ converges to the derivative $\partial f/\partial x_i$. Specifically,

$$\left(\frac{\partial f_\nu}{\partial x_i}, \varphi\right) = \left(f_\nu, -\frac{\partial \varphi}{\partial x_i}\right) .$$

The expression on the right side converges to

$$\left(f, -\frac{\partial \varphi}{\partial x_i}\right) = \left(\frac{\partial f}{\partial x_i}, \varphi\right)$$

because of eq. (15).

A series of generalized functions

$$u_1 + u_2 + \ldots + u_\nu + \ldots$$

that converges to a generalized function f can be termwise differentiated. Then, the series of derivatives

$$u_1' + u_2' + \ldots u_\nu' + \ldots$$

converges to the derivative f'. To show this, note that the sequence of partial sums

$$f_\nu = u_1 + u_2 + \ldots + u_\nu$$

converges to f. Consequently, on the basis of the assertion that we just proved, the sequence of partial sums

$$f'_\nu = u'_1 + u'_2 + \dots + u'_\nu$$

converges to f', as was asserted.

As an example, let us consider the series

$$\sum_{\alpha=1}^{\infty} \alpha^k \cos \alpha x \, , \tag{16}$$

where k is an arbitrary number and x is a point on a straight line. When k is positive, this series diverges. However, the series of the corresponding generalized functions is meaningful. This is true because the series

$$\sum_{\alpha=1}^{\infty} \frac{\cos \alpha x}{\alpha^r} \qquad (r > 1)$$

converges. But the series (16), understood as a sum of generalized functions, is obtained by termwise differentiation of this series and, consequently, converges to some generalized function.

The solving of partial differential equations often leads to convergent series such that each term of the series satisfies the homogeneous equation in question; yet the series cannot be termwise differentiated because the convergence is destroyed thereby. This makes it impossible to be sure that the sum of the series also satisfies that equation. The theory of differentiation of series of generalized functions presented above makes it possible to assert that the sum of these series satisfies the equation in question at least in the generalized sense. For suppose that the series

$$u = \sum_{\alpha=1}^{\infty} u_\alpha \tag{17}$$

converges and that each of its terms satisfies the homogeneous equation $\mathcal{M} u_\alpha = 0$. The series of generalized functions corresponding to the series (17) can always be differentiated infinitely many times. Therefore,

$$(\mathcal{M} u, \varphi) = \sum_{\alpha=1}^{\infty} (\mathcal{M} u_\alpha, \varphi) \, .$$

But each of the terms on the right side is equal to zero. Consequently,

$$(\mathcal{M} u, \varphi) = 0 \, ,$$

as was asserted.

A function $u(x)$ of a point x satisfying the differential equation

$$\mathcal{M} u = f \tag{18}$$

in the generalized sense

$$(\mathcal{M} u - f, \varphi) = 0 \tag{19}$$

is called a generalized solution of this equation. If the function $u(x)$ is a classical solution, it is also a generalized solution.

Eq. (19) can also have singular generalized functions that are not representable in terms of point functions as its solution. These functions can be called singular generalized solutions of eq. (19) (or of eq. (18), with the generalized sense understood).

Later, we shall give a number of examples of generalized solutions.

We note in conclusion that generalized functions were introduced into mathematics by S. L. Sobolev in connection with the problem of solving differential equations (1936).

Problems

1. Show that the mixed derivatives of generalized functions do not depend on the order of differentiation.

2. Show that the definition of the derivative of a generalized function u in the direction of the vector $h(h_1, h_2, h_3)$ as the limit

$$\lim_{\theta \to 0} \left(u, \frac{\varphi(x - \theta h) - \varphi(x)}{\theta h}\right),$$

where θ is some number and h the length of the vector $\boldsymbol{h}$, is equivalent to the definition given in the text.
Method: Use the definition of a translation given in paragraph 7 of the preceding section.

3. Suppose that fundamental functions depend on a single independent variable x (the one-dimensional case). Suppose that u is a generalized function and that $du/dx = 0$. Show that u is a constant.
Solution: Let us represent an arbitrary fundamental function in the form (problem 2 of section 3):

$$\varphi(x) = \varphi_0(x) \int_{-\infty}^{\infty} \varphi(x)\, dx + \varphi_1(x),$$

where

$$\int_{-\infty}^{\infty} \varphi_0(x)\, dx = 1$$

and $\varphi_1(x)$ is the derivative of a fundamental function. By definition,

$$\left(\frac{du}{dx}, \varphi\right) = \left(u, -\frac{d\varphi}{dx}\right).$$

Consequently, it follows from the condition of the problem that

$$(u, \varphi_1) = 0$$

and hence that

$$(u,\varphi) = (u,\varphi_0(x)) \int_{-\infty}^{\infty} \varphi(x)\, dx = \int_{-\infty}^{\infty} C\, \varphi(x)\, dx \ ,$$

where $C = (u, \varphi_0)$ is a constant. Then, it follows by definition (see problem 2 of section 2) that $u = C$. The constant C can be chosen arbitrarily, since

$$\left(\frac{dC}{dx}, \varphi\right) = \left(C, -\frac{d\varphi}{dx}\right) = -C \int_{-\infty}^{\infty} \frac{d\varphi}{dx}\, dx = 0$$

for arbitrary choices of C and $\varphi(x)$.

4. Find (for the one-dimensional case) the solution of the equation

$$du/dx = f \ ,$$

where f is a generalized function.
Solution: By hypothesis,

$$(u, -\varphi') = (f, \varphi) \ .$$

If we represent an arbitrary function $\varphi(x)$ in the same form as in the preceding problem and if we use the condition of the present problem, we obtain

$$(u,\varphi) = (u,\varphi_0) \int_{-\infty}^{\infty} \varphi(x)\, dx - \left(f, \int_{-\infty}^{x} \varphi_1(x)\, dx\right) .$$

The first term on the right side is a constant. By transforming the second term, we obtain

$$(u,\varphi) = (C,\varphi) - \left(f, \int_{-\infty}^{x} \left[\varphi(\xi) - \varphi_0(\xi) \int_{-\infty}^{\infty} \varphi(\zeta)\, d\zeta\right] d\xi\right) .$$

It remains to show that the functional obtained is linear and continuous.
Remark: The solution u is called the indefinite integral of the generalized function f.

5. *The Dirac delta function*

An important example of a singular generalized function is the delta function defined by Dirac (long before the appearance of the concept of a generalized function as such). In present-day notation, the delta function $\delta(x-\xi)$ can be defined by the equation

$$(\delta(x-\xi), \varphi(x)) = \varphi(\xi) \ , \tag{20}$$

where ξ is an arbitrary fixed point. In other words, to every fundamental function $\varphi(x)$ there corresponds a value $\varphi(\xi)$ at the point ξ.

The carrier of the delta function, $\delta(x-\xi)$, is obviously a point ξ. There-

fore, the value of the functional $(\delta(x-\xi), \varphi(x))$ is the same for all fundamental functions that have the same value at the point $x = \xi$.

Let us rewrite eq. (20) in the form

$$\iiint \delta(x-\xi)\ \varphi(x)\ \mathrm{d}V_x = \varphi(\xi)$$

and let us regard it formally as an integral. By virtue of the remark about the carrier of the delta function, the parts of this integral that are obtained by integrating over regions not containing the point ξ must be equal to zero. Therefore, for all fundamental functions that are equal to unity at the point $x = \xi$, we may formally write

$$\iiint_V \delta(x-\xi)\ \mathrm{d}V = \begin{cases} 1 & \text{when } \xi \text{ is inside } V, \\ 0 & \text{when } \xi \text{ is outside } V. \end{cases} \tag{21}$$

This relationship sometimes serves as the definition of the delta function in physics. This definition, however, is not altogether rigorous because there is no function in the classical sense that satisfies eq. (21) (see problem 1).

Let us define a point source in terms of the delta function. If the sources are distributed continuously with density ρ, the amount of the substance (a fluid, for example) formed in the region V will be equal to the integral

$$\iiint_V \rho\,\mathrm{d}V.$$

However, if the source is a point source, this amount will be equal either to the output of the source, when the latter is placed inside the region V, or equal to zero, when the source is placed outside it. Thus, we are dealing with a situation of the same nature as in formula (21). Therefore, it is obvious that the point source must be identified with the delta function multiplied by a constant equal to the output of the source.

Problems

1. Show that the delta function is singular.

   Method: It is necessary to show that the equation

   $$\iiint f(x)\ \varphi(x)\ \mathrm{d}V = \varphi(0)\ , \tag{*}$$

   where $\varphi(x)$ is an *arbitrary* fundamental function, cannot be valid for any locally integrable function $f(x)$. For the fundamental functions $\varphi(|\xi|, \eta)$ (see problem 3 of section 3), we would have

   $$\iiint f(x)\ \varphi(|\xi|, \eta)\ \mathrm{d}V = \frac{1}{e} = \text{constant.}$$

   To solve the problem, it remains to show that no matter what the choice of $f(x)$, the left side of this equation will depend on η whereas the right

side is constant. The contradiction shows that the equation (*) can not be valid for any integrable function.

2. Derive a formula defining the derivative of the delta function.
Answer:

$$(\delta^{(n)}(x-\xi),\varphi(x)) = (-1)^n\,\varphi^{(n)}(\xi)\ .$$

3. Show that in the one-dimensional case, the delta function is a derivative of the function

$$\delta(x-\xi) = \begin{cases} 0 & \text{for} \quad x<\xi\ , \\ 1 & \text{for} \quad x>\xi\ . \end{cases}$$

4. Suppose that $f(x)$ is a piecewise continuous one-dimensional function of the point x with a piecewise continuous derivative. Find the derivative of the generalized function f.
Solution: The function $f(x)$ can be written in the form

$$f(x) = f_1(x) + \sum_\alpha \Delta_\alpha \theta(x-x_\alpha)\ ,$$

where $f_1(x)$ is a function with a continuous derivative, Δ_α is the discontinuity of the function $f(x)$ at the point x_α and $\theta(x\text{-}x_\alpha)$ is the function defined in the preceding problem. The summation should be carried out over all points of discontinuity. Therefore,

$$f'(x) = f_1'(x) + \sum_\alpha \Delta_\alpha \delta(x-x_\alpha)\ .$$

6. *Convolutions of generalized functions*

The integral

$$\int\int\int f(\xi)\ g(x-\xi)\ \mathrm{d}V_\xi\ , \tag{22}$$

which depends on the coordinates of the point x and on the parameters, is called the convolution $f(x) * g(x)$ of the two (integrable) functions $f(x)$ and $g(x)$.

Particular solutions of differential equations can be represented in the form of convolutions. For example, the Newtonian potential

$$u(x) \equiv \int\int\int \frac{\rho(\xi)}{r(x-\xi)}\,\mathrm{d}V_\xi \equiv \rho(x) * \frac{1}{r(x)} \qquad (r(x) \equiv \sqrt{x_1{}^2 + x_2{}^2 + x_3{}^2}) \tag{23}$$

represents the solution of Poisson's equation

$$\Delta u = -4\pi\rho\ . \tag{24}$$

The convolution

$$u(x,t) = \int\int\int f(\xi)\, \frac{\exp\left[-|x-\xi|^2/4k^2t\right]}{(4\pi k^2 t)^{\frac{3}{2}}}\, dV_\xi \equiv f(x) * \frac{\exp\left[-|x|^2/4k^2t\right]}{(4\pi k^2 t)^{\frac{3}{2}}} \tag{25}$$

is the solution of the heat-flow equation

$$\Delta u = \frac{1}{k^2}\frac{\partial u}{\partial t}$$

with the initial condition $u(x,0) = f(x)$. In the one-dimensional case, the convolution

$$u(x,t) = \int f(\xi)\, \frac{\exp\left[-(x-\xi)^2/4k^2t\right]}{(2\pi k^2 t)^{\frac{1}{2}}}\, d\xi = f(x) * \frac{\exp\left[-x^2/4k^2t\right]}{(2\pi k^2 t)^{\frac{1}{2}}} \tag{26}$$

represents the solution of the heat-flow equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k^2}\frac{\partial u}{\partial t}$$

for an infinitely long rod with the initial condition $u(x,0) = f(x)$.

To the convolution (22) we can assign the generalized function

$$(f(x) * g(x), \varphi(x)) = \int\int\int\left[\int\int\int f(\xi)\, g(x-\xi)\, dV_\xi\right] \varphi(x)\, dV_x$$

$$= \int\int\int\int\int\int f(\xi)\, g(\zeta)\, \varphi(\xi+\zeta)\, dV_\xi\, dV_\zeta = (f(\xi), (g(\zeta), \varphi(\xi+\zeta))) \;, \tag{27}$$

which we shall call the convolution of the generalized functions f and g.

To give a general definition of a convolution, we need to give a rule mapping arbitrary generalized functions (f,φ) and (g,φ) into their convolution $(f*g,\varphi)$. In the general case, we cannot give such a rule because the function $\varphi(\xi+\zeta)$ appearing on the right side of eq. (27) does not vanish outside a bounded region of variation of ξ and ζ and hence is not fundamental. Therefore, knowing the functionals (f,φ) and (g,φ), does not make it possible for us to determine the functionals $(g,\varphi(\xi+\zeta))$ and $(f,(g,\varphi(\xi+\zeta)))$. Nevertheless, a definition of the convolution of two singular generalized functions is possible within certain limits.

Let us suppose that at least one of the generalized functions f and g vanishes outside some finite interval V^*. Let us introduce the set of functions $\varphi(x,\xi)$ that are infinitely many times differentiable with respect to the coordinates of the points x and ξ (when both points x and ξ lie outside some bounded region containing the region V^*) and that are different from the functions $\varphi(x+\xi)$ when at least one of the points x and ξ belong to the region V^*. The functions $\varphi(x,\xi)$ are fundamental with respect to the coordinates of each of the points x and ξ. Consequently, if we regard the point x, for example, as a paremeter and use the rule defining the functional (g,φ), we may calculate the functional $(g,\varphi(x,\xi))$. This functional, being a function of the point x, vanishes outside some finite region and, as is easily shown, is infinitely many times differentiable with respect to the coordinates of the point x. Consequently, since it is a function of the point x, it is a funda-

mental function and, by virtue of the definition of the functional (f,φ), we can evaluate the functional

$$(f*g,\varphi) = (f,(g,\overset{*}{\varphi}(x,\xi))) \;, \tag{28}$$

which we shall call the *convolution of the generalized functions* f and g. The consistency of our definition follows from the fact that, for regular generalized functions generated by the locally integrable functions f and g, this definition leads to the integral formula (27).

The following properties of convolutions are easily established:

$$f*g = g*f \qquad \text{(commutativity)} \;, \tag{29}$$

$$(f*g)*h = f*(g*h) \qquad \text{(associativity)} \;, \tag{30}$$

$$\delta(x)*f = f \,. \tag{31}$$

Let us prove formula (31). We have

$$(\delta(x)*f,\varphi) = (\delta(x),(f,\overset{*}{\varphi}(x,\xi)) = (f,\overset{*}{\varphi}(0,\xi)) \,.$$

In the present case, the region V^* is reduced to the coordinate origin, since outside a neighbourhood of the coordinate origin the delta function $\delta(x)$ is equal to zero. But when x belongs to V^*, we have, by definition, $\overset{*}{\varphi}(x,\xi) = \varphi(x+\xi)$. Therefore, $\overset{*}{\varphi}(0,\xi) = \varphi(\xi)$, so that

$$(\delta*f,\varphi) = (f,\varphi(\xi)) = (f,\varphi) \;,$$

as was asserted.

Let us prove the lemma asserting the continuity of a convolution:

LEMMA. *Suppose that f_ν (for $\nu = 1,2,3,\ldots$) is sequence of generalized functions that approaches the generalized function f with increasing ν and that outside a bounded region V^* either the generalized function g or the functions f_ν for all values of ν approach zero. Then, as ν increases, the convolution $f_\nu * g$ approaches the convolution $f*g$.*

Proof: By the definition of the convolution.

$$(f_\nu * g,\varphi) = (f_\nu(\xi),(g(x),\overset{*}{\varphi}(x,\xi))) \,.$$

But, as we have seen, the functional $(g(x),\overset{*}{\varphi}(x,\xi))$, treated as a function of ξ is a fundamental function. The conclusion of the lemma then follows on the basis of the definition of convergence of generalized functions (section 3, paragraph 8).

Let us form the differential expression $\mathcal{M}(f*g)$ with constant coefficients from the convolution $f*g$. By using the definitions of differentiation and convolution, we obtain

$$(\mathcal{M}(f*g),\varphi) = (f*g,\mathcal{N}\varphi) = (f,(g,\mathcal{N}\varphi)) = (f,(\mathcal{M}g,\varphi)) = (f*\mathcal{M}g,\varphi)$$

Because of the commutativity of a convolution, the functions f and g can reverse places so that

$$(\mathcal{M}(f*g),\varphi) = (\mathcal{M}f*g,\varphi)$$

or

$$\mathcal{M}(f*g) = \mathcal{M}f*g = f*\mathcal{M}g \,. \tag{32}$$

Let us suppose that the generalized function $f = f_t$ depends on the parameter t and has a derivative $\partial f_t/\partial t$. Suppose also that outside a bounded region V^* either the function f_t (for all values of the parameter t) or the function g vanishes. Then, the general definition of a convolution (28) is valid, and we have the following formula for differentiating the convolution with respect to a parameter:

$$\frac{\partial}{\partial t}(f_t * g) = \frac{\partial f_t}{\partial t} * g \, . \tag{33}$$

This is true because, under the assumptions that were made, we may again define a function $\overset{*}{\varphi}(x, \xi)$ and evaluate expression (28). Since the function $\psi(x) = (g, \overset{*}{\varphi}(x, \xi))$ is fundamental, the general rule (see section 3, paragraph 9) for differentiating with respect to a parameter is applicable to the functional

$$(f_t * g, \varphi) = (f_t, \psi) \, .$$

The validity of formula (33) follows.

By using the definition and properties of a convolution, we may obtain a number of relationships encompassing the delta function and the generalizing formulae found by classical means.

As a preliminary, we note that property (31) can be used to define the delta function. For, on the basis of formulae (28), (29), and (31),

$$(\delta * f, \varphi) = (f(\xi), (\delta(x), \overset{*}{\varphi}(x, \xi))) \, .$$

But as we saw above,

$$(\delta * f, \varphi) = (f, \varphi) \, .$$

Since these relations are valid for an arbitrary choice of the function f, it follows that

$$(\delta(x), \overset{*}{\varphi}(x, \xi)) = \varphi(\xi) \, .$$

For functions $\varphi(x, \xi)$, that are equal to $\varphi(x + \xi)$ in some neighbourhood of the point ξ, we then obtain

$$(\delta(x), \varphi(x + \xi)) = \varphi(\xi) \, .$$

But according to section 3, paragraph 7, the left side of this equation represents the translation $(\delta(x - \xi), \varphi(\xi))$ of the generalized function $\delta(x)$ by the vector ξ. Therefore, we obtain

$$(\delta(x - \xi), \varphi(x)) = \varphi(\xi) \, ,$$

which coincides with the definition of the delta function (20).

Let us turn now to the convolution (23). When considered as a convolution of generalized functions, it represents a generalized function corresponding to the Newtonian potential. We shall call it the generalized Newtonian potential.

As we note, if $\rho(x)$ is a function with continuous first derivatives, the Newtonian potential $u(x)$ satisfies Poisson's equation. If we apply the Laplacian operator to both sides of eq. (23), using the rules for differentiating a convolution, we obtain, on the basis of eq. (24),

$$\rho = \rho * \left(-\frac{1}{4\pi} \Delta \frac{1}{r}\right) .$$

Since the function ρ is subject only to very general requirements of smoothness and is basically arbitrary, we find, by virtue of eqs. (29) and (31), that

$$-\frac{1}{4\pi} \Delta \frac{1}{r(x)} = \delta(x) . \tag{34}$$

On the other hand, if we keep this equation in mind and recall that ρ is an arbitrary generalized function, we obtain on the basis of eq. (31),

$$\Delta u = \rho * \Delta \frac{1}{r} = -4\pi\rho * \delta = -4\pi\rho ;$$

that is, the generalized Newtonian potential also satisfies Poisson's equation.

Turning to eq. (25) and noting that, from what was said above,

$$\lim_{t\to 0} u(x, t) = f(x) ,$$

we can, on the basis of the lemma on the continuity of a convolution, write

$$f(x) = f(x) * \lim_{t\to 0} \frac{\exp\left[-|x|^2/4k^2t\right]}{(4\pi k^2 t)^{\frac{3}{2}}} ,$$

so that, by using the same reasoning as in deriving formula (34) we find that

$$\lim_{t\to 0} \frac{\exp\left[-|x|^2/4k^2t\right]}{(4\pi k^2 t)^{\frac{3}{2}}} = \delta x . \tag{35}$$

Problems

1. Prove that convolutions are commutative.
   Method: Show that

$$(g, (f, \varphi)) - (f, (g, \varphi)) = (f - g, (f - g, \varphi)) - (f - g, (f - g, \varphi)) .$$

2. Derive the formula

$$\mathcal{M}\delta * f = \mathcal{M} f ,$$

   where $\mathcal{M} f$ is a differential expression, with constant coefficients, of the generalized function f.

3. Show that the definition of a convolution (28) is meaningful even when $f = 0$ for $|x| < a$, and $g = 0$ for $|x| < b$, where a and b are finite positive numbers and $|x|$ is the distance between the point x and the coordinate origin.

4. Show that, in the one-dimensional case,

$$\lim_{t\to 0} \frac{\exp[-x^2/4k^2t]}{(2\pi k^2 t)^{\frac{1}{2}}} = \delta(x) .$$

7. *The concept of fundamental solutions*

Let us consider the Cauchy problem for a differential equation of the parabolic type:

$$\mathcal{M}u = \partial u/\partial t , \qquad t > 0 . \tag{36}$$

$$u|_{t=0} = u_0 , \tag{37}$$

where $\mathcal{M}u$ is a differential expression of the elliptic type with constant coefficients, and u_0 is a generalized function that vanishes outside a finite region. We shall call a solution G_t of this Cauchy problem for a particular value $u_0 = \delta(x)$ the *fundamental* solution. The convolution

$$u = G_t * u_0$$

then represents the solution of the Cauchy problem for an arbitrary initial condition (37). For if t is positive,

$$\mathcal{M}u - \frac{\partial u}{\partial t} = \mathcal{M}(G_t * u_0) - \frac{\partial}{\partial t}(G_t * u_0) = \mathcal{M}G_t * u_0 - \frac{\partial G_t}{\partial t} * u_0 = \left(\mathcal{M}G_t - \frac{\partial G_t}{\partial t}\right) * u_0 .$$

But

$$\mathcal{M}G_t - \frac{\partial G_t}{\partial t} = 0 ,$$

since the fundamental solution G_t satisfies eq. (36). Consequently,

$$\mathcal{M}u - \frac{\partial u}{\partial t} = 0$$

and eq. (36) is satisfied. Furthermore,

$$\lim_{t\to 0} u \equiv (\lim_{t\to 0} G_t) * u_0 = \delta(x) * u_0 = u_0$$

and the initial condition (37) is also satisfied.

The function

$$G_t = \frac{\exp[-|x|^2/4k^2t]}{(4\pi k^2 t)^{\frac{1}{2}}} ,$$

which for positive values of t satisfies the heat-flow equation

$$\Delta u = \frac{1}{k^2} \frac{\partial u}{\partial t} ,$$

and, according to eq. (35), tends to the delta function as t tends to zero, serves as an example of a fundamental solution of the Cauchy problem.

Let us now consider the Cauchy problem for an equation of the hyperbolic type:

$$\mathfrak{M}u = \partial^2 u/\partial t^2 \,, \qquad t > 0 \,, \tag{38}$$

$$u|_{t=0} = u_0 \,, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1 \,, \tag{39}$$

where u_0 and u_1 are finite generalized functions. The solution G_t of the Cauchy problem (38) - (39) satisfying the initial conditions

$$G_t|_{t=0} = 0 \,, \qquad \left.\frac{\partial G_t}{\partial t}\right|_{t=0} = \delta(x)$$

is called the fundamental solution. The convolution

$$u = G_t * u$$

is then the solution of the Cauchy problem with the initial conditions

$$u|_{t=0} = 0 \,, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1 \,.$$

The proof of this statement is left to the reader.

Suppose that u^* is a solution of eq. (38) satisfying the initial conditions

$$u|_{t=0} = 0 \,, \qquad \left.\frac{\partial u^*}{\partial t}\right|_{t=0} = u_1^* \,.$$

The function $v \equiv \partial u^*/\partial t$ is then a solution of eq. (38) satisfying the initial conditions

$$v|_{t=0} = u_1^* \,, \qquad \left.\frac{\partial v}{\partial t}\right|_{t=0} = \left.\frac{\partial^2 u^*}{\partial t^2}\right|_{t=0}$$

From this it is clear that if we find two solutions u_{I} and u_{II} of eq. (38) satisfying the initial conditions

$$u_{\mathrm{I}}|_{t=0} = 0 \,, \qquad \left.\frac{\partial u_{\mathrm{I}}}{\partial t}\right|_{t=0} = u_0 \,,$$

$$u_{\mathrm{II}}|_{t=0} = 0 \,, \qquad \left.\frac{\partial u_{\mathrm{II}}}{\partial t}\right|_{t=0} = u_{\mathrm{I}} - \left.\frac{\partial^2 u_{\mathrm{I}}}{\partial t^2}\right|_{t=0} \,,$$

we can construct the solution of the Cauchy problem (38) - (39):

$$u = \frac{\partial u_{\mathrm{I}}}{\partial t} + u_{\mathrm{II}} = \frac{\partial G_t}{\partial t} * u_0 + G_t * u_{\mathrm{I}} - G_t * \left.\frac{\partial^2 G_t}{\partial t^2}\right|_{t=0} * u_0 \,. \tag{40}$$

As an example, let us examine the problem

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \,, \tag{41}$$

$$u|_{t=0} = u_0 \,, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1 \tag{42}$$

for the vibration of an infinitely long string under the action of an initial

disturbance characterized by eq. (42). We have already had occasion to deal with this problem in the first part of this book. However, now the symbols u_0 and u_1 denote arbitrary generalized functions.

Let us show that the finite function

$$G_t = \begin{cases} 1/2c & \text{for} \quad |x| < ct\,, \\ 0 & \text{for} \quad |x| > ct\,, \end{cases} \tag{43}$$

can be chosen as a fundamental solution of the Cauchy problem (41)-(42). To prove this, let us evaluate the derivatives. By definition,

$$\left(\frac{\partial G_t}{\partial x}, \varphi\right) = -\left(G_t, \frac{d\varphi}{dx}\right) = -\int G_t \frac{d\varphi}{dx}\,dx\,.$$

Keeping eq. (43) in mind, let us transform this integral:

$$-\int G_t \frac{d\varphi}{dx}\,dx = -\frac{1}{2c}\int_{-ct}^{ct} d\varphi = \frac{1}{2c}[\varphi(-ct) - \varphi(ct)]\,.$$

But

$$\frac{1}{2c}[\varphi(-ct) - \varphi(ct)] \equiv \frac{1}{2c}\int \delta(x+ct)\,\varphi(x)\,dx - \frac{1}{2c}\int \delta(x-ct)\,\varphi(x)\,dx$$

$$= \frac{1}{2c}(\delta(x+ct), \varphi) - \frac{1}{2c}(\delta(x-ct), \varphi)\,.$$

Thus,

$$\left(\frac{\partial G_t}{\partial x}, \varphi\right) = \left(\frac{1}{2c}\delta(x+ct) - \frac{1}{2c}\delta(x-ct), \varphi\right)$$

or

$$\frac{\partial G_t}{\partial x} = \frac{1}{2c}\delta(x+ct) - \frac{1}{2c}\delta(x-ct)\,.$$

If we again differentiate and multiply by c^2, we obtain

$$c^2\frac{\partial^2 G_t}{\partial x^2} = \tfrac{1}{2}c\delta'(x+ct) - \tfrac{1}{2}c\delta'(x-ct)\,. \tag{44}$$

Furthermore, if we differentiate the functional

$$(G_t, \varphi) = \int G_t\,\varphi\,dx = \frac{1}{2c}\int_{-ct}^{ct}\varphi\,dx$$

with respect to the parameter t, we obtain

$$\left(\frac{\partial G_t}{\partial t}, \varphi\right) = \tfrac{1}{2}[\varphi(ct) + \varphi(-ct)] = \tfrac{1}{2}([\delta(x-ct) + \delta(x+ct)], \varphi)\,,$$

or

$$\frac{\partial G_t}{\partial t} = \tfrac{1}{2}\delta(x+ct) + \tfrac{1}{2}\delta(x-ct)\,, \tag{45}$$

so that

$$\frac{\partial^2 G_t}{\partial t^2} = \tfrac{1}{2}c\delta'(x+ct) - \tfrac{1}{2}c\delta'(x-ct) \ . \tag{46}$$

If we compare eqs. (44) and (46), we see that the function G_t satisfies eq. (41). Also, it follows from eqs. (43) and (45) that

$$G_t|_{t=0} = 0 \ , \qquad \left.\frac{\partial G_t}{\partial t}\right|_{t=0} = \delta(x) \ .$$

Consequently, the function G_t is indeed a fundamental solution of the problem (41) - (42) and its general solution can be written in the form (40).

If u_0 and u_1 are locally integrable functions,

$$\frac{\partial G_t}{\partial t} * u_0 = \tfrac{1}{2}\delta(x+ct) * u_0 + \tfrac{1}{2}\delta(x-ct) * u_0 = \tfrac{1}{2}u_0(x+ct) + \tfrac{1}{2}u_0(x-ct) \ ,$$

$$G_t * u_1 = \int_{-\infty}^{\infty} G_t(\xi)\, u_1(x-\xi)\, \mathrm{d}\xi = \tfrac{1}{2}\int_{-ct}^{ct} u_1(x-\xi)\, \mathrm{d}\xi = \tfrac{1}{2}\int_{x-ct}^{x+ct} u_1(\xi)\, \mathrm{d}\xi \ .$$

On the basis of eq. (46) for the problem in question,

$$\left.\frac{\partial^2 G_t}{\partial t^2}\right|_{t=0} = 0 \ .$$

If we combine the expressions that we have obtained for the convolutions, we obtain the classical d'Alembert solutions:

$$u = \frac{u_0(x+ct) + u_0(x-ct)}{2} + \tfrac{1}{2}\int_{x-ct}^{x+ct} u_1(\xi)\, \mathrm{d}\xi \ .$$

Finally, let us examine the non-homogeneous equation of the elliptic type

$$\mathfrak{M}u = f \ , \tag{47}$$

where f is a finite generalized function. We shall refer to the function L which satisfies this equation at the particular value $f = \delta(x)$ as its fundamental solution. The convolution

$$u = f * L \tag{48}$$

is, in this case, a solution of eq. (47) for an arbitrary value of f.

Let us consider, for example, the Helmholtz equation

$$\Delta u + k^2 u = -4\pi\rho \ . \tag{49}$$

As we know (Chapter V, section 6), if a function ρ is sufficiently smooth, the oscillational potential

$$u = \iiint \rho\, \frac{e^{ikr}}{r}\, \mathrm{d}V = \rho * \frac{e^{ikr}}{r} \qquad (r(x) = |x|)$$

will satisfy Helmholtz' equation; that is,

$$\rho * (\Delta + k^2) \frac{e^{ikr}}{r} = -4\pi\rho \ .$$

From this, it is clear that

$$-\frac{1}{4\pi} (\Delta + k^2) \frac{e^{ikr}}{r} = \delta(x) \ ;$$

that is, the function

$$L = -\frac{1}{4\pi} \frac{e^{ikr}}{r}$$

is a fundamental solution of Helmholtz' equation. Consequently, if ρ is a generalized finite function, the convolution

$$u = \rho * \frac{e^{ikr}}{r}$$

will be a generalized solution of the Helmholtz equation (49).

In previous chapters, we have frequently encountered fundamental solutions of equations of the elliptic type. In particular, we saw (Chapter XXVII, section 4) that to every boundary-value problem it is possible to assign a fundamental solution – Green's function – satisfying properly chosen boundary conditions. Furthermore by means of this fundamental solution, the solution of the problem can immediately be written in integral form.

For example, if $G(\xi, x)$ is the Green's function of the self-conjugate boundary-value problem

$$\mathcal{M}u = f \quad \text{when} \quad x \in V - \mathcal{F}V \ ; \qquad u = \psi \quad \text{when} \quad x \in \mathcal{F}V \ , \tag{50}$$

the solution of the problem can be given by means of the integral formula (18) (Chapter XXVII) in the form

$$u(x) = -\iint\limits_{\mathcal{F}V} \psi(\xi)\, Q_\xi\, G(\xi, x)\, dS_\xi - \iiint\limits_{V} f(\xi)\, G(\xi, x)\, dV_\xi \ , \qquad x \in V - \mathcal{F}V \ . \tag{51}$$

We denote by V_x an arbitrarily closed neighbourhood of the point x within the region V. In the region $V - V_x$, the Green's function $G(\xi, x)$ and its first two derivatives are continuous. Consequently, the order of differentiation and integration can be reversed in this region. If we differentiate formula (51) and remember that $\mathcal{M}_x G(\xi, x) = 0$ when $x \in V - \mathcal{F}V - \xi$, we obtain

$$\mathcal{M}u(x) = -\mathcal{M} \iiint\limits_{V_x} f(\xi)\, G(\xi, x)\, dV_\xi \ .$$

If we *formally* reverse the order of differentiation and integration on the right side of this equation and note that $\mathcal{M}u = f$ when $x \in V - \mathcal{F}V$, we obtain the formal identity

$$-\iiint\limits_{V_x} f(\xi)\, \mathcal{M}\, G(\xi, x)\, dV_\xi = f(x) \ .$$

We can give this identity a meaning by considering the fact that the set of functions $f(x)$ coincides with the set of fundamental functions $\varphi(x)$. Then, it is obvious that

$$\mathcal{M}G(\xi, x) = -\delta(x-\xi) \quad \text{when} \quad x \in V - \mathcal{F}V \; . \tag{52}$$

Thus, we arrive at the equation for Green's function formulated in terms of the theory of generalized functions. It can be verified by a more rigorous method.

As we saw in section 4 of Chapter XXVII, a Green's function that solves a given selfconjugate boundary-value problem satisfies the homogeneous boundary condition with the same left member as in the boundary condition for the problem. If we combine this homogeneous boundary condition with eq. (52), we arrive at a formulation of the boundary-value problem determining Green's function in terms of the theory of generalized functions.

Like formula (50), eq. (52) is also valid in the case in which the coefficients in the expression $\mathcal{M}u$ are variable. However, here the concept of a convolution is not defined, and it is impossible to give the solution in the form (48) within the framework of the theory that we have expounded.

Problem

Construct Green's function of the one-dimensional boundary-value problem

$$\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu = f \; , \qquad u|_{x=0} = s_0 \; , \qquad u|_{x=1} = s_1 \; ,$$

where $p(x)$ is a continuous function with continuous first derivatives that does not vanish when $0 \leq x \leq 1$ and where s_0 and s_1 are constants.
Solution: The Green's function $G(\xi, x)$ is a solution of the boundary-value problem

$$\frac{\partial}{\partial x}\left(p\frac{\partial G}{\partial x}\right) + qG = -\delta(x-\xi) \; , \qquad G|_{x=0} = G|_{x=1} = 0 \; . \tag{*}$$

Since the delta function $\delta(x-\xi)$ is an arbitrary function equal to zero when x is less than ξ and equal to unity when x is is greater than ξ (see problem 3 of section 5), the equation (*) for $x = \xi$ is satisfied if

$$G(\xi, \xi-0) = G(\xi, \xi+0) \; , \qquad p(\xi)\frac{\partial G}{\partial x}\bigg|_{x=\xi-0} - p(\xi)\frac{\partial G}{\partial x}\bigg|_{x=\xi+0} = -1 \; . \tag{**}$$

We set

$$G(\xi, x) = \begin{cases} v_0(\xi)u_0(x) & \text{for} \quad x < \xi \; , \\ -v_1(\xi)u_1(x) & \text{for} \quad x > \xi \; , \end{cases}$$

where $u_0(x)$ and $u_1(x)$ are linearly independent solutions of the homogeneous equation

$$\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu = 0 ,$$

the first of which vanishes at $x = 0$ and the second at $x = 1$. Therefore, for all $x \neq \xi$, the requirements of the problem (*) are satisfied and $v_0(\xi)$ and $v_1(\xi)$ are functions that we define in such a way that the relations (**) are satisfied. Substitution into (**) yields

$$v_0(\xi)\, u_0(\xi) - v_1(\xi)\, u_1(\xi) = 0 , \qquad v_0(\xi)\, u_0'(\xi) - v_1(\xi)\, u_1'(\xi) = -\frac{1}{p(\xi)} .$$

This system has a solution since its determinant

$$\Delta(\xi) = -\begin{vmatrix} u_0(\xi) & u_1(\xi) \\ u_0'(\xi) & u_1'(\xi) \end{vmatrix}$$

is equal to the Wronskian determinant of the system of linearly independent solutions $u_0(x)$ and $u_1(x)$ and, hence, is non-zero. For the desired functions, we obtain the expressions:

$$v_0(\xi) = \frac{u_1(\xi)}{p(\xi)\,\Delta(\xi)} , \qquad v_1(\xi) = -\frac{u_0(\xi)}{p(\xi)\,\Delta(\xi)}$$

But the Wronskian determinant can be written in the form *

$$\Delta(\xi) = \Delta(0) \exp\left[-\int_0^{\xi} \frac{p'}{p}\, d\xi\right] ,$$

so that

$$p'(\xi)\,\Delta(\xi) + p(\xi)\,\Delta'(\xi) = [p(\xi)\,\Delta(\xi)]' = 0 .$$

Consequently,

$$p(\xi)\,\Delta(\xi) = \text{constant} .$$

If we multiply the solutions $u_0(x)$ and $u_1(x)$ by constant factors, it is easy to arrange for the constant on the right side to be equal to 1. Green's function again takes the form

$$G(\xi, x) = \begin{cases} u_1(\xi)u_0(x) & \text{for} \quad x < \xi , \\ u_0(\xi)u_1(x) & \text{for} \quad x > \xi . \end{cases}$$

8. *The concept of a generalized Fourier transform*

Various generalized integral transforms can be constructed. We give only the generalized Fourier transform.

To do this, we extend the concept of a linear continuous functional presented in section 2. Instead of real fundamental functions, we introduce complex fundamental functions $\varphi(x)$, that is, finite infinitely-many-times

* See V.I.Smirnov [1], Vol. 2, p. 24.

differentiable point functions that can take complex values. To every complex locally integrable function $f(x)$ we assign a linear continuous functional (a complex generalized function)

$$(f,\varphi) = \int\int\int f^*(x)\ \varphi(x)\ \mathrm{d}V\ , \tag{53}$$

where the asterisk denotes the complex conjugate function. Obviously,

$$(f^*,\varphi) = (f,\varphi^*)^* \tag{54}$$

and

$$(\alpha(x)f(x),\varphi(x)) = (f(x),\ \alpha^*(x)\varphi(x))\ , \tag{55}$$

where $\alpha(x)$ is an infinitely many times differentiable function. We can extend these relations to the case of an arbitrary complex generalized function defined by a rule f.

All the results of the preceding section are valid for complex generalized functions with the obvious modifications of statements that follow from formulae (53) - (55).

Let us set up the Fourier transform

$$\overline{\varphi}(\gamma) = \int_{-\infty}^{\infty} \varphi(x)\ \mathrm{e}^{-\mathrm{i}\gamma x}\ \mathrm{d}x \tag{56}$$

of a one-dimensional fundamental function $\varphi(x)$. Since the function $\varphi(x)$ is finite, this integral is infinitely many times differentiable with respect to γ for arbitrary complex values $\gamma \equiv \gamma' + \mathrm{i}\gamma''$. If we integrate by parts, we obtain

$$\overline{\varphi}^{(n)}(\gamma) \equiv \int_{-\infty}^{\infty} \varphi^{(n)}(x)\ \mathrm{e}^{-\mathrm{i}\gamma x}\ \mathrm{d}x = (\mathrm{i}\gamma)^n\ \overline{\varphi}(\gamma) \qquad (n = 0,1,2,\ldots)\ , \tag{57}$$

from which we obtain the inequalities

$$|\gamma^n\overline{\varphi}(\gamma)| = \left|\int_{-\infty}^{\infty} \varphi^{(n)}(x)\ \mathrm{e}^{-\mathrm{i}\gamma' x}\ \mathrm{e}^{\gamma'' x}\ \mathrm{d}x\right| \leqslant C_n\ \mathrm{e}^{|\gamma''|\,b} \qquad (n = 1,2,\ldots)\ , \tag{58}$$

where the

$$C_n = \left|\int_{-\infty}^{\infty} \varphi^{(n)}(x)\ \mathrm{e}^{-\mathrm{i}\gamma' x}\ \mathrm{d}x\right|$$

are finite numbers and b is the largest value of $|x|$ in the region in which the fundamental function $\varphi(x)$ is non-zero. It follows from these inequalities that the Fourier transforms $\overline{\varphi}(\gamma)$ of the fundamental functions $\varphi(x)$ approach zero more rapidly than any power of $1/\gamma$ as Re γ approaches ∞. This is due to the presence, on the right side of these inequalities, of finite numbers that are independent of Re $\gamma = \gamma'$. Therefore, the left sides of the inequalities are also bounded as $\gamma \to \infty$, from which our assertion follows.

Let us now show, conversely, that if any function $\psi(\gamma)$ of a complex

variable γ is infinitely many times differentiable and satisfies the inequalities

$$|\gamma^n \psi(\gamma)| \leqslant C_n\, e^{|\gamma''| b} \qquad (n = 0, 1, 2, \ldots)\ , \tag{59}$$

where the C_n and b are finite positive numbers, it is the Fourier transform of some finite function

$$\varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(\gamma)\, e^{i\gamma x}\, d\gamma\ . \tag{60}$$

To show this, let us note that the function $\varphi(x)$ defined by this equation is infinitely many times differentiable with respect to the parameter x since, for arbitrary n, the integrals

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (i\gamma)^n\, \psi(\gamma)\, e^{i\gamma x}\, d\gamma\ ,$$

obtained by formal differentiation under the integral sign of the right side of eq. (60), converge absolutely on the basis of inequalities (59) and, consequently, are the corresponding derivatives of $\varphi(x)$. Furthermore, on the basis of a Cauchy formula *, integration over the real axis can be replaced with integration over a straight line that is parallel to it; that is,

$$\begin{aligned}\varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(\gamma)\, e^{i\gamma x}\, d\gamma &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(\gamma' + i\gamma'')\, e^{i\gamma' x}\, e^{-\gamma'' x}\, d\gamma' \\ &= \frac{1}{2\pi}\, e^{-\gamma'' x} \int_{-\infty}^{\infty} \psi(\gamma' + i\gamma'')\, e^{i\gamma' x}\, d\gamma'\ ,\end{aligned} \tag{61}$$

where γ'' is an arbitrary real number. By means of the inequalities

$$|\psi(\gamma)| \leqslant C_0\, e^{|\gamma''| b} \quad \text{and} \quad |\psi(\gamma)| \leqslant \frac{C_2\, e^{|\gamma''| b}}{|\gamma|^2}\ , \tag{62}$$

obtained from inequalities (59) for $n = 0$ and $n = 2$, we may find a positive number C such that

$$|\psi(\gamma)| \leqslant \frac{2\pi C\, e^{|\gamma''| b}}{1 + |\gamma|^2} \leqslant \frac{2\pi C\, e^{|\gamma''| b}}{1 + \gamma'^2}\ ,$$

and therefore

$$|\varphi(x)| \leqslant C\, e^{|\gamma''| b - \gamma'' x} \int_{-\infty}^{\infty} \frac{e^{i\gamma' x}}{1 + \gamma'^2}\, d\gamma'\ .$$

Since the integral on the right side converges, there exists a positive number C' such that

$$|\varphi(x)| \leqslant C'\, e^{|\gamma''| b - \gamma'' x}\ .$$

* See V. I. Smirnov 1), Vol. 3, p. 7.

If $|x| > b$, the number γ'' can be chosen so that the right side of the inequality will be less than an arbitrary given number. Since the left side does not depend on the number γ'' the function $\varphi(x)$ vanishes whenever $|x| > b$. Consequently, the function $\varphi(x)$ is not only infinitely-many-times differentiable, but is also finite; that is, it is a fundamental function, as was asserted.

Thus, the Fourier transform $\varphi(\gamma)$ of an arbitrary function $\varphi(x)$ represents a function that is infinitely many times differentiable and that satisfies inequalities of the form (59). Conversely, an arbitrary function $\psi(\gamma)$ that satisfies these conditions is the Fourier transform of some function belonging to the set of functions $\varphi(x)$. On the basis of Fourier's integral theorem, this correspondence is then one-to-one *. Thus, the Fourier transform sets up a one-to-one correspondence between the sets (spaces) of the functions $\varphi(x)$ and $\psi(\gamma)$.

This ensures the existence of a correspondence or (as they are called) of dual relationships between the operations in both spaces (in particular, between operations involving limiting processes).

It follows from formula (57) that multiplication of the functions $\psi(\gamma)$ by $i\gamma$ corresponds to differentiation of the functions $\varphi(x)$ with respect to x. If we differentiate formula (56) with respect to γ, we easily see that, conversely, multiplication of the functions $\varphi(x)$ by $-ix$ corresponds to differentiation of the functions $\psi(\gamma)$ with respect to γ.

It follows from the relationship

$$\int_{-\infty}^{\infty} e^{-i\gamma x}\, \varphi(x-h)\, dx = \int_{-\infty}^{\infty} e^{-i\gamma(\zeta+h)}\, \varphi(\zeta)\, d\zeta$$

$$= e^{-i\gamma h} \int_{-\infty}^{\infty} e^{-i\gamma\zeta}\, \varphi(\zeta)\, d\zeta = e^{-i\gamma h}\, \overline{\varphi}(\gamma)$$

that the mapping $\overline{\varphi}(\gamma) \to e^{-i\gamma h}\, \overline{\varphi}(\gamma)$ in the space of the functions $\psi(\gamma)$ corresponds to the translation $\varphi(x) \to \varphi(x-h)$ in the space of the functions $\varphi(x)$. Conversely, the mapping $\overline{\varphi}(\gamma) \to \overline{\varphi}(\gamma+h)$ corresponds to the mapping $\varphi(\gamma) \to e^{-ixh}\, \varphi(x)$. The proof is analogous to the preceding proof.

Dual relationships can also be set up for other linear operations.

Let us consider the identity

$$e^{-ixh}\, \varphi(x) \equiv \varphi(x) \sum_{\nu=0}^{\infty} \frac{h^\nu}{\nu!} (-ix)^\nu = \sum_{\nu=0}^{\infty} \frac{h^\nu}{\nu!} (-ix)^\nu\, \varphi(x) .$$

By using the relationships given above, we find its dual formula:

$$\psi(\gamma+h) = \sum_{\nu=0}^{\infty} \frac{h^\nu}{\nu!} \psi^{(\nu)}(\gamma) , \tag{63}$$

* Different continuous functions have different Fourier transforms and, conversely, if the Fourier transform of two continuous functions do not coincide, the functions are different. See, for example, Smirnov [1], Vol. 2, p. 160.

which is simply Taylor's series. Consequently, the function $\psi(\gamma)$ can be expanded in a Taylor series about an arbitrary point γ at an arbitrary complex value h.

It is possible to construct linear continuous functionals, generalized functions, in the space of the functions $\psi(\gamma)$, just as it is in the space of functions $\varphi(x)$. In the space $\psi(\gamma)$ a generalized function is called regular if it can be represented by an expression of the form

$$(g, \psi) = \int_{-\infty}^{\infty} g(\gamma)\ \psi(\gamma)\ d\gamma \tag{64}$$

using some absolutely integrable function $g(\gamma)$ of a complex variable γ.

We denote the sets of generalized functions in the spaces of the functions $\varphi(x)$ and $\psi(\gamma)$ by K' and Z', respectively. The symbols $f \in K'$ and $g \in Z'$ then indicate which of the sets K' or Z' the generalized functions f and g belong to.

A number of the results that we have obtained above for the functions $f \in K'$ can be carried over to the generalized functions $g \in Z'$. In particular, the definitions of convergence and differentiation can be carried over. It follows from the last remark that the generalized functions $g \in Z'$ are infinitely many times differentiable. Furthermore, they are also *analytic*; that is, they can be expanded in a generalized Taylor series:

$$g(\gamma + h) = \sum_{\alpha=0}^{\infty} \frac{h^\alpha}{\alpha!} g^{(\alpha)}(\gamma) , \tag{65}$$

where $g(\gamma + h)$ represents the displacement of the generalized function $g(\gamma)$ by an amount h. This is true because

$$\Big(\sum_{\alpha=0}^{\infty} \frac{h^\alpha}{\alpha!} g^{\alpha}(\gamma), \psi(\gamma)\Big) = \Big(g(\gamma), \sum_{\alpha=0}^{\infty} \frac{(-h)^\alpha}{\alpha!} \psi^{\alpha}(\gamma)\Big) .$$

The series in the functional on the right side converges to $\psi(\gamma - h)$ because, as was shown above, the function $\psi(\gamma)$ can be expanded in a Taylor series. Also, from the definition of a translation, $(g(\gamma), \psi(\gamma - h)) = (g(\gamma + h), \psi(\gamma))$, from which formula (65) follows.

We are now able to define a Fourier transformation of generalized functions. As above, we must find a definition that (in the case of generalized functions derived from formula (64) by using point functions having classical Fourier transforms), follows from the conventional definition and from the rules of classical integration. If a function $f(x)$ has a Fourier transform $\bar{f}(\gamma)$, we have

$$(f,\varphi) \equiv \int_{-\infty}^{\infty} f^*(x)\,\varphi(x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(x)\left[\int_{-\infty}^{\infty} \bar{\varphi}(\gamma)\,e^{ix\gamma}\,d\gamma\right]dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\varphi}(\gamma)\left[\int_{-\infty}^{\infty} f^*(x)\,e^{ix\gamma}\,dx\right]d\gamma = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\varphi}(\gamma)\,[f(x)\,e^{-ix\gamma}\,dx]^*\,d\gamma$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}^*(\gamma)\,\bar{\varphi}(\gamma)\,d\gamma \equiv \frac{1}{2\pi}(\bar{f},\bar{\varphi}) .$$

The relationship

$$(\bar{f},\bar{\varphi}) = 2\pi(f,\varphi) \tag{66}$$

is called Parseval's equation. Since it is a relationship between generalized functions, it relates the generalized function $\bar{f} \in Z'$ to the generalized function $f \in K'$. It can also be taken as the general definition of a Fourier transform of the generalized function f. Specifically, we shall call the generalized function $\bar{f} \in Z'$ defined by eq. (66) the Fourier transform of the generalized function $f \in K'$. We note that when the *left* side of eq. (66) is given, it defines the inverse Fourier transform for generalized functions.

Let us take some examples. Let us find the Fourier transform of the generalized function 1. By definition

$$(\bar{1},\bar{\varphi}) = 2\pi(1,\varphi) = 2\pi\int_{-\infty}^{\infty}\varphi(x)\,dx = 2\pi\left[\int_{-\infty}^{\infty}\varphi(x)\,e^{-ix\gamma}\,dx\right]_{\gamma=0}$$

$$= 2\pi\bar{\varphi}(0) + 2\pi(\delta(\gamma),\bar{\varphi}(\gamma)),$$

where $\delta(\gamma) \in Z'$ is the delta function of the set of functions $\psi(\gamma)$. Therefore,

$$\bar{1} = 2\pi\delta(\gamma) . \tag{67}$$

Let us find the Fourier transform $\bar{\delta}(\gamma)$ of the delta function $\delta(x) \in K'$:

$$(\bar{\delta},\bar{\varphi}) = 2\pi(\delta,\varphi) = 2\pi\varphi(0) = \left[\int_{-\infty}^{\infty}\bar{\varphi}(\gamma)\,e^{i\gamma x}\,d\gamma\right]_{x=0} = \int_{-\infty}^{\infty}\bar{\varphi}(\gamma)\,d\gamma = (1,\bar{\varphi}) .$$

Therefore,

$$\bar{\delta}(\gamma) = 1 . \tag{68}$$

Let us find the Fourier transform $\overline{x^n}$ of the generalized function x^n (where n is a non-negative integer):

$$(\overline{x^n}, \overline{\varphi}) = 2\pi \int_{-\infty}^{\infty} x^n \varphi(x)\, dx = 2\pi \left[\int_{-\infty}^{\infty} x^n \varphi(x)\, e^{-i\gamma x}\, dx\right]_{\gamma=0}$$

$$= 2\pi\, i^n \left[\int_{-\infty}^{\infty} (-ix)^n \varphi(x)\, e^{-i\gamma x}\, dx\right]_{\gamma=0}$$

$$= 2\pi\, i^n \left[\frac{d^n}{d\gamma^n} \int_{-\infty}^{\infty} \varphi(x)\, e^{-i\gamma x}\, dx\right]_{\gamma=0}$$

$$= 2\pi\, i^n \left.\frac{d^n \overline{\varphi}}{d\gamma^n}\right|_{\gamma=0} = 2\pi\, (-i)^n\, (\delta^{(n)}(\gamma), \overline{\varphi}(\gamma)) .$$

Therefore,

$$\overline{x^n} = 2\pi \left(-i \frac{d}{d\gamma}\right)^n \delta(\gamma) . \tag{69}$$

Finally, let us find the Fourier transform $\overline{e^{\zeta x}}$ of the generalized exponential function $e^{\zeta x}$ (where ζ is an arbitrary complex number):

$$(\overline{e^{\zeta x}}, \overline{\varphi}) = 2\pi(e^{\zeta x}, \varphi) = 2\pi \left(\sum_{\alpha=0}^{\infty} \frac{(\zeta x)^\alpha}{\alpha!}, \varphi\right) = \left(\sum_{\alpha=0}^{\infty} \overline{\frac{(\zeta x)^\alpha}{\alpha!}}, \overline{\varphi}\right)$$

$$= 2\pi \left(\sum_{\alpha=0}^{\infty} \frac{(-i\zeta)^\alpha}{\alpha!} \delta^{(\alpha)}(\gamma), \overline{\varphi}\right) .$$

The sum

$$\sum_{\alpha=0}^{\infty} \frac{(-i\zeta)^\alpha}{\alpha!} \delta^{(\alpha)}(\gamma)$$

represents the Taylor series for the delta function with $h = -i\zeta$. Since the generalized functions in the set Z' are analytic; the Taylor series converges to $\delta(\gamma - i\zeta)$. Consequently,

$$\overline{e^{x}} = 2\pi\delta(\gamma - i\zeta) . \tag{70}$$

Whereas the ordinary Fourier transform exists only for functions that approach zero sufficiently rapidly when their arguments increase without bound in absolute value, the generalized Fourier transform can be applied to functions of arbitrary rate of increase. This is clear from the fact that the integrals

$$(f, \varphi) = \int_{-\infty}^{\infty} f(x)\, \varphi(x)\, dx ,$$

are, because of the finiteness of the function $\varphi(x)$ meaningful independently of the nature of the increase of the function $f(x)$ within infinite limits. Therefore, the generalized Fourier transform is a convenient base for the construction of an operational calculus, which up to now has usually been

developed on the basis of the classical Laplace and Fourier transforms. The use of the generalized Fourier transform does not essentially change the formal apparatus of the operational calculus. The basic change consists only in the introduction of the delta function and its derivative, which makes it possible to construct transformations of functions that increase without bound at infinity. We shall confine ourselves to these remarks on the subject, since this development is beyond the scope of our book.

We also emphasize the following fact. The generalized Fourier transform is applicable to all generalized functions without exception. Therefore, the results that are obtained using it are always valid and lead at least to generalized solutions.

The theory of generalized Fourier transforms can easily be extended to the multi-dimensional case, though we shall not pursue the subject.

Problems

1. Show that the delta function defined by the equation

$$(\delta(\gamma - h), \psi(\gamma)) = (\delta(\gamma), \psi(\gamma + h)) = \psi(h) ,$$

is a generalized function belonging to the set Z'.

2. Expand the delta function $\delta(\gamma - h)$ in a series of powers of h.
Answer:

$$\delta(\gamma - h) = \sum_{\alpha=0}^{\infty} \frac{(-h)^\alpha}{\alpha!} \delta^{(\alpha)}(\gamma) .$$

3. Find the Fourier transform of the generalized function $P(d/dx)f(x)$, where $P(d/dx)$ is a polynomial with constant coefficients in d/dx.
Answer:

$$\overline{P(d/dx)\, f(x)} = P(i\gamma)\, \bar{f}(\gamma) .$$

4. Show that the Fourier transform of a periodic generalized function $f(x)$ with period 2π is of the form

$$\bar{f}(\gamma) = \sum_{\alpha=-\infty}^{\infty} C_\alpha\, \delta(\gamma + \alpha) ,$$

where the C_α are the coefficients of the expansion of the function $f(x)$ in a complex Fourier series

$$f(x) = \sum_{\alpha=-\infty}^{\infty} C_\alpha\, e^{i\alpha x} .$$

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